Original scientific paper
Accepted 27. 10. 2016.

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3rd Class Circular Curves in Quasi-Hyperbolic Plane Obtained by Projective Mapping

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ABSTRACT

The metric in the quasi-hyperbolic plane is induced by an absolute figure $\mathcal{F}_{\mathbb{QH}} = \{F, \mathbf{f_1}, \mathbf{f_2}\}$, consisting of two real lines $\mathbf{f_1}$ and $\mathbf{f_2}$ incident with the real point F. A curve of class n is circular in the quasi-hyperbolic plane if it contains at least one absolute line.
The curves of the 3rd class can be obtained by projective mapping, i.e. obtained by projectively linked pencil of curves of the 2nd class and range of points. In this article we show that the circular curves of the 3rd class of all types, depending on their position to the absolute figure, can be constructed with projective mapping.

Key words: projectivity, circular curve of the 3rd class, quasi-hyperbolic plane

MSC2010: 51M15, 51N25

Cirkularne krivulje 3. razreda u kvazihiperboličnoj ravnini dobivene projektivnim preslikavanjem

SAŽETAK

U kvazihiperboličnoj ravnini metrika je inducirana s apsolutnom figurom $\mathcal{F}_{\mathbb{QH}} = \{F, \mathbf{f_1}, \mathbf{f_2}\}$ koja se sastoji od dva realna pravca $\mathbf{f_1}$ i $\mathbf{f_2}$ sa sjecištem u realnoj točki F. Za krivulju razreda n kažemo da je cirkularna u kvazihiperboličnoj ravnini ako sadrži barem jedan apsolutni pravac.
Krivulje 3. razreda se mogu dobiti projektivnim pridruživanjem između pramena krivulja 2. razreda i niza točaka. U ovom ćemo članku pokazati kako se svi tipovi cirkularnih krivulja 3. razreda mogu konstruirati projektivnim preslikavanjem.

Ključne riječi: projektivitet, cirkularna krivulja 3. razreda, kvazihiperbolična ravnina

1 Introduction

In the 19th century F. Klein founded the basis of the modern approach to geometry by defining it as the study of the properties of a space which are invariant under a given group of transformations. Later on this was know as *Erlangen program* according to the fact that Klein gave his first lecture on this subject at the University of Erlangen, [5]. There exist nine plane geometries with projective metric on a line and on a pencil of lines which can be parabolic, hyperbolic or elliptic. Due to Cayley's influence on Klein the geometries are denoted as Cayley-Klein projective metrics. Furthermore, each of these projective metrics can be embedded in the projective plane $\mathcal{P}_2 = \{\mathcal{P}, \mathcal{L}, \mathbf{I}\}$ where then an absolute figure, given as a proper or singular conic, induces the metric in the plane, [6, 7, 13] (for n-dimension see [12]).

The *quasi-hyperbolic plane*, denoted as $\mathbb{QH}_2$, is a projective plane where the metric is induced by the absolute figure $\mathcal{F}_{\mathbb{QH}} = \{F, \mathbf{f_1}, \mathbf{f_2}\}$ consisting of a pair of real lines $\mathbf{f_1}$, $\mathbf{f_2}$ intersecting at a real point F, [8, 10, 13]. The point F is called the *absolute point* and lines $\mathbf{f_1}$, $\mathbf{f_2}$ are called the *absolute lines*. In the Cayley-Klein model of the quasi-hyperbolic plane only the geometric objects inside of one projective angle between absolute lines are observed, while the points, lines and line segments inside the other angle are omitted. We observe the projectively extended quasi-hyperbolic plane where all points and lines of the projective plane are included as in [10].

In the sense of the Erlangen program, for the fundamental group of transformations in $\mathbb{QH}_2$ we use the *4-parameter general quasi-hyperbolic group of similarities* $\mathfrak{G}_4$, [8]. Transformations are of the form

$$[u_0,u_1,u_2] \mapsto [\alpha_0 u_0, \alpha_1 u_0 + \alpha_2 u_1 + \alpha_3 u_2, \alpha_4 u_0 \pm \alpha_3 u_1 \pm \alpha_2 u_2],$$
$$\alpha_i \in \mathbb{R}, \quad i = \{0 \dots 4\}, \quad \pm\alpha_2^2 \pm \alpha_3^2 \neq 0,$$

whereby the absolute figure $\mathcal{F}_{\mathbb{QH}}$ is determined by

$$F = (1,0,0),\ \mathbf{f_1} = [0,1,1],\ \mathbf{f_2} = [0,-1,1].$$

Definition 1 *A line passing through the absolute point F is called isotropic line and a point incident with the absolute line $\mathbf{f_1}$ or $\mathbf{f_2}$ is called isotropic point.*

For some further results on the basic notions in $\mathbb{QH}_2$ see [10].

Definition 2 *If the intersection of the curve ζ of the class n and the pencil (F), in $\mathbb{QH}_2$, is the absolute line $\mathbf{f_1}$ with the intersection multiplicity t and the absolute line $\mathbf{f_2}$ with the intersection multiplicity r, than ζ is said to be a $(t+r)$-circular curve or circular curve of type (t,r). $t+r$ is the degree of circularity, and if $t+r=n$ then the curve ζ is entirely circular.*

In further classification we will not distinguish circular curve of the type (t,r) from the one of the type (r,t) since the possibility of constructing one of them implies the possibility of constructing the other.

In accordance to the group $\mathfrak{G}_4$, proper curves of the 2nd class in $\mathbb{QH}_2$ are classified into nine types, see [1, 10]. They can also be classified in accordance to its degree and type of circularity as following:

i) non-circular curves of the 2nd class: ellipses (e), hyperbolas (h_1,h_2,h_3), parabolas (p);

ii) 1-circular curves of the 2nd class: special hyperbolas $(h_{s1},h_{s2}$, type of circularity $(1,0))$;

iii) 2-circular curves of the 2nd class: circles $(c$, type of circularity $(1,1))$, special parabolas $(p_s$, type of circularity $(2,0))$.

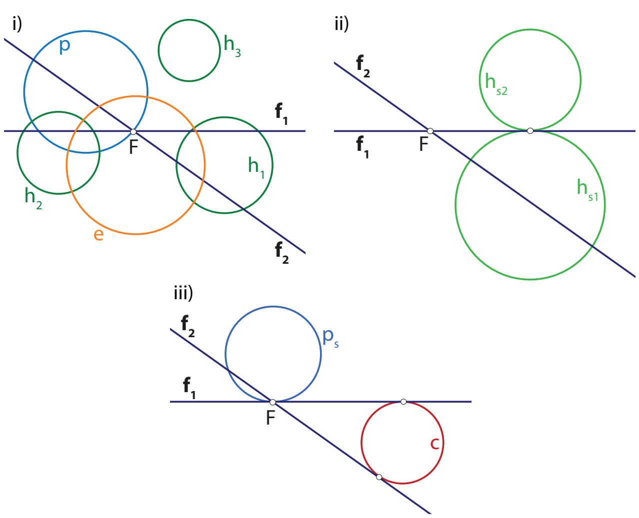

Figure 1: *Classification of the curves of the 2nd class in $\mathbb{QH}_2$ according to their degree of circularity*

Remark 1 *In all figures of the article the class curves are drawn as point objects as we are used to, although they are line envelopes in the quasi-hyperbolic plane.*

The circular curves of the 3rd class can be classified, according to their position with respect to $\mathcal{F}_{\mathbb{QH}}$, into the following types and subtypes:

- 1-circular curves of the 3rd class
  - type of circularity (1,0)
    - a) the curve contains the absolute line $\mathbf{f_1}$ and two isotropic lines that are conjugate imaginary;
    - b) the curve contains the absolute line $\mathbf{f_1}$ and two isotropic lines that are real and distinct;
    - c) the curve contains the absolute line $\mathbf{f_1}$ and two isotropic lines that coincide;
    - d) the curve contains the absolute line $\mathbf{f_1}$ and an isotropic double line with two conjugate imaginary tangent points (isolated double line);
    - e) the curve contains the absolute line $\mathbf{f_1}$ and an isotropic double line with two real and distinct tangent points (double tangent line);
    - f) the curve contains the absolute line $\mathbf{f_1}$ and an isotropic double line with two tangent points that coincide (inflection line);

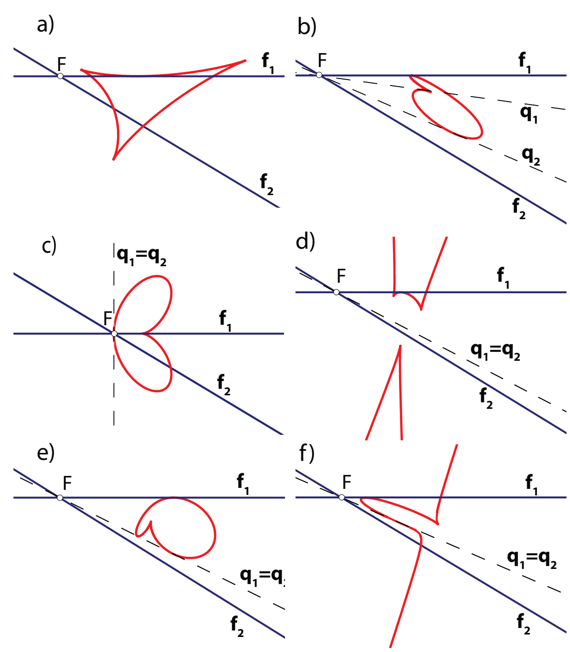

Figure 2: *Classification of the 1-circular curves of the 3rd class in $\mathbb{QH}_2$*

- 2-circular of the 3rd class
  - type of circularity (1,1)
    - a) the curve contains both absolute lines $\mathbf{f_1}$ and $\mathbf{f_2}$;
  - type of circularity (2,0)
    - b) the curve contains the absolute line $\mathbf{f_1}$ where the absolute point F is the tangent point;
    - c) the absolute line $\mathbf{f_1}$ is an isolated double line of the curve;
    - d) the absolute line $\mathbf{f_1}$ is a double tangent line of the curve;
    - e) the absolute line $\mathbf{f_1}$ is an inflection line of the curve;

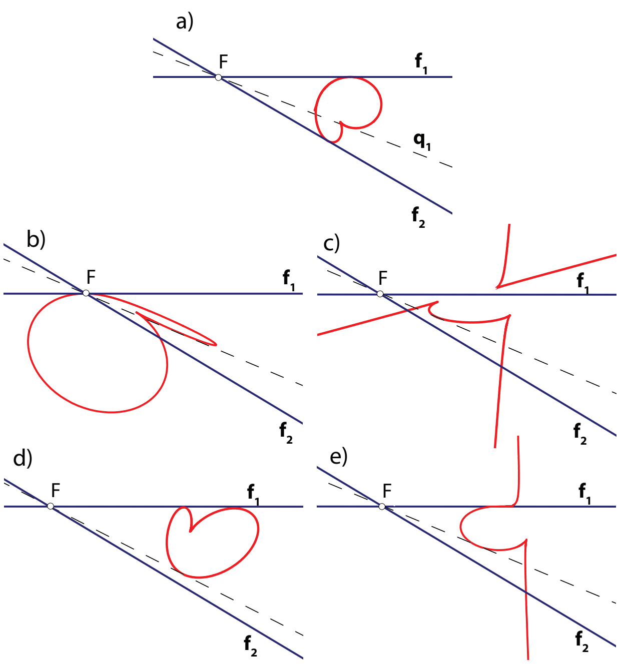

Figure 3: *Classification of the 2-circular curves of the 3rd class in* $\mathbb{QH}_2$

- 3-circular curves of the 3rd class
  - type of circularity (2,1)
    - a) the curve contains both absolute lines $\mathbf{f_1}$, $\mathbf{f_2}$ and the absolute point F is the tangent point of the line $\mathbf{f_1}$;
    - b) the curve contains both absolute lines $\mathbf{f_1}$, $\mathbf{f_2}$ such that $\mathbf{f_1}$ is an isolated double line;
    - c) the curve contains both absolute lines $\mathbf{f_1}$, $\mathbf{f_2}$ such that $\mathbf{f_1}$ is a double tangent line
    - d) the curve contains both absolute lines $\mathbf{f_1}$, $\mathbf{f_2}$ such that $\mathbf{f_1}$ is an inflection line;
  - type of circularity (3,0)
    - e) the absolute line $\mathbf{f_1}$ is a double tangent with one tangent point at the absolute point F;
    - f) the absolute line $\mathbf{f_1}$ is an inflection line with the tangent point at the absolute point F;
    - g) the curve contains the absolute line $\mathbf{f_1}$ and has a cusp at the absolute point F.

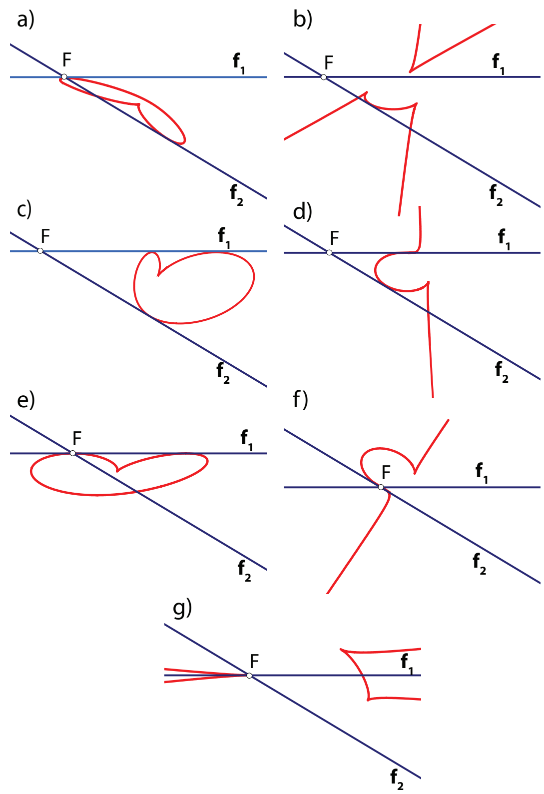

Figure 4: *Classification of the 3-circular or entirely circular curves of the 3rd class in* $\mathbb{QH}_2$

The aim of this article is to construct every type of circular curves of the 3rd class in the quasi-hyperbolic plane by using projective mapping. The classification of circular curves, according to their position with respect to the absolute figure, obtained by projective mapping in some other Cayley-Klein projective metrics can be found in [2, 3, 4, 11].

2 Projective mapping

Let points P_1, P_2 and curves of the 2nd class ζ_1, ζ_2 be given. The associated symmetric bilinear form for the 2nd class curves is given with

$$\zeta_1 \ldots \quad f_{\zeta_1}(\mathbf{u},\mathbf{v}) := \mathbf{u}^T C_1 \mathbf{v} = 0,$$
$$\zeta_2 \ldots \quad f_{\zeta_2}(\mathbf{u},\mathbf{v}) := \mathbf{u}^T C_2 \mathbf{v} = 0,$$

and in the following the the curves ζ_1 and ζ_2 will be identified with its corresponding matrix representation C_1 and C_2. The result of a projective mapping

$$\pi : [C_1, C_2] \mapsto [P_1, P_2],$$
$$\pi(C_1 + \lambda C_2) = P_1 + \lambda P_2, \quad \forall \lambda \in \mathbb{R} \cup \infty,$$

between the pencil of the 2nd class curves $[C_1, C_2]$ and the range of points $[P_1, P_2]$ is a curve of the 3rd class k_π^3 given by the equation

$$k_\pi^3 \ldots \; F(\mathbf{u}) \equiv \mathbf{u}^T C_1 \mathbf{u} \cdot P_2^T \mathbf{u} - \mathbf{u}^T C_2 \mathbf{u} \cdot P_1^T \mathbf{u} = 0. \qquad (1)$$

The curve k_π^3 contains the following nine lines: four basic lines of the pencil $[C_1, C_2]$, basic line of the range $[P_1, P_2]$, two intersection lines of ζ_1 and (P_1), two intersection lines of ζ_2 and (P_2). It is known that the number of lines required for determination of a curve of the 3rd class is nine, but nine lines do not determine a single curve of the 3rd class in every case, [9]. For defining the projectivity we need three pairs of elements (ζ_1, P_1), (ζ_2, P_2) and (ζ_3, P_3). Furthermore, we should point out that although the proportional matrices C_1, C_2, P_1, P_2 and αC_1, βC_2, γP_1, δP_2 represent the same two curves of the 2nd class and two points, the corresponding curves of the 3rd class are different, but they properties of circularity stay the same.

Let us observe a line $\mathbf{v} \in k_\pi^3$, such that the curve k_π^3 is obtained by a projective mapping π and without loss of generality we can assume $\mathbf{v} \in C_1$, $P_1 \in \mathbf{v}$ thus

$$\mathbf{v}^T C_1 \mathbf{v} = 0, \; P_1^T \mathbf{v} = 0$$

is valid. The behaviour of the line $\mathbf{v}$ can be studied by observing the intersection lines of curve k_π^3 and a pencil (X) such that $X \in \mathbf{v}$. Therefore an arbitrary point X on the line $\mathbf{v}$ can be given as

$$X \ldots \quad \mathbf{v} + t\mathbf{w}, \; t \in \mathbb{R} \cup \infty,$$

hence intersection lines of k_π^3 and (X) are determined by the roots of the following polynomial

$$F(\mathbf{v} + t\mathbf{w}) = F_1(\mathbf{v},\mathbf{w}) + t^2 F_2(\mathbf{v},\mathbf{w}) + t^3 F_3(\mathbf{v},\mathbf{w}), \qquad (2)$$

where

$$F_1(\mathbf{v},\mathbf{w}) = 2P_2^T \mathbf{v} \cdot \mathbf{v}^T C_1 \mathbf{w} - P_1^T \mathbf{w} \cdot \mathbf{v}^T C_2 \mathbf{v},$$
$$F_2(\mathbf{v},\mathbf{w}) = P_2^T \mathbf{v} \cdot \mathbf{w}^T C_1 \mathbf{w} + 2P_2^T \mathbf{w} \cdot \mathbf{v}^T C_1 \mathbf{w} - 2P_1^T \mathbf{w} \cdot \mathbf{v}^T C_2 \mathbf{w},$$
$$F_3(\mathbf{v},\mathbf{w}) = P_2^T \mathbf{w} \cdot \mathbf{w}^T C_1 \mathbf{w} - P_1^T \mathbf{w} \cdot \mathbf{w}^T C_2 \mathbf{w}.$$

From (2) we can conclude the following statements:

- tangent point on the regular line $\mathbf{v}$ of the curve k_π^3 is given by the equation
$$F_1(\mathbf{v},\mathbf{w}) = 0; \qquad (3)$$
- necessary condition to gain $\mathbf{v}$ as a double line of the curve k_π^3 is
$$F_1(\mathbf{v},\mathbf{w}) = 0, \quad \forall \mathbf{w}; \qquad (4)$$
- tangent points on a double line $\mathbf{v}$ of the curve k_π^3 are given by the equation
$$F_2(\mathbf{v},\mathbf{w}) = 0; \qquad (5)$$
- necessary condition to gain a cusp at X on the line $\mathbf{v}$ for the curve k_π^3 is if the equation (5) is valid for every line $\mathbf{w}$ such that $X \in \mathbf{w}$.

Remark 2 *Generally there are three possible positions for a curve of the 2nd class ζ_1 and its line $\mathbf{v}$:*

a) the curve ζ_1 is a proper curve and the equation $\mathbf{v}^T C_1 \mathbf{w} = 0$ is its the tangent point on the line $\mathbf{v}$;

b) the curve ζ_1 is a singular curve, but $\mathbf{v}$ is not its singular line, i. e. $\zeta_1 = (Z_1) \cup (\hat{Z}_1)$, $Z_1 \in \mathbf{v}$, $\hat{Z}_1 \notin \mathbf{v}$. The point Z_1 is the tangent point at $\mathbf{v}$ and its equation is $\mathbf{v}^T C_1 \mathbf{w} = 0$;

c) the curve ζ_1 is a singular curve and $\mathbf{v}$ is its singular line, i. e. $\zeta_1 = (Z_1) \cup (\hat{Z}_1)$, $Z_1, \hat{Z}_1 \in \mathbf{v}$. The equation $\mathbf{v}^T C_1 \mathbf{w} = 0$ is valid for every line $\mathbf{w}$.

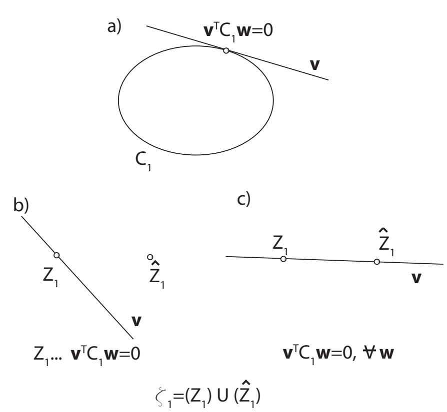

Figure 5: *Positions of the 2nd class curve ζ_1 and its line $\mathbf{v}$*

Furthermore, in respect to the basic elements of the mapping π there are four different positions for a line $\mathbf{v}$ of the curve k_π^3 such that $\mathbf{v} \in \zeta_1$, $P_1 \in \mathbf{v}$:

- $\mathbf{v} \notin \zeta_2, \quad P_2 \notin \mathbf{v}$;
- $\mathbf{v} \in \zeta_2, \quad P_2 \notin \mathbf{v}$;
- $\mathbf{v} \notin \zeta_2, \quad P_2 \in \mathbf{v}$;
- $\mathbf{v} \in \zeta_2, \quad P_2 \in \mathbf{v}$.

Taking in consideration the remark 2 we could discuss all these cases, but in the next section we will present only some of them. By selecting different corresponding pairs (ζ_1, P_1), (ζ_2, P_2) of the projective mapping π we can obtain circular curves of the same type. Therefore, for every type we will present one construction.

Figure 6 represents an example of the entirely circular curve of the 3rd class obtained by the projective mapping π where the corresponding pairs of the mapping are (ζ_1, P_1), (ζ_2, P_2), (ζ_3, P_3), such that curves $\zeta_1 = (Z_1) \cup (\hat{Z}_1)$ and $\zeta_2 = (Z_2) \cup (\hat{Z}_2)$ are singular. The red curve is obtained as a set of tangent points of the curve k^3_π calculated in the software *Wolfram Mathematica*, and the figure is drawn in dynamic software *Geometer's Sketchpad*. As mentioned earlier in Remark 1 it is customary to represent curves as point objects, therefore on the remaining figures in the article curves will be presented in this way.

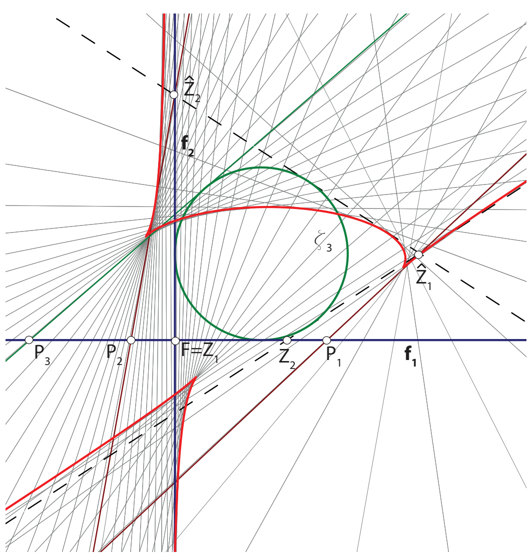

Figure 6: *Circular curve of the 3rd class with the circularity type (2,1) in $\mathbb{QH}_2$*

3 1-circular curves of the 3rd class in $\mathbb{QH}_2$

From the equation (1), as we already mentioned, the curve k^3_π obtained by projective mapping $\pi : [C_1, C_2] \mapsto [P_1, P_2]$ contains nine specific lines, therefore only by picking certain pencils of the 2nd class curves or ranges of points we can ensure the circularity of the curve k^3_π. For instance, if one basic line of the pencils of the 2nd class curves or the basic line of the point range is the absolute line $\mathbf{f_1}$ then the obtained curve k^3_π is 1-circular curve of type $(1,0)$.

Let us observe the case $\mathbf{v} \in \zeta_1$, $P_1 \in \mathbf{v}$, $\mathbf{v} \notin \zeta_2$, $P_2 \notin \mathbf{v}$ when the curve ζ_1 is a proper curve of the 2nd class. From the equation (3) we can conclude that if P_1 is the tangent point of the curve ζ_1 then P_1 is also a tangent point of the curve k^3_π.

Theorem 1 *Let $[C_1, C_2]$ be a pencil of 2nd class curves and $[P_1, P_2]$ a range of isotropic points in $\mathbb{QH}_2$. The result of the projective mapping $\pi : [C_1, C_2] \mapsto [P_1, P_2]$ gives a 1-circular curve of the 3rd class k^3_π of type (1,0) or (0,1). If the curve of the 2nd class corresponding to the absolute point F is an ellipse, hyperbola or parabola then the remaining two isotropic lines of k^3_π are conjugate imaginary, real and distinct or coincide respectively.*

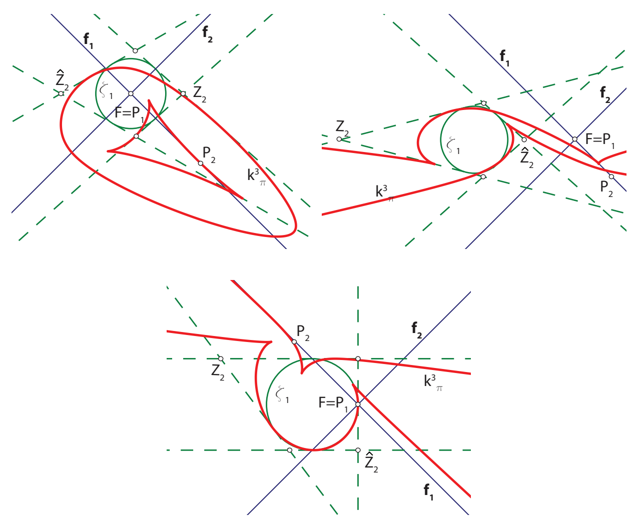

Figure 7: *1-circular curves of the 3rd class of type a, b and c*

Let us observe the case $\mathbf{v} \in \zeta_1, \zeta_2$, $P_1 \in \mathbf{v}$, $P_2 \notin \mathbf{v}$ when the curve ζ_1 is a singular curve of the 2nd class with a singular line $\mathbf{v}$, $\zeta_1 = (Z_1) \cup (\hat{Z}_1)$, $Z_1, \hat{Z}_1 \in \mathbf{v}$. The curves of the pencil $[C_1, C_2]$ are touching at some point on the line $\mathbf{v}$ and the condition (4) is fulfilled, hence the line $\mathbf{v}$ is a double line of the curve k^3_π. The tangent points of the double line are given with the equation (5).

Theorem 2 *Let $[C_1, C_2]$ be a pencil of 2nd class curves with a common tangent point on the isotropic line $\mathbf{v}$, $[P_1, P_2]$ a range of isotropic points on the absolute line $\mathbf{f_1}$ and the curve k^3_π the result of the projective mapping $\pi : [C_1, C_2] \mapsto [P_1, P_2]$ in $\mathbb{QH}_2$. If the absolute point F is the corresponding point to the singular 2nd class curve with the singular line $\mathbf{v}$, then the curve k^3_π is a 1-circular curve of the 3rd class of type (1,0) with the double line $\mathbf{v}$.*

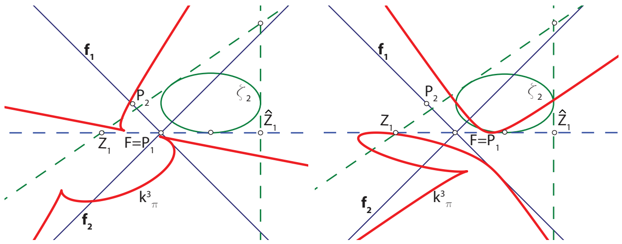

Figure 8: *1-circular curves of the 3rd class of type d and e*

Let us observe the case $P_1, P_2 \in \mathbf{v}$, $\mathbf{v} \in \zeta_1, \zeta_2$. The condition (4) is fulfilled, thus the line $\mathbf{v}$ is the double line of k^3_π. Tangent points on the line $\mathbf{v}$ are given with the equation (5) which in this case is

$$P_2^T \mathbf{w} \cdot \mathbf{v}^T C_1 \mathbf{w} - P_1^T \mathbf{w} \cdot \mathbf{v}^T C_2 \mathbf{w} = 0. \tag{6}$$

One tangent point at the line $\mathbf{v}$ of the curve k^3_π coincides with the tangent point of the curve ζ_1 if and only if P_1 is the tangent point of ζ_1 or curve ζ_1 and ζ_2 are touching.

If the latter case, if the curves ζ_1, ζ_2 are touching then the whole pencil $[C_1, C_2]$ has a common tangent point on the line $\mathbf{v}$. Furthermore, there exists a singular 2nd class curve with the singular line $\mathbf{v}$ and with out loss of generality we can assume it is the curve ζ_1. The equation (6) is of the form

$$P_1^T \mathbf{w} \cdot \mathbf{v}^T C_2 \mathbf{w} = 0,$$

hence one tangent point on the line $\mathbf{v}$ of the curve k^3_π is the common tangent point of $[C_1, C_2]$ while the other one is the point of the range that corresponds to the singular 2nd class curve of $[C_1, C_2]$ with the singular $\mathbf{v}$. These two tangent points can coincide and in that case the line $\mathbf{v}$ is an inflection line of the curve k^3_π.

Theorem 3 *Let $[C_1, C_2]$ be a pencil of special hyperbolas of type $(1,0)$ with a common tangent point on the isotropic line $\mathbf{v}$, $[P_1, P_2]$ a range of points on $\mathbf{v}$ and the curve k^3_π the result of the projective mapping $\pi : [C_1, C_2] \mapsto [P_1, P_2]$ in $\mathbb{QH}_2$. If the corresponding point to the singular 2nd class curve with the singular line $\mathbf{v}$ is the common tangent point of the pencil $[C_1, C_2]$, then the curve k^3_π is 1-circular curve of type (1,0) with the inflection line $\mathbf{v}$.*

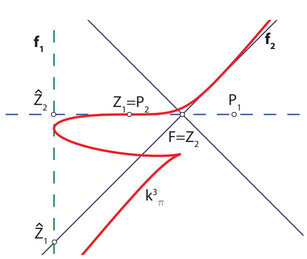

Figure 9: *1-circular curve of the 3rd class of type f*

3.1 2-circular curves of the 3rd class

Theorem 4 *Let $[C_1, C_2]$ be a pencil of circles and $[P_1, P_2]$ a range of points in $\mathbb{QH}_2$. The result of the projective mapping $\pi : [C_1, C_2] \mapsto [P_1, P_2]$ gives a 2-circular curve of the 3rd class k^3_π of type (1,1).*

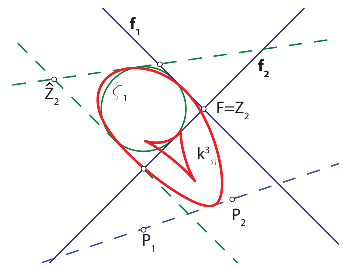

Figure 10: *2-circular curve of the 3rd class of type a*

If in the case $P_1 \in \mathbf{v}$, $P_2 \notin \mathbf{v}$, $\mathbf{v} \in \zeta_1, \zeta_2$ we assume that the curve ζ_1 is a proper curve, then the equation (3) is of the form

$$P_2^T \mathbf{v} \cdot \mathbf{v}^T C_1 \mathbf{w} = 0.$$

Hence, the conclusion is that the tangent point at the line $\mathbf{v}$ of the curve k^3_π coincides with the tangent point of the curve ζ_1. Specially, if the curves of the pencil $[C_1, C_2]$ are touching at a point on the line $\mathbf{v}$ then this common tangent point of the pencil $[C_1, C_2]$ is also the tangent point of the curve k^3_π.

Theorem 5 *Let $[C_1, C_2]$ be a pencil of special parabolas of type (2,0), $[P_1, P_2]$ a range of points and the curve k^3_π the result of the projective mapping $\pi : [C_1, C_2] \mapsto [P_1, P_2]$ in $\mathbb{QH}_2$. Then the curve k^3_π is a 2-circular curve of type (2,0) where the absolute point F is the tangent point at the absolute line* $\mathbf{f_1}$.

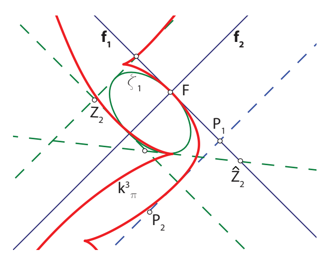

Figure 11: *2-circular curve of the 3rd class of type b*

From the observations before Theorem 2 follows also

Theorem 6 *Let $[C_1, C_2]$ be a pencil of special parabolas of type $(2,0)$, $[P_1, P_2]$ a range of points and the curve k^3_π the result of the projective mapping $\pi : [C_1, C_2] \mapsto [P_1, P_2]$ in $\mathbb{QH}_2$. If the isotropic point of the range $[P_1, P_2]$ incident with the absolute line $\mathbf{f_1}$ corresponds to the singular curve with the singular line $\mathbf{f_1}$ of the pencil $[C_1, C_2]$, then the curve k^3_π is a 2-circular curve of the 3rd class of type (2,0) with the double line* $\mathbf{f_1}$.

In this case the double line of the curve k^3_π can only be an isolated double line or a double tangent.

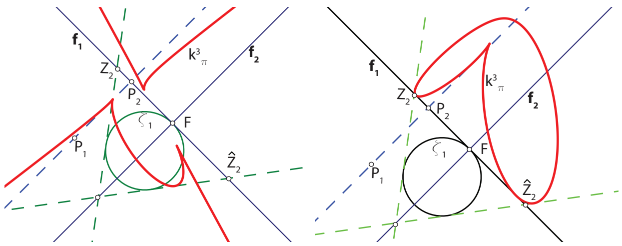

Figure 12: *2-circular curves of the 3rd class of type c and d*

From the observation before Theorem 3 we can ensure that the double line of the curve k^3_π is an inflection line:

Theorem 7 *Let $[C_1,C_2]$ be a pencil of special hyperbola of type $(1,0)$ with a common tangent point on the absolute line $\mathbf{f_1}$, $[P_1,P_2]$ a range of isotropic points on the absolute line $\mathbf{f_1}$ and the curve k^3_π the result of the projective mapping $\pi:[C_1,C_2]\mapsto[P_1,P_2]$ in $\mathbb{QH}_2$. If the corresponding point to the singular curve with the singular line $\mathbf{f_1}$ is the common tangent point of the pencil $[C_1,C_2]$, then the curve k^3_π is a circular curve of type $(2,0)$ with the inflection line $\mathbf{f_1}$.*

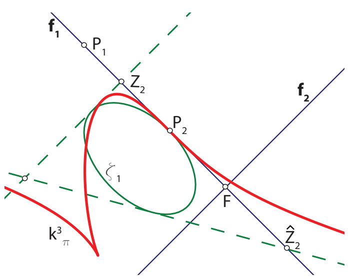

Figure 13: *2-circular curve of the 3rd class of type e*

3.2 3-circular curves or entirely circular curves

Generally, we already concluded that if there exists a point of the range $[P_1,P_2]$ which is the tangent point of its corresponding curve of the 2nd class in the pencil $[C_1,C_2]$, then this point is also a tangent point for k^3_π. Thus, the following theorem is valid:

Theorem 8 *Let $[C_1,C_2]$ be a pencil of the 2nd class curves, $[P_1,P_2]$ a range of isotropic points on the absolute line $\mathbf{f_1}$ and the curve k^3_π the result of the projective mapping $\pi:[C_1,C_2]\mapsto[P_1,P_2]$ in $\mathbb{QH}_2$. If the pencil $[C_1,C_2]$ contains a special parabola of type $(0,2)$ whose corresponding point is the absolute point F, then the curve k^3_π is a 3-circular curve of type $(1,2)$. The absolute point F is the tangent point at the line $\mathbf{f_2}$ of the curve k^3_π.*

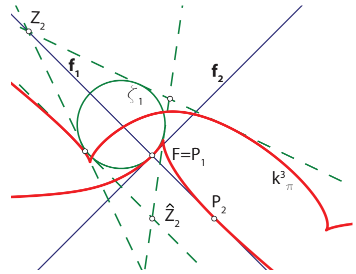

Figure 14: *1-circular curve of the 3rd class of type a*

From the observations before Theorem 3 follows also

Theorem 9 *Let $[C_1,C_2]$ be a pencil of circles, $[P_1,P_2]$ a range of isotropic points on the absolute line $\mathbf{f_1}$ and the curve k^3_π the result of the projective mapping $\pi:[C_1,C_2]\mapsto[P_1,P_2]$ in $\mathbb{QH}_2$. Then the curve k^3_π is an entirely circular curve of the circularity type $(2,1)$, where the absolute line $\mathbf{f_1}$ is a double isolated line or a double tangent line.*

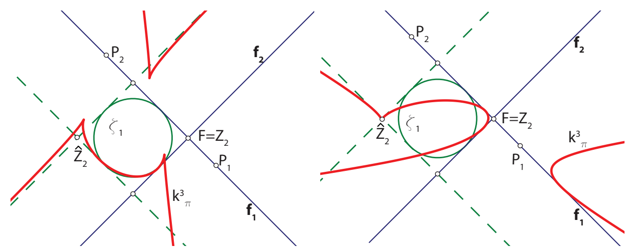

Figure 15: *3-circular curves of the 3rd class of type b and c*

Theorem 10 *Let $[C_1,C_2]$ be a pencil of circles with a common tangent point on the absolute line $\mathbf{f_1}$, $[P_1,P_2]$ a range of isotropic points on the absolute line $\mathbf{f_1}$ and the curve k^3_π the result of the projective mapping $\pi:[C_1,C_2]\mapsto[P_1,P_2]$ in $\mathbb{QH}_2$. If the corresponding point to the singular 2nd class curve with the singular line $\mathbf{f_1}$ is the common tangent point, then the curve k^3_π is a 3-circular curve of type $(2,1)$, where the absolute line $\mathbf{f_1}$ is a inflection line.*

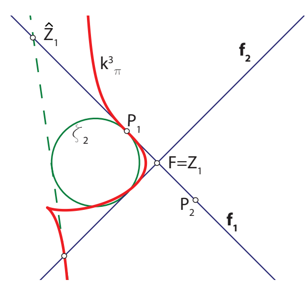

Figure 16: *3-circular curve of the 3rd class of type d*

Theorem 11 *Let $[C_1,C_2]$ be a pencil of special parabolas of type $(2,0)$, $[P_1,P_2]$ a range of isotropic points on the absolute line $\mathbf{f_1}$ and the curve k^3_π the result of the projective mapping $\pi:[C_1,C_2]\mapsto[P_1,P_2]$ in $\mathbb{QH}_2$. The curve k^3_π is* *an entirely circular curve of the 3rd class of the circularity type* $(3,0)$ *with the double line* $\mathbf{f_1}$*. The absolute point* F *is one tangent point on the double line* $\mathbf{f_1}$*, and the other tangent point is the point of the range* $[P_1,P_2]$ *that corresponds to the singular curve of* $[C_1,C_2]$ *with the singular line* $\mathbf{f_1}$*. Specially, if this latter point coincides with* F *then line* $\mathbf{f_1}$ *is an inflection line.*

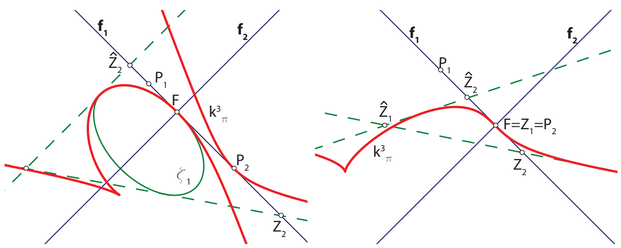

Figure 17: *3-circular curves of the 3rd class of type e and f*

From the Theorem 5 and the observation before we can also conclude

Theorem 12 *Let* $[C_1,C_2]$ *be a pencil of special parabolas of type* $(2,0)$*,* $[P_1,P_2]$ *a range of points and the curve* k_π^3 *the result of the projective mapping* $\pi : [C_1,C_2] \mapsto [P_1,P_2]$ *in* $\mathbb{QH}_2$*. If the corresponding point to the singular curve whose one pencil is* (F) *is the isotropic point of* $[P_1,P_2]$ *incident with the line* $\mathbf{f_1}$*, then the curve* k_π^3 *is a 3-circular curve of type* $(3,0)$ *with a cusp at the point* F*.*

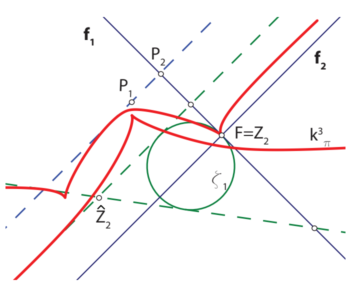

Figure 18: *3-circular curve of the 3rd class of type g*

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