Introduction to Planimetry of Quasi-Elliptic Plane

1 Introduction

This paper begins the study of the quasi-elliptic plane from the constructive and synthetic point of view. We will see although the geometry denoted as quasi-elliptic is dual to Euclidean geometry it is a very rich topic indeed and there are many new and unexpected aspects.

In this paper some basic notations concerning the quasi-elliptic conic and some selected constructions and a theorem will be presented. It is known that there exist nine geometries in plane with projective metric on a line and on a pencil of lines which are denoted as Cayley-Klein projective metrics and they have been studied by several authors, such as [2], [3], [4], [8], [9], [10], [13], [14], [15], [16].

The quasi-elliptic geometry, further in text qe-geometry, has elliptic measure on a line and parabolic measure on a pencil of lines. In the quasi-elliptic plane, further in text qe-plane, the metric is induced by the absolute figure $F_{QE} = \{j_1, j_2, F\}$ consisting of a pair of conjugate imaginary lines $j_1$ and $j_2$, intersecting at the real point $F$. Some basic geometric notions, definitions, selected constructions and a theorem in the quasi-elliptic plane will be presented.

Key words: quasi-elliptic plane, perpendicular points, central line, qe-conic classification, hyperosculating qe-circle, envelope of the central lines

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An elliptic involution on the pencil $(F)$ the absolute triple $F_{QE} = \{j_1, j_2, F\}$ can be given as follows:

- An elliptic involution on the pencil $(F)$ is determined by two arbitrary chosen pairs of corresponding lines $a_1, a_2$; $b_1, b_2$. An elliptic involution $(F)$ has the absolute lines $j_1$ and $j_2$ for double lines ([1], p.244-245, [6], p.46).
- Notice that the absolute point $F$ can be finite (Figure 1a) or at infinity (Figure 1c).
- In this paper the model were involutory pair of corresponding lines are perpendicular to each other in Euclidean sense (Figure 1b) or at infinity (Figure 1c).

The absolute point $F$ is inside the conic $k$. Pairs of conjugate lines with respect to a conic $k$ determine aforementioned elliptic involution $(F)$. The absolute lines $j_1$ and $j_2$ are double lines for the involution $(F)$ and in this case they are a pair of imaginary tangent lines to $k$ from the absolute point $F$ (Figure 1d).
2 Basic notation and selected constructions in the quasi-elliptic plane

For the points and the lines in the qe-plane the following are defined:

- **isotropic lines** - the lines incident with the absolute point $F$,
- **isotropic points** - the imaginary points incident with one of the absolute line $j_1$ or $j_2$,
- **parallel points** - two points incident with the same isotropic line,
- **perpendicular lines** - if at least one of two lines is an isotropic line,
- **perpendicular points** - two points $A, A_1$ that lie on a pair of corresponding lines $a, a_1$ of an elliptic (absolute) involution $(F)$. 

**Remark.** The perpendicularity of points in qe-plane is determine by the absolute involution, therefore an elliptic involution $(F)$ is a circular involution in the qe-plane. ([7], p.75)

Notice that the absolute point $F$ is parallel and perpendicular to each point in the qe-plane. Furthermore, in the qe-plane there are no parallel lines.

**A brief review of some basic construction**

**Example 1** Let the absolute figure $F_{QE}$ of the qe-plane be given with the involutory pencil $(F)$ (Figure 1b). Let $A$ be the point and $p$ the line which is not incident with the point $A$ in the qe-plane (Figure 2). Construct the point $A_1$ which is perpendicular to the point $A$ and incident with the given line $p$.

Points $A, A_1$ are perpendicular if they lie on a pair of corresponding lines $a, a_1$ of an absolute elliptic involution $(F)$, i.e. if they lie on a pair of perpendicular lines in a Euclidean sense ([7], p.71-75).

**Example 2** Let the absolute figure $F_{QE}$ of the qe-plane be given with the involutory pencil $(F)$. Construct the midpoints $P_i$ and the bisectors $s_i$ of a given line segment $AB$ ($i = 1, 2$) (Figure 3).
The midpoint of a segment in the qe-plane is dual to an angle bisector in the Euclidean plane, consequently a segment in a qe-plane has two perpendicular midpoints $P_1$ and $P_2$ that are in harmonic relation with the points $A$ and $B$. A line segment $AB$ in the qe-plane has two isotropic bisectors $s_1$ and $s_3$ that are a common pair of corresponding lines of two involutions $(F)$ with the center $F$, denoted as $I_1$, $I_2$. In order to construct the midpoints and bisectors we observe aforementioned involutions $(F)$, a circular involution $I_1$ is determined by perpendicular corresponding lines in a Euclidean sense and the second hyperbolic involution $I_2$ is determined by isotropic lines $a = AF$, $b = BF$ as its double lines. The construction is based on the Steiner’s construction ([6], p.26, [7], p.74-75). These two pencils will be supplemented by the same Steiner’s conic $s$, which is an arbitrary chosen conic through $F$. The involutions $I_1$ and $I_2$ determine two involutions on the conic $s$. Let the points $O_1$ and $O_2$ be denoted as the centers of these involutions, respectively. The line $O_1O_2$ intersects the conic $s$ at two points. Isotropic lines $s_1$ and $s_2$ through these points are a common pair of these two involutions $(F)$. The intersection points $P_1$ and $P_2$ of bisectors $s_1$ and $s_2$ with the line $AB$ are midpoints of the line segment $AB$.

**Example 3** Let the absolute figure $\mathcal{F}_{QE}$ of the qe-plane be given with the involutory pencil $(F)$. Let two non-isotropic lines $a$, $b$ be given. Construct an angle bisector between given rays $a$, $b$ (Figure 4).

The angle bisector in the qe-plane is dual to a midpoint of a segment in the Euclidean plane. Let $V$ be the vertex of an angle $\angle(a,b)$. Let the isotropic line $VF$ be denoted as $f$. The angle bisector $s$ is a line in a pencil $(V)$ that is in harmonic relation with triple $(a,b,f)$. The isotropic line $f$ is an isotropic bisector.

**Example 4** Let the absolute figure $\mathcal{F}_{QE}$ of the qe-plane be given with the involutory pencil $(F)$. Let the lines $a$, $b$, $c$ determine a trilateral $\triangle ABC$ with the vertices $A$, $B$, $C$. Construct the ortocentar line of the given trilateral (Figure 5).

The ortocentar line $o$ of the trilateral in the qe-plane is dual to the orthocenter of a triangle in the Euclidean plane. The points $A_1$, $B_1$, $C_1$ are incident with lines $a$, $b$, $c$ and perpendicular to the opposite vertices $A$, $B$, $C$, respectively. The points $A_1$, $B_1$, $C_1$ are collinear and determine a unique ortocentar line.

**Example 5** Let the absolute figure $\mathcal{F}_{QE}$ of the qe-plane be given with the involutory pencil $(F)$. Let the lines $a$, $b$, $c$ determine a trilateral $\triangle ABC$ with the vertices $A$, $B$, $C$. Construct the centroid line of a trilateral (Figure 6).

The centroid line $o$ of a trilateral in the qe-plane is dual to the centroid of a triangle in the Euclidean plane. The angel bisectors $s_a$, $s_b$, $s_c$ of trilateral intersect opposite sides $a$, $b$, $c$ of the trilateral at the points $S_A$, $S_B$, $S_C$, respectively. The points $S_A$, $S_B$, $S_C$ are collinear and determine a unique centroid line.
3 Qe-conic classification

There are four types of the second class curves classified according to their position with respect to the absolute figure (Figure 7):

- **qe-hyperbola** \((h)\) - a curve of the second class that has a pair of real and distinct isotropic lines.
- **Equilateral qe-hyperbola** \((h_{EQ})\) - a curve of the second class that has isotropic lines as a corresponding lines for the absolute involution \((F)\).
- **qe-ellipse** \((e)\) - a curve of the second class that has a pair of imaginary isotropic lines.
- **qe-parabola** \((p)\) - a curve of the second class where both imaginary isotropic lines coincide.
- **qe-circle** \((k)\) - is a special type of qe-ellipse for which the isotropic lines coincide with the absolute lines \(j_1\) and \(j_2\). In a model of an absolute figure that is used in this paper each qe-conic that has an absolute point \(F\) as its Euclidean foci is a qe-circle.

In the projective model of the qe-plane every type of a qe-conic can be represented with every type of Euclidean conics without loss of generality.

Each pair of conjugate points incident with the central line with respect to a qe-circle are perpendicular, consequently a qe-circle has infinitely many pairs of qe-centers. The **isotropic (the minor) diameters** are the lines joining a qe-center to the absolute point \(F\). A qe-ellipse and a qe-hyperbola have two isotropic diameters.

The lines incident with qe-centers of a qe-conic are called the **vertices lines** of a qe-conic in the qe-plane. A qe-hyperbola has two real vertices lines, while a qe-ellipse has four real vertices lines.

A hyperosculating qe-circle of a qe-conic can be constructed only at the vertices lines of a qe-conic.

The intersection points of a qe-conic and vertices lines are called **co-vertices points**.

4 Some construction assignments

**Exercise 1** Construct a qe-circle \(k\) determined with the given central line \(c\) and the line \(p\) (Figure 8).

In order to construct the qe-circle as a line envelope, a perspective collineation that maps arbitrary chosen qe-circle \(k_1\) into qe-circle \(k\) is used. The construction is carried out in the following steps:

1. The absolute point \(F\) is selected for the center of the collineation. Let \(k_1\) be an arbitrary chosen qe-circle with the center \(F\). A polar line \(c_1\) of \(F\), is the central line for chosen qe-circle \(k_1\). Notice that \(c_1\) is the line at infinity.
2. The lines \(c\) and \(c_1\) are corresponding lines for the perspective collineation with the center \(F\). Let the point \(S\) be the intersection point of the lines \(p\) and \(c_1\). To determine an axis \(o\) of the perspective collineation, the point \(R\) that is perpendicular to the point \(S\) and incident with the line \(p\) is constructed. A ray \(FR\) of the collineation intersect the qe-circle \(k_1\) at the points \(R_1\) and \(R_2\). Let the line \(p_1\) touch the qe-circle \(k_1\) at a point \(R_1\). The lines \(p\) and \(p_1\) are corresponding lines for the perspective collineation with a center \(F\). The axis \(o\) passes through the intersection point \(S_1\) of the lines \(p_1\) and \(p\), and it is parallel to \(c\).
In order to construct the envelope of the central lines of the envelope of the central lines in the given pencil.

Exercise 2 Construct the hyperosculating qe-circle of a qe-hyperbola $h_1$ (Figure 9).

Let the qe-hyperbola $h_1$ be given and its central line be denoted as $c$. A hyperosculating qe-circle of the qe-hyperbola $h_1$ can be constructed only at the vertices lines. A qe-hyperbola $h_1$ has two real vertices lines $q_1$ and $q_2$. Let the points $T_1$ and $T_2$ be co-vertices points. Let the line $t_2$ and the point $T_2$ be observed. In order to construct a hyperosculating qe-circle, the point $S_2$ that is perpendicular to $T_2$, and incident with the line $t_2$ is constructed. The central line $c_h$ of a hyperosculating qe-circle is incident with $S_2$. In order to construct $c_h$, let the line $y_1$ of the qe-hyperbola $h_1$ be arbitrary chosen. The intersection point of $h_1$ and the line $y_1$ is denoted as $Y_1$. The intersection point of lines $t_2$ and $y_1$ is denoted as $K$. The point $K_1$ is perpendicular to $K$ and incident with the line $T_2Y_1$. The line $S_2K_1$, denoted as $c_h$, is a central line of a hyperosculating qe-circle. The central line $c_h$ and the line $t_2$ determine a hyperosculating qe-circle and to construct it the same principle as in Exercise 1 is used.

Exercise 1 is used.

If one of the base lines in a pencil is isotropic line, the pencil will be supplemented by the Steiner’s conic $s$, which is an arbitrary chosen conic through $F$. Let the point $O$ be denoted as a center of the involution $(F)$.

Let the point $O$ be outside the conic $s$, involutory pencil $(F)$ contains real double lines, and the envelope $\delta_1$ is a qe-hyperbola. If the point $O$ is on the conic $s$, double lines of involution $(F)$ coincide, and the envelope $\delta_1$ is a qe-parabola.

If one of the base lines in a pencil is isotropic line, the pencil of qe-conics contains qe-hyperbolas and one qe-parabola. The envelope $\delta_1$ is a qe-parabola if the pencil contain one qe-parabola.

The envelope $\delta_1$ is an qe-ellipse if the pencil does not contain qe-parabolas (Figure 10). Double lines for the elliptic involution $(F)$ are imaginary lines. Pencil will be supplemented by the Steiner’s conic $s$, which is an arbitrary chosen conic through $F$. Let the point $O$ be denoted as a center of the involution $(F)$.

If the point $O$ is inside the conic $s$, involutory pencil $(F)$ contains imaginary double lines, and the envelope $\delta_1$ is an qe-ellipse (Figure 10).

If one of the base lines in a pencil are isotropic lines, the envelope $\delta_1$ degenerates into a point.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9.png}
\caption{Qe-hyperbola - an envelope of the central lines}
\end{figure}
Corollary 1 Let any two degenerated qe-conics in a pencil of qe-conics be given as a pair of perpendicular points i.e. the pencil of equilateral qe-hyperbolas. Than the envelope of the central line is a qe-circle.

References


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