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Graph Colouring and its Application within Cartography

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ABSTRACT

The problem of colouring geographical political maps has historically been associated with the theory of graph colouring. In the middle of the 19th century the following question was posed: how many colours are needed to colour a map in a way that countries sharing a border are coloured differently. The solution has been reached by linking maps and graphs. It took more than a century to prove that 4 colours are sufficient to create a map in which neighbouring countries have different colours.

Key words: graph, graph colouring, map, map colouring, the four colour theorem

MSC2010: 05C15, 05C90, 86A30, 68R10

O problemu bojanja grafova s primjenom u kartografiji

SAŽETAK

Problem bojanja geografskih političkih karata povijesno je vezan uz teoriju bojanja grafova. Polovicom 19. stoljeća nametnulo se pitanje koliko je boja potrebno da bi se dana geografska karta obojila tako da zemlje koje graniče budu obojane različitim bojama. Do rješenja se došlo povezivanjem karata i grafova. Bilo je potrebno više od jednog stoljeća kako bi se dokazalo da su četiri boje dovoljne za obojiti (geografsku) kartu na takav način da susjedna područja (države) imaju različitu boju.

Ključne riječi: graf, bojanje grafa, karta, bojanje karte, teorem o 4 boje

1 Introduction

The problem of colouring geographical political maps has historically been associated with the theory of graph colouring. In the middle of the 19th century the following question was posed: how many colours are needed to colour a map in a way that countries sharing a border are coloured differently. The solution has been reached by linking maps and graphs. It took more than a century to prove that 4 colours are sufficient to create a map in which neighbouring countries have different colours.

In graph theory, graph colouring is a special case of graph labelling. It is about assigning a colour to graph elements: vertices, edges, regions, with certain restrictions.

With this paper we would like to assess the elements of the theory of graph colouring with an emphasis on its application on practical problems in the field of cartography.

A mathematical basis for map colouring will be given along with the chronology of proving *The Four Colour Theorem*. In addition, world political map will be shown, to determine the minimum number of colours needed to colour a map properly in practice.

2 Elements of Graph Theory

Graph Theory is a special branch of combinatorics closely related to applied mathematics, optimization theory and computer sciences. The simplest and most frequently applied combinatorical structure is a graph, and exactly the simplicity of this structure provides easy transfer and modelling of practical problems in graph terms, as well as the application of known proved theoretical concepts, algorithms and abstract ideas to particular graphs.

2.1 Historical overview of Graph Theory

Graph Theory has rather precise historical aspects. The first paper in the Graph theory was the article "Solutio problematis ad geometriam situs pertinentis", i.e. "The Solution of a Problem Relating to the Geometry of Position" by a well known Swiss mathematician Leonhard Euler (1707-1783) from the year 1736. In this paper, "the Königsberg Bridge Problem" was defined and solved. The Prussian city Königsberg, now Kaliningrad (Russia) occupies both banks of the river Pregolya. The river divides the city into four territories, two river islands and two coastal areas mutually connected with seven bridges.

The inhabitants of Königsberg sought to solve the issue that troubled them for many years: *Is it possible to walk around the city crossing each of the seven bridges once and only once and to finish the walk through the city at the starting point?*

Leonhard Euler eliminated all features of the terrain except the land masses and bridges presenting it by means of a graph (Fig. 1); the points represent the coastal parts (B and C) and the islands (A and D), while the bridges are presented as graph edges, i.e. as connections of points. His answer to the question mentioned above was clear: such walk is not possible if each part of mainland is not connected with the other parts with an even number of bridges (see Chapter 2.2).

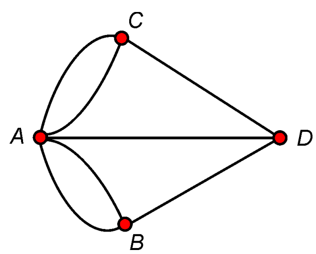


Figure 1: *The Königsberg bridges problem presented by a graph*

Although the origins of the Graph Theory date back as far as the 18th century, it started to develop in the second half of the 20th century. The first book dealing with the topic of Graph Theory was written in 1936 by the Hungarian mathematician D. König, and it is considered to be the beginning of the development of Graph Theory as a separate mathematical discipline. König unified and systematised the earlier results offering the list of 110 published papers where the term graph had appeared explicitly. Among the authors of these papers are famous names like G. Kirchhoff (1824 - 1887) and A. Cayley (1821 -1895). Ever since, graph has become a generally accepted term [6].

Greater development of research in the field of Graph Theory and its applications started in the 60-ties of the twentieth century and has been continuing parallel with the development of information technologies up to the present day.

2.2 Graph Theory basic concepts and definitions

Definition 1 *A graph G consists of a finite non-empty set $V = V(G)$ whose elements are called* ***vertices, points*** *or* ***nodes*** *of G and a finite set $E = E(G)$ of unordered pairs of distinct vertices called* ***edges*** *of G.*

Such a graph we denote $G(V,E)$ when emphasizing the two parts of G, (Fig. 2).

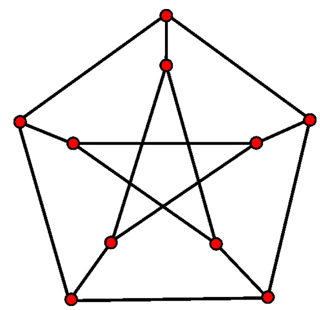


Figure 2: *Example of a simple graph - The Petersen graph*

Definition 2 *An edge $e = u,v$ is said to* ***join*** *the vertices u and v, and is usually abbreviated to $e = uv$. In such a case, u and v are called* ***endpoints*** *and they are said to be* ***adjacent****. Further, vertices u and v are said to be* ***incident*** *on e and vice versa, the edge e is said to be* ***incident*** *on each of its endpoints u and v. Similarly, two distinct edges e and f are adjacent if they have a vertex in common.*

Remark 1 *If two or more edges connect the same endpoints, we call them* ***multiple edges****. An edge is called a* ***loop*** *if its endpoints are the same vertex. The former definition of a graph permits neither multiple edges nor loops. In some texts the term* ***simple graph*** *refers to the graph without multiple edges and loops while the one permitting them is called a* ***multigraph****. Most often it does not matter whether we deal with a simple graph or a multigraf, and if necessary, will be specially emphasized.*

Definition 3 *If the vertex set of a graph G can be split into two disjoint sets X and Y so that each edge of G joins a vertex of X and a vertex of Y, then G is said to be* ***bipartite****. A* ***complete bipartite graph*** *is a bipartite graph in which each vertex in X is joined to each vertex in Y by just one edge. If r is the number if vertices in X and s is the number of vertices in Y we denote this graph $K_{r,s}$, (Fig. 3).*

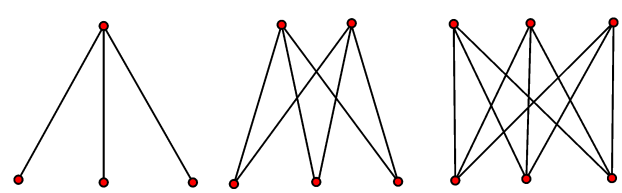


Figure 3: *Complete bipartite graphs $K_{1,3}$, $K_{2,3}$, $K_{3,3}$*

Definition 4 *For the two disjoint graphs $G = (V(G),E(G))$ and $H = (V(H),E(H))$, their union $G \cup H$ is defined by $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$.*

Definition 5 *A graph G is **connected** if it cannot be represented as the union of two graphs. Otherwise, it is **disconnected**. Any disconnected graph can be represented as the union of connected graphs called **connected components** of G. A graph is said to be **finite** if it has a finite number of vertices and a finite number of edges, otherwise it is **infinite**.*

Graphs within this article shall be finite.

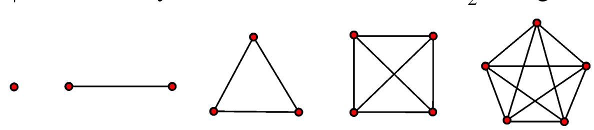
Definition 6 *The **degree** of a vertex v in G, written deg (v), is equal to the number of edges in G incident with v. It shall be taken conventionally that a loop contributes 2 to the degree of v. A vertex of degree zero is called an **isolated vertex** and a vertex of degree 1 is an **end-vertex**.*

Definition 7 *Consider the graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$. H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is said to be a **spanning subgraph** of G if $V(H) = V(G)$.*

Subgraphs are often obtained from a given graph by deleting its vertices and edges. Specifically, if v is a vertex in G, $G - v$ is a subgraph of G obtained by deleting v and all edges incident with v. Similarly, if e is an edge in G, $G - e$ is a subgraph of G obtained by deleting e from G. However, it is easily seen that contracting an edge of a graph does not give a subgraph. **Contracting** an edge e from G means removing it and identifying its ends u and v so that the resulting vertex is incident with those edges that were originally incident with u or v. Such a graph is denoted by $G|e$.

Definition 8 *A graph G is said to be **complete** if every vertex in G is adjacent to every other vertex in G. A complete graph with n vertices is denoted by K_n, (Fig. 4).*

$K_n : s$ is used to denote a complete graph with $|V| = n$ and $|E| = s$. It is easy to check that K_n has $s = \frac{n(n-1)}{2}$ edges.



$K_1 : 0$ $K_2 : 1$ $K_3 : 3$ $K_4 : 6$ $K_5 : 10$

Figure 4: *Some complete graphs*

Definition 9 *A **walk** in a graph G is an alternating sequence of vertices and edges of the form $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$, where each edge e_i contains the vertices v_{i-1} and v_i, $1 \leq i \leq k$. In a simple graph a walk is determined by a sequence $v_0, v_1, \ldots, v_k$, of vertices; v_0 being the **initial vertex** and v_k the **final vertex**. We say a walk is **from** v_0 **to** v_k, or **connects** v_0 **to** v_k. A walk is **closed** if the initial and final vertices are identified. The number k of edges in a walk is called its **length**. A **trail** is a walk such that all of the edges are distinct. A **path** is a walk such that all of the vertices and edges are distinct. A **circuit** is a closed trail, while a **cycle** is a closed path.*

Definition 10 *A connected graph G is called **Eulerian** if there exists a closed trail containing every edge of G. Such a trail is called an **Eulerian trail**. A non-Eulerian graph G is **semi-Eulerian** if there exists a trail containing every edge of G.*

Let us now observe the theorem that solves the problem of the Königsberg bridges.

Theorem 1 (Euler, 1736) *A connected graph is Eulerian if and only if each vertex has even degree.*

For the proof see e.g. [20].

Considering now a graph given in Figure 1 in the light of the above theorem, we conclude that the closed trail that meets the required conditions does not exist.

From the proof of Theorem 1 arises,

Corollary 1 *Any connected graph with two odd vertices is semi-Eulerian. A trail may begin at either odd vertex and will end at the other odd vertex.*

3 Graph colouring

Definition 11 *Consider a graph G. A (**vertex**) **colouring** of G is an assignment of colours to the vertices of G such that adjacent vertices have different colours. It is a mapping $c : V(G) \to S$. The elements of S are called **colours**. If $|S| = k$, we say that c is a **k-colouring**. A colouring is **proper** if adjacent vertices have different colours. A graph is **k-colourable** if it has a proper k - colouring.*

Each graph with n vertices is n - colourable, since each vertex may be coloured with a different colour. Consequently, the question is: what is the minimum necessary number of colours to colour the graph properly.

Definition 12 *If a graph G is k-colourable, but not $(k-1)$-colourable, it is said that G is **k-chromatic**. The minimum number of colours needed to colour G is called the **chromatic number** of G and is denoted by $\chi(G)$, $\chi(G) \leq |V|$.*

3.1 Planar graphs and maps

3.1.1 About planar graphs

Although graphs are usually presented two-dimensionally, i.e. in a plane, on paper or screen, it should be noted that each graph can always be presented in three-dimensional Euclidean space without the edges being crossed. The proof of this property is simple and can be found for example in [14]. In this section we deal with requirements needed for a graph drawn in a plane to have the specified property.

Definition 13 *A graph is said to be* ***planar*** *if it can be drawn in a plane so that its edges do not cross, (Fig. 5).*

From the above definition it can be deduced that each subgraph of a planar graph is planar, and that each graph with a nonplanar subgraph is nonplanaran.

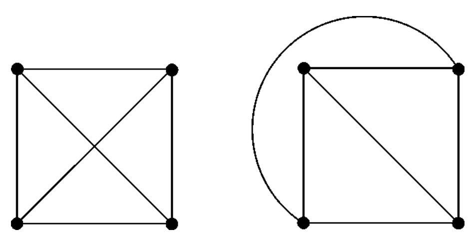


Figure 5: *The complete graph K_4 is a planar graph and a map*

Definition 14 *A* ***map*** *is a connected planar graph where all vertices have degree at least 3. A map divides the plane into a number of* ***regions*** *or* ***faces*** *(one of them infinite). The term* ***degree*** *of a region, written deg(r), refers to the length of the cycle that surrounds it. Regions are said to be* ***adjacent*** *if they share an edge, not just a point.*

Definition 15 *Graphs G and H are said to be* ***isomorphic*** *$(G \approx H)$ if there is a one-to-one correspondence between their vertices and their edges so that adjacent vertices are maped in adjacent ones.*

Definition 16 *Two graphs are said to be* ***homeomorphic*** *if they are isomorphic or one from another can be obtained by removing or inserting vertices of degree 2.*

Note that homeomorphism preserves planarity, i.e. inserting vertices of degree 2 does not have impact on planarity.

3.1.2 Some results related to planar graphs

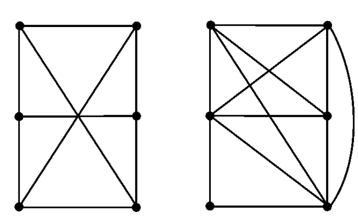
Some Euler's results and their consequences are listed below. Here we state some of the proofs, however, the other proofs can be found in e.g. [14], [20].

1. In any map K the sum of degrees of all regions equals to twice the number of edges in K.
2. "Euler's formula": Let $G = (V,E)$ be a connected planar graph with $v = |V|$, $e = |E|$, and let r denotes the number of its regions. Then, $v - e + r = 2$.
3. If $G = (V,E)$ is a simple connected planar graph with the v vertices, $e \geq 3$ edges and r regions, then $3r \leq 2e$ and $e \leq 3v - 6$. If G does not contain triangles, i.e. the degree of each region is at least 4, then $e \leq 2v - 4$.
4. The graph K_5 is not planar.
   Proof: Indeed, in K_5 we have $v = 5$ and $e = 10$, hence $3v - 6 = 9 < e = 10$, which is in contradiction with the previous result.
5. The graph $K_{3,3}$ is not planar.
   Proof: As $K_{3,3}$ does not contain triangles, to be planar $e \leq 2v - 4$ must be fulfilled, which in this case leads to a contradiction, i.e. $9 \leq 2 \cdot 6 - 4 = 8$.
6. Every simple planar graph contains a vertex of degree lower than 6.

The following important result gives a necessary and sufficient condition for a graph to be planar.

Theorem 2 (Kazimierz Kuratowski, 1930) *A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.*

For example, graphs given in Fig. 6 are not planar.



(a) (b)

Figure 6: *Examples of non-planar graphs*

Indeed, graph (a) is graph $K_{3,3}$, (Fig. 7):

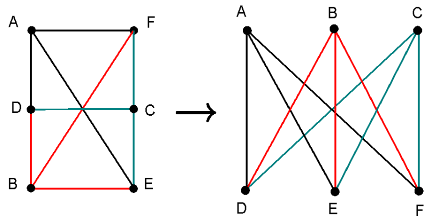


Figure 7: *Graph $K_{3,3}$*

while graph (b) is homeomorphic to graph K_5, (Fig. 8):

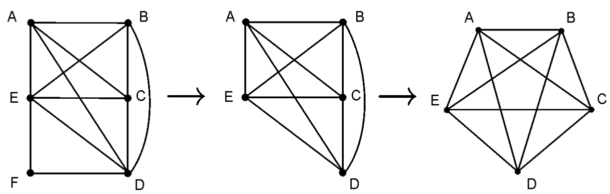


Figure 8: *A graph homeomorphic to graph K_5*

3.1.3 Every planar graph is 6-colourable

The claim will be proved by induction on the number of vertices of a graph.

For the base case we take the claim every graph with at most 6 vertices is 6-colourable. Let G be a simple planar graph with n vertices, and let all the simple planar graphs with $n-1$ vertices be 6-colourable. We consider graph G has a vertex of degree at most 5, say $v.G-v$ is a graph with $n-1$ vertices and as such is 6-colourable. As v has five neighbours, simply colouring it with the remaining colour out of 6, a proper 6-colouring of G is obtained.

3.1.4 Every planar graph is 5-colourable (Kempe, Heawood)

For the proof, we use again induction on the number of vertices of a graph.

The result holds trivially if G has one vertex. Let us assume G is a simple planar graph with n vertices, and let all the simple planar graphs with $n-1$ vertices be 5-colourable. We take in account that within G there is a vertex v of degree at most 5. $G-v$ is a graph with $n-1$ vertices and by the induction hypothesis, is 5-colourable. Now we have to assign a colour to v.

The claim of the theorem holds for $\deg(v)=5$ since in that case it would be sufficient to colour v with the one remainin colour.

Hence, we may assume and v has five neighbours coloured differently. For being all mutually adjacent to each other would mean K_5 is a subgraph of G, being in contradiction with the assumption that G is planar. Therefore, at least one pair of vertices is not connected. Let v_1 and v_3 be the vertices in question. Contracting the edges vv_1 and vv_3, we get a graph with $n-2$ vertices being 5-colourable by the induction hypothesis. After performing the colouring, we invert the process, i.e. we stretch the contracted edges. Since v_1 and v_3 are not adjacent it causes no problem if being of the same colour. As for the neighbours of v one needs now 4 colours, we simply colour v with the one remaining colour.

3.2 Dual graph of a map

Definition 17 ***Map colouring*** *is the act of assigning different colours to different regions (faces) of a map in a way that no two adjacent regions (regions with a boundary line in common) have the same colour. We now define a map to be* ***k-colourable*** *if its faces can be coloured with k colours.*

Similarity between the above definition and the one defining graph colouring is obvious. In order to show the colouring of a map to be equivalent to the vertex colouring we need a concept of the **dual map**, also known as **geometrical dual**.

Creating a dual map for a given map K is based on a correspondence (dualism) reflected in a following way:

region $\leftrightarrow$ graph vertex

adjacent regions $\leftrightarrow$ adjacent vertices

map colouring $\leftrightarrow$ colouring graph vertices

The procedure is as follows: A point within each region of a map K needs to be selected. If two regions are adjacent, points need to be connected with a curve. These curves can be drawn so that they do not intersect. The result is a new map K^*, called the **dual** of K (Fig. 9). Any colouring of the regions of a map K correspond to vertex colouring of the dual K^*.

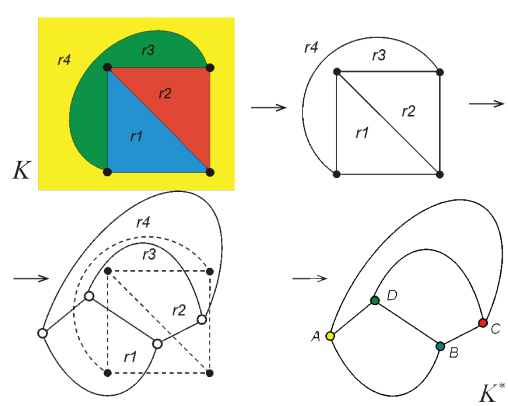


Figure 9: *From map K to its dual K^**

It is easy to see that K being planar and connected entails its geometric dual K^* to be planar and connected graph as well. Even more is fulfilled:

Theorem 3 *Let K be a planar connected graph with v vertices, e bridges and r regions, and let its dual K^* has v^* vertices, e^* bridges and r^* regions. Then, $v^* = r$, $e = e^*$, $r^* = v$.*

Proof: $v^* = r$ follows at once from the definition of a dual graph. As there is a bijection between the edges of K and the edges of K^*, we have $e = e^*$, and as K^* is planar and connected, applying Euler's formula one gets $r^* = 2 - v^* + e^* = v$ □

As mentioned before, any colouring of the regions of a map K correspond to vertex colouring of the dual K^* (Fig. 9). As a concequence, we have the following result.

Theorem 4 *A map K is region (face) k-colourable if and only if the planar graph of its geometrical dual K^* is vertex k-colourable.*

For the proof see [14].

And finally,

Theorem 5 *The four-colour theorem for maps is equivalent to the four-colour theorem for planar graphs.*

For the proof see [20].

4 The four colour theorem

The four colour problem was defined as Francis Guthrie (1831-1899), the student of the University in London in 1852 was given the task to colour the map of English counties with as few colours as possible. He concluded that 4 colours were sufficient to complete the task with the counties sharing a common border being coloured with different colours. He wanted to find out whether each map in a plane or on a sphere can be coloured with 4 colours at the most with the neighbouring countries being coloured with various colours. It implies the fact that each country presents one coherent area. This question shall initiate a great number of attempts to find the answer by mathematicians and laypersons, which shall last for more than a century making this theorem one of the issues remaining unproven for the longest period of time. The main "tool" that the mathematicians will use in solving this problem will be the Graph Theory.

4.1 Historical overview

4.1.1 Francis Guthrie first noticed the problem

Although August Möbius, a German mathematician and astronomer mentioned the four colour problem in one of his lectures held in 1840, it is considered that Francis Guhrie first posed the problem.

Francis Guthrie was a versatile person who was active in many areas. He was a very efficient barrister, acknowledged botanist (two plants were named after him: *Guthriea capensis* and *Erica Guthriei*), but first of all an excellent mathematician. However, since he could not find the solution to the four colour problem, he sent his notes with his brother Frederick to their mutual professor Augustus De Morgan. Augustus De Morgan (1806-1871) was a prominent English mathematician, a professor at the University in London who was very intrigued by this problem. Since he did not know the answer, he wrote a letter on October 23, 1852 to his colleague and friend, Sir William R. Hamilton in Dublin where he presented the statement and gave an example showing that four colours suffice. He wrote in his letter as follows, (Fig. 10):

"A student of mine asked me to give him a reason for a fact which I did not know was fact - and do not yet. He says that if a figure be anyhow divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured – four colours may be wanted. I cannot find an example where five colours are needed. If you retort with some very simple case, I think I must do as the Sphynx did…" [23]

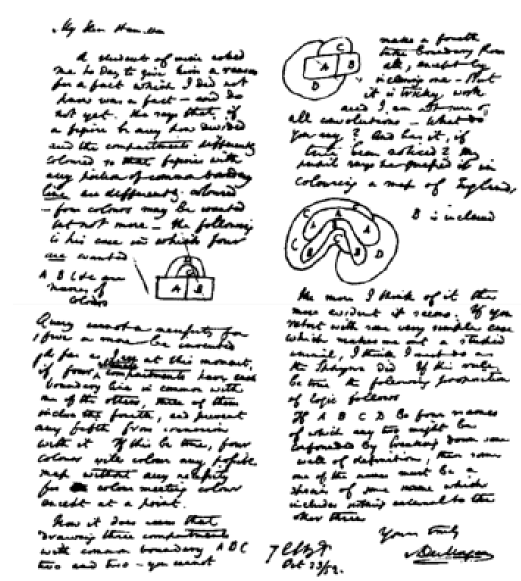


Figure 10: *Display of the original letter [5]*

Sir Hamilton, however, was not interested. Therefore, De Morgan published the problem in 1860 in a literary journal *Athenaeum*. The American mathematician, philosopher and logician Charles Sanders Pierce learned about the problem probably from the journal, and tried to solve it. Although it was said that he managed to solve it, the proof has never been published.

4.1.2 Arthur Cayley refreshes the problem

After the year 1860, in the period of about 20 years, the mathematicians almost completely stopped to be interested in the four colour problem until the British Arthur Cayley (1821 - 1895) "revived" the problem in 1878 at the meeting of the London Mathematical Society. He was namely concerned if anyone of the participants at the meeting managed to find a solution of this problem. Cayley was a mathematician and barrister, as well as a professor at the University in Cambridge. He was the youngest person that was elected a professor at the university in the 19. century. In 1879, he published an article in the journal *Proceedings of the Royal Geographical Society*. In this article, he admitted that he could not prove the statement in spite of the efforts made, but he came to some important conclusions:

– it is sufficient to observe only the maps where exactly three countries meet in each node, so called cubic maps. Namely, if more than three countries meet in some node, then a small circular "patch" is put on that node, the map thus obtained is coloured, and then the patch simply removed, (Fig. 11);

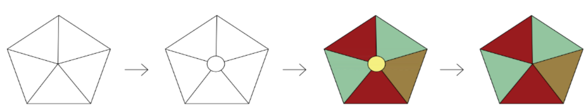


Figure 11: *Putting and removal of a "patch" when colouring a map where five countries meet in a node*

– if the four colour theorem was true, then map colouring could be performed in such a way that all countries located along the map edge are coloured with three colours at the most;

– if an arbitrary map consisting of n countries is already coloured with four colours and if we add one more country to this map, then a new map consisting of n+1 countries can be coloured with four colours.

The previous conclusion inspired Cayley to consider whether the problem could be solved by using the method of mathematical induction.

Hence, if a country is added to a map and correctly coloured, it would refer to proving the induction step: presuming that all maps with n counties can be coloured with 4 colours, it would mean also that all maps with $n+1$ counties can be coloured with 4 colours.

Consequently, the Theorem would thus be proved. However, there are too many combinations and ways in which one country can be added to some map. It is also a problem to attribute a colour to a new country. In some situations, it is trivial. However, there are cases when the colour needs to be changed for a large number of coloured countries in order to colour a new country correctly. For $n = 1,2,3$ and 4, the statement is trivial. Then, it could be derived from the statement for $n = 4$ that the Theorem is valid also when $n = 5$, if it is valid for all maps with 5 countries, it would be valid also for all maps with 6 countries, etc. Thus, the statement would be valid generally for all maps. It was very difficult to find a method to enlarge a map from n to $n+1$ countries that would be generally valid [7]. This is why Cayley decided to try to solve a problem by *contradiction*.

The basic idea when proving by contradiction is to assume that the statement we want to prove, say A, is false, i. e. $\neg A$ is true, and then show that this assumption leads to falsehood. Analogously, if a statement $\neg(A \Rightarrow B)$ leads to contradiction, it follows that $A \Rightarrow B$ is true.

It is first presumed that there are maps that cannot be coloured with 4 colours. A map with the smallest number of countries is selected that can be coloured with 5 or more colours. Such map is defined as the **minimal counterexample**. Then, the following statement is valid: the minimal counterexample cannot be coloured with four colours, but any map with fewer countries can be coloured with four colours. Hence, in order to prove the four colour theorem, it is necessary to prove that the **minimal counterexample does not exist**.

The next figure shows that the minimal counterexample does not contain a country that has only two neighbours. The following procedures is applied: if one edge is removed, a map with one country less is obtained that can be coloured with 4 colours at the most. The removed edge is then brought back to the map. The country with two coloured neighbours can be coloured with one out of two remaining colours (Fig. 12).

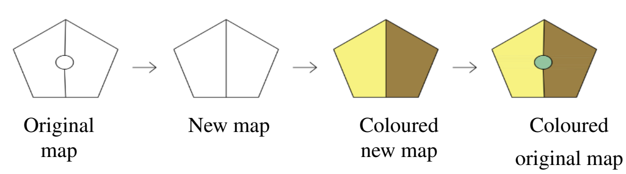


Figure 12: *A procedure applied for a country that has only two neighbours*

Similar proof procedure will also be applied with the country having three neighbours. One edge is removed and 3 countries are obtained out of 4 countries. Such map can be coloured with 4 colours. Three countries are coloured with three various colours, the removed edge is brought back, and the fourth country is coloured with the remaining fourth colour, (Fig. 13).

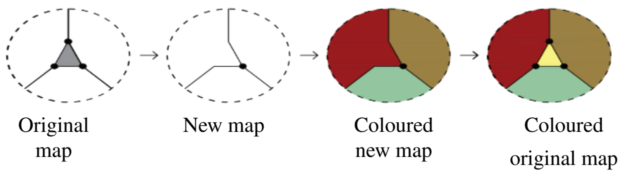


Figure 13: *Colouring countries having three neighbours*

However, there is a problem when applying the method of removing and restoring to the countries with 4, 5 or more neighbors, (Fig. 14 and Fig. 15).

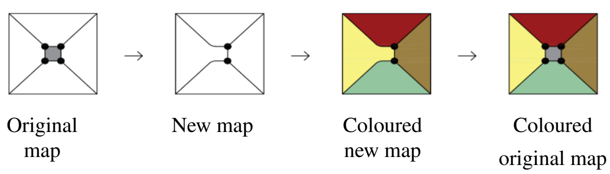


Figure 14: *The case when a map contains a square*

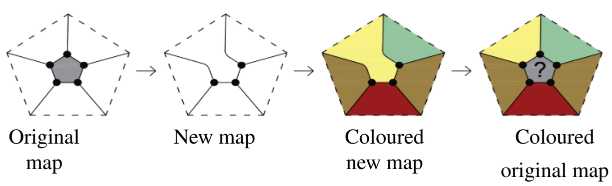


Figure 15: *The case when a map contains a pentagon*

4.1.3 Maps and Euler polyhedra

Leonhard Euler played an important role in proving the four colour conjecture with his findings and research that will later on be used by mathematicians. Dealing with regular polyhedra, he made an important discovery, namely a formula ("Euler formula") that states that:

number of faces $-$ **number of edges** $+$ **number of verteces** $= 2$.

The formula has many applications and can be generalized in various ways with one of them being used to handle planar graphs, i.e. maps (Chapter 3.1.2).

The connection between a map and a polyhedron is achieved by projecting a polyhedron from one point to a plane (Fig. 16). The faces of polyhedron in a plane projection represent countries/regions where one face is observed as the exterior of the projection, and the edges are actually boundary lines of countries. On the other hand, every cubic map, i.e. the map on which exactly three countries meet in each vertex, can be drawn onto a sphere and then, it can be imagined that it presents some polyhedron. In this case, the problem of colouring a spherical map is identical to the problem of colouring a map in the plane [22].

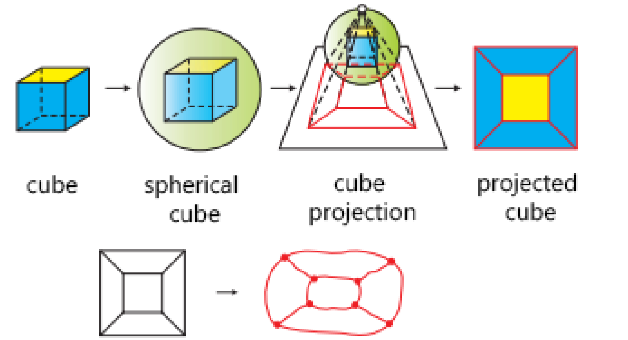


Figure 16: *Projection of a polyhedron from a point into the plane [22]*

A direct consequence of Euler's formula is the so called *enumeration* formula [7]. Using this formula one can count the regions, edges and vertices of a map that has r_2 regions with exactly two neighbors, r_3 regions with exactly 3 neighbors, r_4 regions with exactly 4 neighbors, etc.

Euler used the enumeration formula for proving the"only 5 neighbors" theorem, i.e. that *every cubic map has at least one region with five or fewer neighbors*. In addition, *if a map does not contain any biangle or a triangle and not a single square, it must contain at least 12 pentagons*. Similarly, it can be concluded the following: *if a cubic map consists entirely of pentagons and hexagons, then it must have exactly 12 pentagons* (for the proofs see [23]). Although Arthur Cayley had failed to prove that the minimum counterexamples do not exsist, his idea proved useful because it was used to prove somewhat weaker claim, the six – colour theorem.

4.2 Kempe "solves" the problem

Sir Alfred Bray Kempe (1849-1922) was also a barrister and mathematician. He finished his studies at the Trinity College, Cambridge where he attended the lectures of Arthur Cayley. He was also present at the meeting of the London Mathematical Society where Arthur Cayley spoke about the problem. He succeeded to apprehend the issues of 4 colour theorem, and a year later he published an article in the *American Journal of Mathematics* where he claimed to have managed to solve the problem successfully. The procedure of Kempe's method of colouring any map can be presented in all of the following six steps:

1. Find a country on the map that has 5 or less neighbors (it exists according to Theorem of "only 5 neighbors");

2. Cover the country in question with a piece of blank paper of similar shape, just a little bigger;

3. Extend all borders that touch the "patch" so that they meet at one point on the paper - as if the selected country has been reduced to one point (with this procedure the number of the countries on the map is reduced by 1), (Fig. 17);

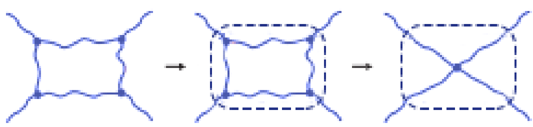


Figure 17: *Reducing the number of countries on a map*

4. Repeat the three previous steps until the initial map is reduced to a map with exactly one country;

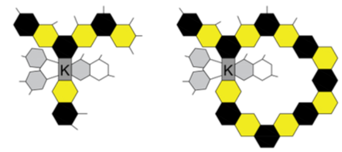
5. Colour the only remaining country with any of the four given colors;

6. Reverse the upper process: remove "patches" all the way back until you get the initial map and colour every "restored" country with different color from the neighbor along the way [8].

Now we face the problem that Cayley couldn't solve, i.e. how to color the country which has 4 or 5 neighbors. Kempe has solved this problem by using the method of chains.

4.2.1 Method of Kempe chain

Kempe assumed that the country K that needs to be coloured has a square form, i.e. borders with 4 countries. He then selected two countries that do not share borders with each other. Fig. 18 presents the country K and two not neighbouring countries that share their border with the country K. They are coloured in black and yellow colour. Now, on both of them, we continue with a line of black-yellow coloured countries. These lines can be connected in such a way that they make a closed circle that is then called a *chain*. Two cases may occur when coloring a map by means of this method, both shown in Figure 18.



(a) (b)

Figure 18: *Two possible outcomes when using the Kempe chain method*

Fig. 18 a) presents the first case of colouring the country K. It can be seen on the figure that the black neighbour of K is not connected with the yellow neighbour of K. Then it is possible to re-colur the black neighbour of the country K, e.g. with yellow colour. Black colour remains then available for the country K so that the map can be in accordance with the theorem. This procedure is shown on Fig. 19.

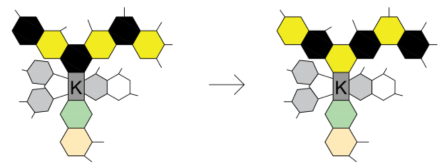


Figure 19: *Colour replacement in the line*

In the case presented on Fig. 18 b), the previous procedure of colour replacement shall not be successful. However, the chain of black and yellow countries makes a loop that starts and ends in the country K. Two other neighbours can be seen on Fig. 20: blue and yellow. It is not possible to join the chains of these two neighbours because they are interrupted by a black-yellow loop. The method is therefore applied similarly as in the first case: if a blue neighbour changes the colour into yellow, and the colours of the entire blue-yellow branch are replaced, the country K can be coloured with the remaining blue colour.

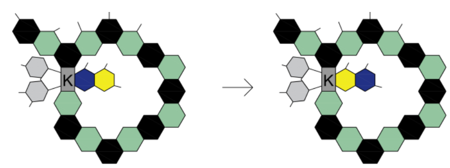


Figure 20: *Replacement of yellow and blue*

It is herewith proved that **no minimal counterexample contains "a square"**. It is namely sufficient to have 4 colours for the "square". It would be necessary to prove furthermore that the minimal counterexample does not contain a pentagon. The pentagon is surrounds by 5 countries that are already coloured with 4 colours. Kempe solved this problem by selecting two neighbours of *P* that do not touch each other: in this case, the yellow and red neighbour "above" and "below" *P* as presented on Fig. 21.

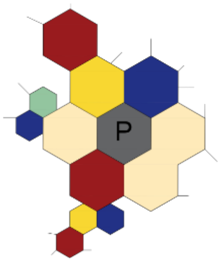


Figure 21: *Pentagon P and its neighbours*

If the above yellow-red line is not connected to the red-yellow line below, then the colours of the neighbours of the country *P* can be replaced, hence the yellow neighbour of *P* becomes red. Thus, yellow colour is left as a possible colour for *P*, as it is presented on Fig. 22.

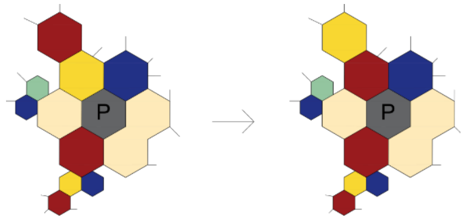


Figure 22: *Yellow neighbour of P becomes red*

If the "above" yellow-red line on Fig. 21 is connected with the red-yellow line "below" the country *P*, then the blue neighbour of the country *P* can be observed, as well as red-blue and blue-red lines. Such example is presented on Fig. 23.

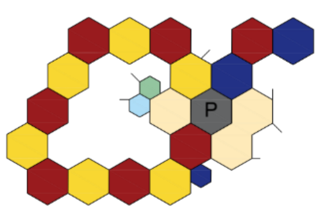


Figure 23: *The red-yellow chain*

Consider the situation given on Fig. 22. If the blue-red line "above" was not connected with the red-blue line "below", the blue neighbour of the country *P* can be coloured with red colour, and all countries in the blue-red line can be re-coloured. In this way, another red-blue line is obtained. Thus, only blue colour is left for the country *P* as presented on Fig. 24.

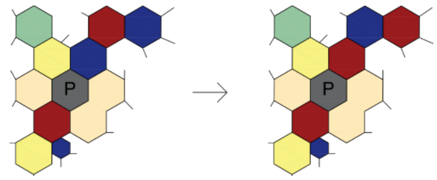


Figure 24: *Blue neighbour of P becomes red*

However, if the chains are linked, then there are two loops together with the previous one. Such situation is presented on the next figure.

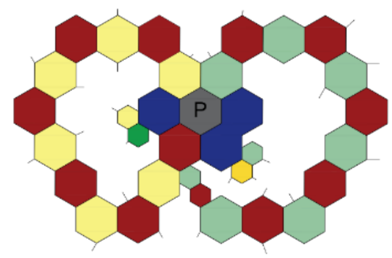


Figure 25: *The blue-yellow and the blue-green chain*

It can be seen on Fig. 25 that blue-yellow line on the left side of the country *P* cannot be connected with the blue-yellow line on the right side of the country *P*. The colours of the blue-yellow line on the right side can then be replaced. The blue-green line on the left side cannot be connected with the blue-green line on the right side, hence, the blue-green line on the left side can be re-coloured. If the lines are re-coloured simultaneously, the country *P* will have the neighbours in yellow, red and green colour, and it can be coloured with blue colour, as presented on Fig. 26.

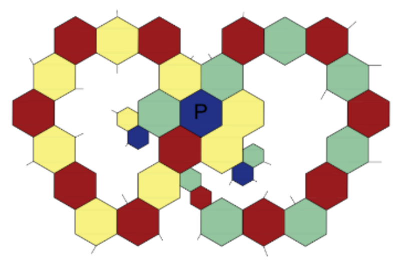


Figure 26: *P is coloured blue*

Hereby, the procedure of colouring the map to which the country *pentagon* has been brought back is completed. Using the above described procedure of colouring the map that has a country with five neighbours, Kempe found a proof that the minimal counterexample **does not contain a pentagon**. It is, however, in contradiction with the "five neighbours" theorem according to which every cubic map contains at least one country with five or less neighbours.

According to Kempe, the four colour theorem would thus be proved. After publishing the article, Kempe was recognized as the person who proved the theorem. His article was published in the *American Journal of Mathematics*.

4.2.2 The flaw in Kempe's proof

It took 11 years to spot the mistake in Kempe's proof. In 1890, Percy John Heawood (1861 - 1955), a professor of mathematics from Durham denied Kempe's theory. After that, Kempe's proof became the most famous inaccurate proof in the history of mathematics.

In his article *Map-colour Theorem*, published in the *Quarterly Journal of Mathematics* in 1890, Heawood explained that the error occurred at the end of the proof in determining the colour of the country with pentagon form. Heawood proved that *there were maps on which it is impossible to make changes of colours in two different chains simultaneously*.

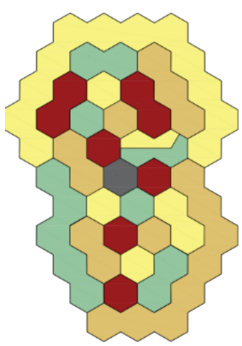


Figure 27: *Map with 25 countries*

On Fig. 27, each of the twenty five countries is coloured with one of the four colours: red, golden, yellow or green, except the central pentagon *P*. It was proved that this map can be coloured only with four colours.

The application of Kempe's methods in determining the pentagon *P* provides the re-colouring of two neighbours of the pentagon *P*.

Each of these two changes is allowed if it is done separately. The problem occurs if it is attempted to make the changes simultaneously.

Two neighbouring countries marked with the letter *A* (yellow colour) and *B* (golden colour) become red as it is presented on Fig. 28. The basic principle of 4 colour problem implying that the neighbouring countries should be coloured with various colours is hereby undermined. Hence, one came to the conclusion that Kempe's method of proof was wrong.

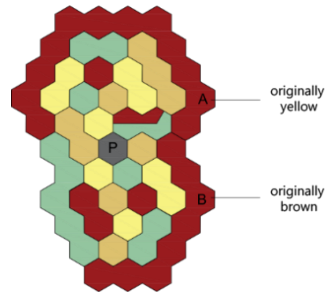


Figure 28: *Contradiction*

In 1891, Kempe admitted publicly that he was wrong. However, he managed to notice the mistake in the proof Heawood did not know how to correct it. In his second paper, Heawood approached the problem using a number theory, but even this attempt to prove the four colour theorem ended unsuccessfully. Using Kempe's ideas, Heawood was able to prove the Five colour theorem. Although the Five colour theorem was weaker than the Four colour theorem, it still represents one more step that will be needed to prove the initial problem.

After his proof had been denied, Kempe approached the problem in somewhat different way; in each country on a map, he highlighted one point (e.g. capital city) and then connected the points representing the neighbouring countries with lines. The new structure matched the structure of a graph. The problem of determining the colours of individual countries was reduced to assigning it to points, but in such a way that the neighbouring points were named differently. The importance of this idea is related to the fact that in such a way the problem of map colouring was transferred into graph theory (Fig. 9). Based on this idea and with the help of computers, the Four colour theorem will finally be proved.

4.3 Heesch, Appel and Haken finally solve the problem

The conjecture on four colours is articulated so simply that it was presumed someone would find an elegant and simple solution one day. However, something completely different happened.

In 1904, a new idea about the proof of this conjecture occurred. This approach started with the search for *unavoidable sets*. Before defining this term, it is necessary to define the terms triangulation and configuration. A plane graph is a *triangulation* if it is connected and every region is a triangle. A *configuration* is a part of triangulation included inside the area (map). The unavoidable set is then defined as a set of configurations with the property that any triangulation must contain one of the configurations in the set. Unavoidable set is actually a set of countries out of which at least one country must be located on every map. Since it is sufficient to observe only cubic maps, Fig. 29 presents some countries that a cubic map must contain.

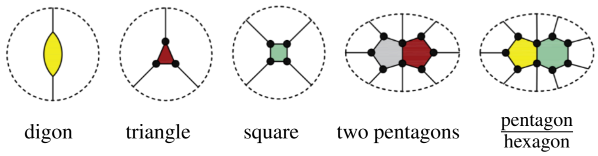


Figure 29: *Unavoidable set for a cubic map*

It can be deduced from the figure that if a map does not contain digon (a country with two neighbours), triangle or rectangle, it must contain either two connected pentagons or connected pentagon and hexagon.

Among the mathematicians who started to search for unavoidable sets, George David Birkhoff (1884 - 1944) made a distinguished contribution in this area. Birkhoff was the first to introduce the concept of *reducibility*. A configuration is reducible if it cannot be contained in a triangulation of the smallest graph which cannot be 4-coloured. It means that reducible configuration is a set of countries that cannot appear in the minimal counterexample. Minimal counterexample is, as it has been defined earlier, a map that contains the smallest number of countries and can be coloured with 5 or more colours. From this follows that the minimal counterexample cannot be coloured with 4 colours, but every map with smaller number of countries can. Hence, in order to prove the Four colour theorem, it is necessary to prove that the minimal counterexample does not exist.

The research and search for unavoidable sets and reducible configurations were developing separately until German mathematician Heinrich Heesch (1906 - 1995) unified those 1960. His goal was namely to find an *unavoidable set of reducible configurations*. If a set is unavoidable, then each map must contain at least one of the configurations from that set, and since every configuration is reducible, it cannot be contained in a minimal counterexample. It would thus be proved that there are no minimal counterexamples, and consequently, the 4 colour theorem would be proved. He therefore developed an algorithm naming it *D-Reduction* that he adapted to computer methods (programming) [24]. This algorithm is used to prove that every graph contains a subgraph from a specific set, i.e. that every map contains some map from an unavoidable set.

Heesch presumed that he would have needed to observe a set of about 8900 configuration. However, certain problems appeared in his approach, as for example the inability to test the reductions of some configurations, mostly because of a large number of vertices within some rings, i. e. configurations that “wrap arround and meet themselves” [17].

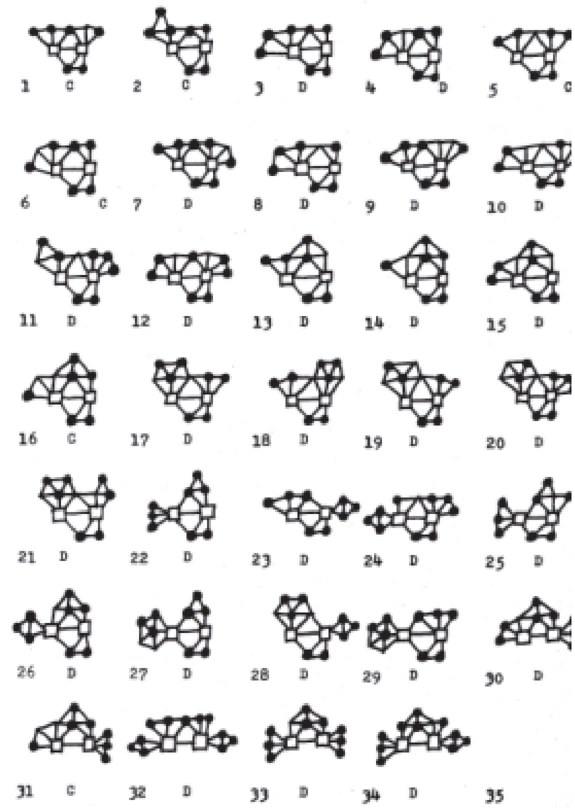


Figure 30: *A set of configurations*

In 1972, Wolfgang Haken, a student of mathematics, physics and philosophy continued after a short collaboration with Heesch to work with a programmer and mathematician Kenneth Appel on upgrading of Heesch’s idea. They were focused on the improvement of Heesch algorithm. After two years, John Koch joined them, and the three of them succeeded together to create the programme to be used in searching for unavoidable sets of reducible configurations. Unlike Heesch, they manage to reduce the number of ring vertices from 18 to 14 avoiding thus the complications and simplifying the counting.

Using the programme for searching an unavoidable set of reducible configurations, Appel and Haken, both from the University of Illinois, managed to prove the assumption in 1976. Since there are too many possible configurations, the proof could not be carried out without computer assistance. The usage of computer in proving this problem caused numerous discussions and disapprovals.

Still, Haken and Appel published the proof on July 22, 1976 that was based on the construction of the unavoidable set of 1936 reducible configurations, and in 1977 all three of them published the proof in *Illinois Journal of Mathematics* with the unavoidable set of 1482 reducible configurations. The proof was published in two parts, and the text was accompanied by the material on microfilm with 450 pages of various diagrams and detailed explanations. However, Ulrich Schmidt found an error in the programme in 1981 that was soon corrected.

Regardless of the difficulties, the four colour problem gained the status of a theorem for the second time. The following comment illustrates the opinion of a great number of mathematicians at that time:

”Good mathematical proof is like a poem - this is a telephone directory”

Appel and Haken published therefore in 1986 an article where they described their methods in details strongly defending the proof and rejecting any doubt, and three years later they also published a book titled *Every Planar Map is Four Colourable*.

Due to the complicated nature of part of the proof that can’t be checked without computer assistance, Paul Seymour, Neil Robertson, Daniel Sanders and Robin Thomas decided to simplify the proof and to eliminate all doubts. However, they gave up soon after they had started to study it. They decided to develop their own proof based on the ideas of Appel and Hesch.

As they [17], wrote in the article, the concept of the proof itself is identical to the concept of Appel and Haken. They tested the set of 633 configurations proving that each of them is reducible. Furthermore, they proved that at least one of the 633 configurations appears in a planar graph with 6 vertices (minimal counterexsample). They have thus proved the unavoidability. This part shows the largest difference between their proof and the one made by Appel and Haken. In order to prove the unavoidability, they used the method of discharging that unlike with Appel and Haken has 32 rules as related to 300+.

The article itself and their proof were presented at the International Congress of Mathematicians in Zürich in 1994 where they finally proved that Appel and Haken were right. The proof itself was also upgraded and improved and contained 633 configurations instead of 1482.

Fig. 31 presents 17 out of 633 configurations that were used in this proof. When drawing the configurations, they used Heesch’s method of marking. The forms of vertices present the degree of a vertex. Black circle represents the vertex of degree 5, the point (shown on the figure without a symbol) has the vertex degree 6, empty circle represents the vertex of the degree 7, the triangle the vertex of degree 9, and the pentagon represents the vertex of the degree 10.

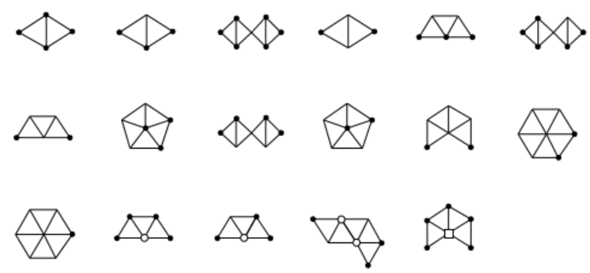


Figure 31: *17 out of 633 configurations [17]*

5 Application within cartography

Each map in the plane can be coloured with four colours such that neighbour areas are in different colours, as it is shown in previous chapters. Application of four colour theorem for colouring political maps is tested in this chapter. We use a world political map as an example. The software used is QGIS [25] with its unofficial plugin TopoColour [26], which implements algorithms for graph colouring.

Geographical maps of administrative units, i.e. political maps, have some specialities that should be considered prior to the application of graph colouring algorithms.

As it is defined in 3.1.1, neighbouring countries (administrative units) are those which shares common boundary line. Existence of common point does not imply neighbours. Geographical maps are abstract and generalised models of reality, and it is possible that some short boundary line in reality is represented as point, due to model or cartographic generalisation. Application is therefore possible only to model of geographical reality, users should be aware of these constraints, and in case of unexpected results, know how to deal with it.

Maritime boundaries are often not shown on political maps, and almost never are administrative areas on the sea coloured with different colours. This is certainly true for world political maps, where colouring is usually applied to land parts of countries. This means that countries that share only maritime boundaries will not be considered as neighbours and could be coloured automatically with the same colour.

Further, countries are often consisted of more or less distant land parts, e.g. islands or exclaves. For example, some countries at certain administrative level contains overseas territories. This could potentially lead to non-planar graphs representing neighbours.

5.1 Methodology and programs used

The data used for the world countries were taken from GADM database of Global Administrative Areas [27].

There is no unique solution for model of countries and its boundaries, and it is often result of the point of view of certain diplomacy. For the purpose of this paper, we will not change the original data, because we will use it for the purpose of demonstration of application of graph colouring, and not for making special world political map. The dataset consists of 256 top-level administrative units, i.e. countries.

Data was first loaded in QGIS as separate vector layer and transformed to Eckert VI map projection, one of map projections suitable for world maps.

TopoColour plugin implements algorithms from graph colouring theory with purpose of colouring polygons in vector layer. It also allows creating graph representing detected neighbours in dataset.

Typical procedure for colouring areas in vector layer is as follows:

– Start the TopoColour plugin and select polygon vector layer and one column in attribute table that has unique value for each administrative unit. Finding of neighbours starts. It can take a while for complex geometries (e.g. up to one hour or more).

– When neighbours are found, user selects "greedy" or "random" algorithm and starts the computing of colours. Number of colours needed is given as a result. For "random" algorithms, successive computations can give different number of colours.

– Save the colour numbers to one column in attribute table and style the layer.

Greedy algorithm gives five colours for political map of the world. It does not guarantee optimum number of colours. Brute force approach for four colours and 256 countries would yield 4^{256} different colour assignments, and it is not feasible even with modern computers. Random algorithm usually gives six or seven colours for this political map.

After computation of colours is done, layer can be styled in order to get nice coloured map (Fig. 32).

It is known that world political map can be coloured with four colours. In order to achieve this we start with automatically defined colours, eliminate less used colour by replacing it with one of the four colours, and rearranging the colours of neighbours. It is not too hard to accomplish that, and result can be seen on Fig. 33. This also means that graph representing neighbouring countries is planar (Fig. 34) and algoritms used are not giving optimal solution.

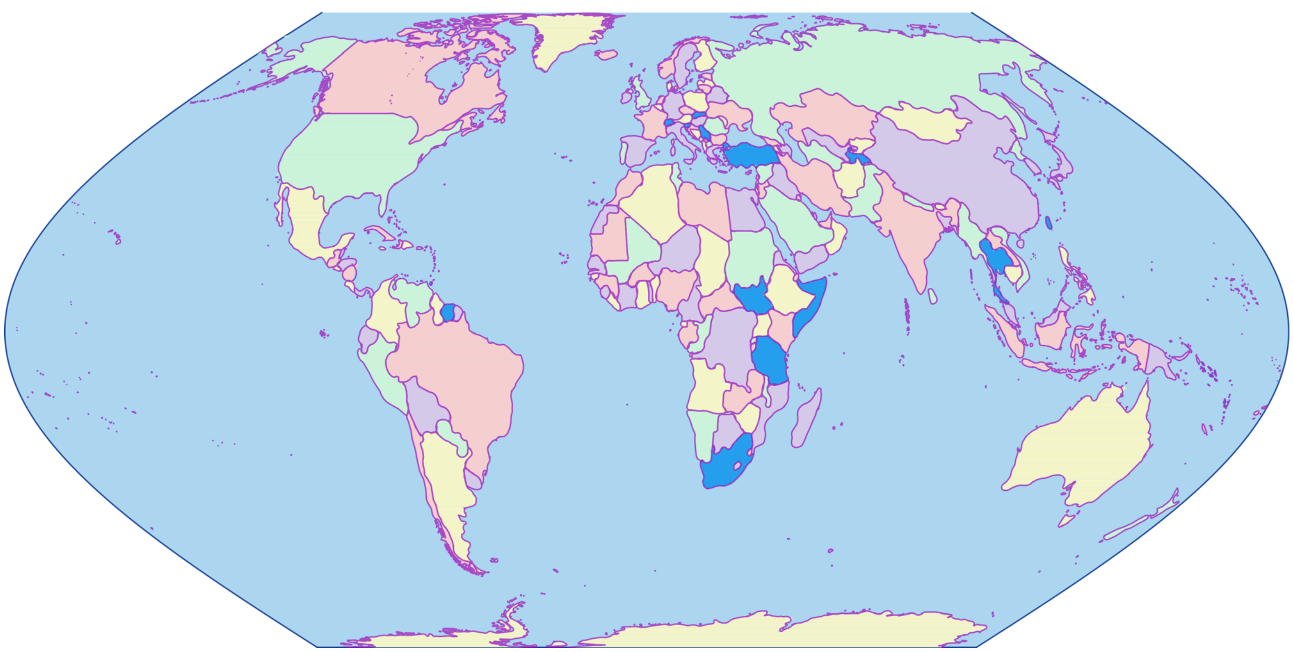


Figure 32: *World political map coloured with five colours obtained by greedy algorithm implemented in QGIS plugin TopoColour. Less used colour is blue, and is a good candidate for manual elimination.*

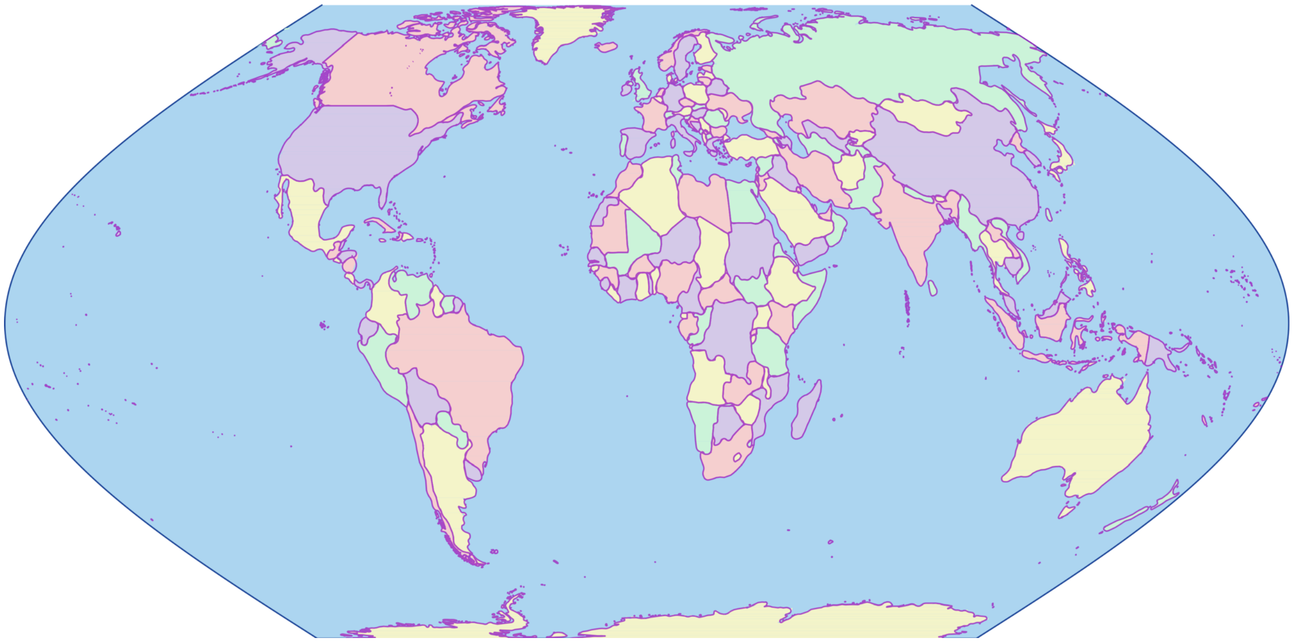


Figure 33: *Manual elimination of the fifth colour from automatically coloured world political map gives a map with four colours.*

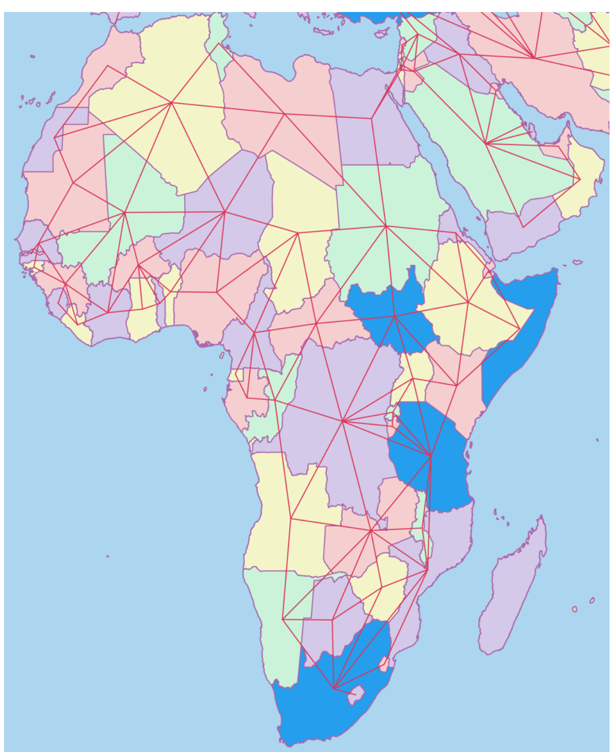


Figure 34: *Graph representing neighbouring countries (clipped to Africa region)*

6 Conclusion

Graph colouring is widely applied in many scientific fields. In this paper, the focus is on the application in cartography. Since the map colouring with four colours is rarely mentioned in cartographic books, we were motivated to research this connection in this paper.

The four colours conjecture has proved to be one of the greatest and long lasting problems in mathematics. The problem itself has attracted the attention both of mathematicians and laypersons. It took more than one century to prove this conjecture, which was achieved only with the development of information science and with computer assistance. It is also the first more significant theorem that has been proved in such a way. The theorem faced a lot of negative comments because of that and was not well accepted by the mathematical public of that time.

Algorithms implementing graph colouring and four colour theorem, which are still not so widely available in cartographic software, provide analysis and processing of map data with aim of colouring administrative units or creating political maps. In this process one should take care of geometry of boundaries because even very small differences in coordinates can give unexpected results. Special care has to be given to model of geographical reality, e.g. definition of administrative entities, maritime booundaries between countries, overseas territories etc. We can conclude that automatization of colouring of administrative units can greatly help mapmakers, but for the final map, manual interventions are still required.

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