FINITE NONABELIAN $p$-GROUPS OF EXPONENT $> p$
WITH A SMALL NUMBER OF MAXIMAL ABELIAN SUBGROUPS OF EXPONENT $> p$

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Abstract. Y. Berkovich has proposed to classify nonabelian finite $p$-groups $G$ of exponent $> p$ which have exactly $p$ maximal abelian subgroups of exponent $> p$ and this was done here in Theorem 1 for $p = 2$ and in Theorem 2 for $p > 2$. The next critical case, where $G$ has exactly $p + 1$ maximal abelian subgroups of exponent $> p$ was done only for the case $p = 2$ in Theorem 3.

Let $G$ be a nonabelian finite $p$-group of exponent $> p$. If $S$ is a minimal nonabelian subgroup in $G$, then $S$ has exactly $p + 1$ maximal subgroups $S_1, S_2, \ldots, S_{p+1}$ and they are abelian and they lie in $p + 1$ pairwise distinct maximal abelian subgroups in $G$. If at least two of $S_i$ are elementary abelian, then $S$ is generated by its elements of order $p$ and then (by Lemma 65.1 in [2]) $S \cong D_8$ or $S \cong S(p^2)$ (the nonabelian group of order $p^2$ and exponent $p > 2$). If all minimal nonabelian subgroups of $G$ are generated by its elements of order $p$, then by Theorem 10.33 in [1] (for $p = 2$) and Proposition 7 in [3] (for $p > 2$), $G$ has only one maximal abelian subgroup $A$ of exponent $> p$, where $A$ is of index $p$ in $G$ and $A = H_p(G)$ (Hughes subgroup). However, if a minimal nonabelian subgroup of $G$ has at most one elementary abelian maximal subgroup, then $G$ has at least $p$ maximal abelian subgroups of exponent $> p$.

From the above follows that a nonabelian $p$-group $G$ of exponent $> p$ has either exactly one maximal abelian subgroup of exponent $> p$ or $G$ has at least $p$ of them. Therefore Y. Berkovich has proposed to classify nonabelian finite $p$-groups of exponent $> p$ which have exactly $p$ maximal abelian subgroups of exponent $> p$ and this was done here in Theorem 1 for $p = 2$ and in Theorem

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2 for \( p > 2 \). By the above, such a group \( G \) possesses a minimal nonabelian subgroup \( S \) which is not isomorphic to \( D_8 \) or \( S(p^3) \). Also, such an \( S \) has exactly one maximal subgroup \( X \) which is elementary abelian so that \( \Phi(S) = \text{Z}(S) \) is elementary abelian and \( |S : \Phi(S)| = p^2 \). Let \( a \in S \setminus X \) and \( b \in X \setminus \Phi(S) \) so that \( o(a) \leq p^2 \), \( o(b) = p \) and \( S = \langle a, b \rangle \), where \( \Phi(S) = \langle a^p, [a, b] \rangle \). If \( |\Phi(S)| = p \), then \( |S| = p^3 \) and \( \simeq M_p^3 \) (the nonabelian group of order \( p^3 \) and exponent \( p^2 \), where \( p > 2 \)). If \( |\Phi(S)| = p^2 \), then \( S \simeq M_p(2, 1, 1) \), where

\[
M_p(2, 1, 1) = \langle a, b \mid a^{p^2} = b^p = 1, [a, b] = c, e^p = [c, a] = [c, b] = 1 \rangle.
\]

Suppose that \( S \) is non-normal maximal subgroup of \( H \) of exponent \( > p \). Set \( K = N_S(H) \) so that \( |G : K| = p \), \( H < K \) and \( H^G \leq K \). All elements in \( G - K \) are of order \( p \). If \( p = 2 \), then \( K \) is abelian (by a result of Burnside), a contradiction. Hence in this case we must have \( p > 2 \). For any \( g \in G - K \), \( H^g \leq K \) and so \( H \) and \( H^g \) normalize each other.

Y. Berkovich has proposed to consider also the next critical case, where \( G \) has exactly \( p + 1 \) maximal abelian subgroups of exponent \( > p \). However, we have been able to classify such \( p \)-groups only in case \( p = 2 \) in Theorem 3.

**Theorem 1.** Let \( G \) be a nonabelian 2-group with exactly 2 maximal abelian subgroups of exponent \( > 2 \). Then \( G = M \times V \), where

\[
M \cong M_2(2, 1, 1) = \langle a, b \mid a^4 = b^2 = 1, [a, b] = c, c^2 = [c, a] = [c, b] = 1 \rangle
\]

and \( \exp(V) < 2 \).

**Proof.** Let \( G \) be a nonabelian 2-group with exactly 2 maximal abelian subgroups of exponent \( > 2 \). Let \( H_1 \) and \( H_2 \) be the two maximal abelian subgroups of exponent \( > 2 \), where we know that \( H_1 \) and \( H_2 \) are normal in \( G \). If \( H_1 H_2 < G \), then all elements in \( G - (H_1 H_2) \) are involutions and then (by a result of Burnside) \( H_1 H_2 \) would be abelian, a contradiction. Hence \( H_1 H_2 = G \) and \( H_1 \cap H_2 = \text{Z}(G) \) so that \( G \) is of class 2 and all elements in \( G - (H_1 \cup H_2) \) are involutions. Indeed, all elements of order \( > 2 \) lie in \( H_1 \) or \( H_2 \) (by our hypothesis). If \( g \in G - (H_1 \cup H_2) \), then a maximal abelian subgroup \( H \) containing \( \langle g \rangle \) is elementary abelian implying that \( \text{Z}(G) \) is elementary abelian.

Since \( H \unlhd G \), Lemma 57.1 in [2] implies that for any \( x \in G - H \) there is \( h \in H \) such that \( \langle x, h \rangle \) is minimal nonabelian. Since \( \langle x, h \rangle \cong D_8 \) or \( M_2(2, 1, 1) \), it follows that \( \exp(\langle x, h \rangle) = 4 \) and so \( o(x) \leq 4 \). We have proved that \( \exp(G) = 4 \).

For any \( x, y \in G \), \( [x^2, y] = [x, y]^2 = 1 \) and so we get \( \hat{O}_4(G) \leq \text{Z}(G) \).

Suppose that both \( H_1 \) and \( H_2 \) are not maximal subgroups in \( G \). Then \( |H_i : \text{Z}(G)| \geq 4 \) for \( i = 1, 2 \) and let \( t_i \in H_i - \text{Z}(G) \) be an element of order 4 \( (i = 1, 2) \) so that \( 1 \neq t_i^2 \in \text{Z}(G) \). Let \( H_i^* \) be a maximal subgroup of \( H_i \) which contains \( \text{Z}(G) \langle t_i \rangle \), \( i = 1, 2 \). Then \( M_1 = H_1 H_1^* \) and \( M_2 = H_2 H_2^* \) are distinct maximal subgroups of \( G \) containing \( H_1 \) and \( H_2 \), respectively. Since all elements in \( G - (H_1 \cup H_2) \) are involutions, it follows that all elements in
$G - (M_1 \cup M_2)$ are involutions. Let $g \in G - (M_1 \cup M_2)$ and $m \in M_1 \cap M_2$. Then $g$ and $gm \in G - (M_1 \cup M_2)$ are involutions and so we get

$$1 = (gm)^2 = gmgm = g^2m^2m = m^2m$$

It follows that $g$ inverts each element in $M_1 \cap M_2$ so that a result of Burnside implies that $M_1 \cap M_2$ is abelian. In particular, $\langle h_1, h_2 \rangle$ is abelian. Let $Y$ be a maximal abelian subgroup in $G$ containing $\langle h_1, h_2 \rangle$. By our hypothesis, $Y = H_1$ or $Y = H_2$, a contradiction. We have proved that we may assume $|G: H_1| = 2$ and so $H_1$ is a maximal subgroup in $G$.

Let $H_1^*$ be a maximal subgroup of $H_1$ containing $\Omega_1(H_1)$. Then $M_2 = H_2H_1^*$ is a maximal subgroup of $G$ and all elements in $G - (H_1 \cup M_2)$ are involutions. If $g \in G - (H_1 \cup M_2)$, then for any $x \in H_1^* = H_1 \cap M_2$, $gx \in G - (H_1 \cup M_2)$ is an involution. This implies $x^g = x^{-1}$ and so $g$ inverts each element in $H_1^*$. In particular, $g$ centralizes $\Omega_1(H_1)$. It follows that $\Omega_1(H_1) \leq Z(G)$ and so $\Omega_1(H_1) = Z(G) = H_1 \cap H_2$ and therefore all elements in $H_1 - Z(G)$ are of order 4.

Suppose that $Z(G)$ is not a maximal subgroup in $H_1$. Note that all elements in $G - (H_1 \cup H_2)$ are involutions and all elements in $H_2 - H_1$ and in $H_1 - Z(G)$ are of order 4. Let $v \in H_1 - Z(G)$ so that $v^2 \in Z(G)$ and let $H_1^*$ be a maximal subgroup of $H_1$ containing $Z(G)\langle v \rangle$ so that $M_2^* = H_1^*H_2$ is a maximal subgroup in $G$. If $g \in G - (H_1 \cup M_2^*)$, then $g$ and $gv \in G - (H_1 \cup M_2^*)$ are involutions implying that $v^g = v^{-1}$. Then each element in $G - H_1$ also inverts $\langle v \rangle$. Hence each element in $G - H_1$ inverts each element of order 4 in $H_1$ and since it also centralizes $Z(G)$, it follows that each element in $G - H_1$ inverts each element in $H_1$. But then $G$ is quasidihedral and so in particular all elements in $G - H_1$ must be involutions, a contradiction. We have proved that $Z(G) = H_1 \cap H_2$ is a maximal subgroup in $H_1$ and so $H_2$ is also a maximal subgroup in $G$.

If each minimal nonabelian subgroup in $G$ is isomorphic to $D_8$, then by Theorem 10.33 in [1] our group $G$ is quasidihedral and so $G$ has only one maximal abelian subgroup of exponent $> 2$, a contradiction. Hence $G$ possesses a minimal nonabelian subgroup

$$M \cong M_2(2, 1, 1) = \langle a, b \mid a^4 = b^2 = 1, [a, b] = c, c^2 = [c, a] = [c, b] = 1 \rangle.$$ 

Then $M$ covers $G/H_1$ and $H_1/Z(G)$ and $M \cap H_1$ is abelian of type $(4, 2)$, where we have $M \cap Z(G) \cong E_4$. Indeed, if $M$ does not cover $G/H_1$ or $H_1/Z(G)$, then $M$ would be abelian, a contradiction. Let $V$ be a complement of $M \cap Z(G)$ in $Z(G)$. Then $G = M \times V$ and our theorem is proved.

**Theorem 2.** Let $G$ be a nonabelian $p$-group of exponent $> p$, where $p > 2$. Suppose that $G$ has exactly $p$ maximal abelian subgroups $H_1, H_2, \ldots, H_p$ of exponent $> p$. Then $\exp(G) = p^2$, $Z(G)$ is elementary abelian, each $H_i$ normalizes each $H_j$ ($i, j = 1, 2, \ldots, p$), $H = H_1H_2\cdots H_p = H_p(G)$ (Hughes subgroup) and $\Omega_1(G) \leq Z(H) = H_1 \cap H_2\cdots \cap H_p$.
Proof. Let $G$ be a $p$-group, $p > 2$, satisfying the assumptions of Theorem 2. It is easy to see that $G$ possesses at least one minimal nonabelian subgroup $M$ which is isomorphic to $M_{p^3}$ or $M_p(2,1,1)$. Suppose that this is false. Then all minimal nonabelian subgroups of $G$ are isomorphic to $S(p^3)$ and so by Proposition 7 in [3] $G$ has an abelian subgroup $A$ of exponent $> p$ and index $p$ such that $A = H_p(G)$. But then $G$ has only one maximal abelian subgroup of exponent $> p$, a contradiction. Hence there is such $M$ as above. Any two maximal subgroups of $M$ lie in two distinct maximal abelian subgroups in $G$. In this way we get $p$ pairwise distinct maximal abelian subgroups in $G$ of exponent $> p$ and one maximal abelian subgroup which is elementary abelian. In particular, $Z(G)$ is elementary abelian.

We want to show that $\exp(G) = p^2$. Let $H_1, H_2, \ldots, H_p$ be the set of all $p$ maximal abelian subgroups in $G$ which are of exponent $> p$. Set $\exp(G) = p^e$, where $e \geq 2$ and let $g$ be an element of order $p^e$ so that $g \in H = H_1 H_2 \cdots H_p$, where we know that each $H_i$ normalizes each $H_j$ (see the paragraph preceding Theorem 1). If $g$ is not contained in all $H_i$ $(i = 1, 2, \ldots, p)$, say $g \notin H_1$, then by Lemma 57.1 in [2], there is $h_1 \in H_1$ such that $\langle g, h_1 \rangle$ is minimal nonabelian. Since all minimal nonabelian subgroups of $G$ are of exponent $\leq p^2$, we get $e = 2$. So suppose that $g \in H_i$ for all $i = 1, 2, \ldots, p$. In particular, $g \in H_1 \cap H_2$. Since $\langle H_2 - H_1 \rangle = H_2$, there is $h \in H_2 - H_1$ such that $o(h) = p^e$. By Lemma 57.1 in [2], there is $k \in H_1$ such that $\langle h, k \rangle$ is minimal nonabelian. This implies again $e = 2$. We have proved that $\exp(G) = p^2$. If $H < G$, then all elements in $G - H$ are of order $p$ and so $H = H_p(G)$. Now, $Z(H)$ centralizes all $H_i$ and so $Z(H) \leq H_1 \cap H_2 \cap \cdots \cap H_p$. But $H_1 \cap H_2 \cap \cdots \cap H_p \leq Z(H)$ and so we get $Z(H) = H_1 \cap H_2 \cap \cdots \cap H_p$.

Let $g$ be any element of order $p^2$ in $G$. Then $g \in H = H_1 H_2 \cdots H_p$, where $H_i \leq H$ for all $i = 1, 2, \ldots, p$. We have either $g \in H_1$ (and then also $g^p \in H_i$) or (by Lemma 57.1 in [2]) there is $h_i \in H_i$ such that $M = \langle g, h_i \rangle$ is minimal nonabelian, where $M \cong M_{p^3}$ or $M \cong M_p(2,1,1)$. Then we know that $M$ contains exactly one maximal subgroup $X$ of exponent $p^2$ such that $X \leq H_i$. This implies that $g^p \in X \leq H_i$. Hence in any case we get $g^p \in H_i$ for all $i = 1, 2, \ldots, p$. Hence $g^p \in H_1 \cap H_2 \cdots \cap H_p = Z(H)$ and so $U_1(G) \leq Z(H)$. Our theorem is proved.

Theorem 3. Let $G$ be a nonabelian 3-group with exactly 3 maximal abelian subgroups $H_1, H_2, H_3$ of exponent $> 2$. Then $G = H_1 H_2 H_3$ and $Z(G) = H_1 \cap H_2 \cap H_3$.

(a) If $H_1$ is conjugate in $G$ to (say) $H_2$, then $\exp(H_2) = 4$, $H_2$ is of index 2 in $G$ with $\exp(H_3) \leq 8$, $Z(G)$ is elementary abelian and $G$ has a maximal subgroup which is quasidihedral of exponent 4.

(b) If all $H_i$ are normal in $G$, $i = 1, 2, 3$, then $G$ is of class 2, $U_1(G) \leq Z(G)$ and so $G'$ is elementary abelian.
Proof. Let $G$ be a nonabelian 2-group with exactly 3 maximal abelian subgroups $H_1, H_2, H_3$ of exponent $> 2$. Set $H = \langle H_1, H_2, H_3 \rangle$ so that $H \trianglelefteq G$. If $H < G$, then all elements in $G - H$ are involutions. But then (by a result of Burnside) $H$ is abelian, a contradiction. Hence we have $G = \langle H_1, H_2, H_3 \rangle$ and then obviously $Z(G) = H_1 \cap H_2 \cap H_3$.

(i) First we consider the case where some $H_i$ are not normal in $G$.

Then we may assume that $H_1$ and $H_2$ are conjugate in $G$ and then $H_3 \trianglelefteq G$. We set $K = N_G(H_1)$ so that $|G : K| = 2$, $H_1 < K$ and $K = N_G(H_2)$. For any $g \in G - K$, $H_2 = H_1^g$ and $H_1 H_2 = H_1^{g^2}$. Then $H_3$ covers $G/(H_1 H_2)$ so that $G = (H_1 H_2)H_3$. All elements in $G - (K \cup H_3)$ are involutions and so for each involution $i \in G - (K \cup H_3)$, a maximal abelian subgroup in $G$ containing $i$ is elementary abelian. In particular, $Z(G)$ is elementary abelian.

Set $G_1 = H_1 H_3$ and let $g \in G - K$ so that $H_2 = H_1^g \leq G_1$. It follows $G_1 = G$ and set $H_3' = H_3 \cap K$ so that $H_3'$ normalizes $H_1$. We have $H_1 \cap H_3 = Z(G)$ is elementary abelian and so $H_2 \cap H_3 = Z(G)$. Then $K = H_1 H_3'$ and $K' \leq H_1 \cap H_3' = Z(G) \leq Z(K)$ so that $K$ is of class 2 and $K'$ is elementary abelian. For any $k_1, k_2 \in K$ follows $[k_1, k_2] = [k_1, k_2] = 1$ and so $\mathcal{O}_1(K) \leq Z(K)$. We have $Z(K) < H_1$ and if $Z(K) > Z(G)$, then $Z(K)H_3'$ is contained in a maximal abelian subgroup in $G$ distinct from $H_1, H_2$ and $H_3$ and so $Z(K)H_3'$ must be elementary abelian. We have proved that in any case $Z(K)$ is elementary abelian and so $\exp(K) = 4$ and $4 \leq \exp(H_3) \leq 8$.

Assume, by way of contradiction, that $Z(K) > Z(G)$. Since $Z(K) < H_1$, it follows that $L = Z(K)H_3$ is a proper subgroup of $G$. We know that all elements in $G - (K \cup L)$ are involutions. Let $i \in G - (K \cup L)$ and $x \in K \cap L$. Then $ix \in G - (K \cup L)$ and so

$$1 = (ix)^2 = ixix \text{ implying } x^i = x^{-1}.$$ 

Since $i$ inverts each element in $K \cap L$, it follows that $i$ centralizes $Z(K)$ (noting that $Z(K)$ is elementary abelian). But then $Z(K) \leq Z(G)$, a contradiction. We have proved that $Z(K) = Z(G)$ and so in particular, $\mathcal{O}_1(H_1) \leq Z(G)$.

Suppose, by way of contradiction, that $H_3$ is not a maximal subgroup in $G$. Let $v$ be an element of order 4 in $H_1$ so that $v^2 \in Z(K) = Z(G)$ and we set $R = H_3(v)$. Since $[R : H_3] = 2$, it follows that $R$ is a proper subgroup of $G$ and all elements in $G - (K \cup R)$ are involutions. If $i \in G - (K \cup R)$ and $y \in K \cap R$, then $iy \in G - (K \cup R)$ so that $iy$ is an involution implying $y^i = y^{-1}$. Thus $i$ inverts each element in $K \cap R = \langle v \rangle H_3'$ implying that $K \cap R$ is abelian. Let $X$ be a maximal abelian subgroup of $G$ containing $K \cap R$. Since $X$ is obviously distinct from each $H_i$, $i = 1, 2, 3$, and $\exp(X) > 2$, we have a contradiction. We have proved that $H_3$ is a maximal subgroup in $G$.

All elements in $G - (K \cup H_3)$ are involutions, where $K$ and $H_3$ are two distinct maximal subgroups in $G$. Then each involution $i \in G - (K \cup H_3)$
inverts each element in $K \cap H_3 = H_3^1$. In particular, $i$ centralizes $\Omega_1(H_3^1)$ and so $\Omega_1(H_3^1) = H_1 \cap H_3 = Z(G)$. Since $H_1 \cap H_3 < H_3^*$, it follows that $\exp(H_3^*) = 4$. Then $H_3^*(i)$ is quasidihedral of exponent $4$ and $H_3^*(i)$ is a maximal subgroup in $G$. Finally, $H_1 \cap H_3^* = Z(G)$ is a maximal subgroup of $H_1$ and so $\exp(H_1) = 4$ and $G = H_1H_2 = H_1H_2H_3$. We have proved all properties of $G$ stated in part (a) of our theorem.

(ii) Now assume that all $H_i$ are normal in $G$, $i = 1, 2, 3$.

Then we have again $G = H_1H_2H_3$.

(ii) First suppose that $H_1, H_2$ and $H_3$ do not cover $G$.

Then $G - (H_1 \cup H_2 \cup H_3)$ is not empty so that all elements in $G - (H_1 \cup H_2 \cup H_3)$ are involutions. Let $i \in G - (H_1 \cup H_2 \cup H_3)$ and let $A$ be a maximal abelian subgroup in $G$ containing $i$ so that $A$ is distinct from $H_1, H_2$ and $H_3$ implying that $A$ must be elementary abelian. Since $Z(G) < A$, it follows that $Z(G)$ is elementary abelian.

It is easy to see that $\exp(G) = 4$. Suppose that $g \in G$ with $o(g) \geq 8$. For any $i \in \{1, 2, 3\}$, we have either $g \in H_i$ (and then also $g^2 \in H_i$) or $g \in G - H_i$. In the second case Lemma 57.1 in [2] implies that there is $h \in H_i$ such that $M = \langle h, g \rangle$ is minimal nonabelian. Since $\exp(M) \geq 8$, each of the three maximal subgroups $M_i$ ($i = 1, 2, 3$) of $M$ are of exponent $> 2$ and they lie in three pairwise distinct maximal abelian subgroups $H_1, H_2, H_3$ of exponent $> 2$ in $G$. Hence for an $j \in \{1, 2, 3\}$, we have $M_j \subseteq H_j$ and then $g^2 \in M_j \subseteq H_i$. We have proved that in any case $g^2 \in H_i$ for each $i \in \{1, 2, 3\}$ and so $g^2 \in H_1 \cap H_2 \cap H_3 = Z(G)$. But $Z(G)$ is elementary abelian and so $o(g^2) \leq 2$, a contradiction. We have proved that $\exp(G) = 4$.

Suppose that there is $h \in G$ of order $4$ such that $h^2 \not\in Z(G)$. Since all elements of order $4$ in $G$ are contained in $H_1 \cup H_2 \cup H_3$, we may assume that $h \in H_1$. Then interchanging $H_2$ and $H_3$ (if necessary), we may assume that $h^2 \not\in H_2$. Set $K_0 = H_1H_2$ so that $Z(K_0) = H_1 \cap H_2$ and $h^2 \not\in Z(K_0)$. We have $K_0' \leq H_1 \cap H_2 = Z(K_0)$ and so $K_0$ is of class $2$. Suppose, by way of contradiction, that $\exp(Z(K_0)) = 4$. Let $k \in K_0 - (H_1 \cup H_2)$ and let $B$ be a maximal abelian subgroup of $G$ containing $Z(K_0)(k)$ so that we must have $B = H_3$. But then $H_3 \supseteq Z(K_0)$ and so $Z(K_0) = H_1 \cap H_2 \cap H_3 = Z(G)$, a contradiction. Hence $Z(K_0)$ is elementary abelian. But then for all $x \in K_0$, $[h^2, x] = [h, x]^2 = 1$ and so $h^2 \in Z(K_0)$, a final contradiction. We have proved that $\Omega_3(G) \leq Z(G)$ implying that $G'$ is elementary abelian and so we have obtained some 2-groups from part (b) of our theorem.

(iii) Now assume that $G = H_1 \cup H_2 \cup H_3$, i.e., $H_1, H_2, H_3$ cover $G$.

Let $i \neq j$ with $i, j \in \{1, 2, 3\} = \{i, j, k\}$. If $H_iH_j < G$, then $H_k \geq G - (H_iH_j)$ and since $(G - (H_iH_j)) = G$, $G$ would be abelian, a contradiction. Thus

$H_iH_j = G$, $H_i \cap H_j = Z(G)$, $H_k \geq G - (H_i \cup H_j)$ and $H_k \geq Z(G)$. 
Because \( i \neq j \) are arbitrary elements in \( \{1, 2, 3\} \), we also get
\[
H_i \cap H_k = H_j \cap H_k = \mathbb{Z}(G) \quad \text{and so} \quad H_k = (G - (H_i \cup H_j)) \cup \mathbb{Z}(G).
\]
Also, \( G' \leq H_i \cap H_j = \mathbb{Z}(G) \) and so \( G \) is of class 2.

If \( \mathbb{Z}(G) \) is elementary abelian, then for any \( x, y \in G \), \([x^2, y] = [x, y]^2 = 1\) and so \( \mathcal{O}_1(G) \leq \mathbb{Z}(G) \). So assume that \( \text{exp}(\mathbb{Z}(G)) > 2 \). In this case each maximal abelian subgroup of \( G \) contains \( \mathbb{Z}(G) \) and so must be equal to one of \( H_1, H_2, H_3 \). Let \( g \in G \). Then either \( g \in H_i \) (and then also \( g^2 \in H_i \)) or \( g \in G - H_1 \). In the second case, by Lemma 57.1 in [2], there is \( h \in H_i \) such that \( M = (g, h) \) is minimal nonabelian. Then three maximal subgroups \( S_1, S_2, S_3 \) of \( M \) lie in three pairwise distinct maximal abelian subgroups in \( G \) which are equal to \( H_1, H_2, H_3 \). Hence we may assume \( S_1 \leq H_i \) and so \( g^2 \in H_i \). Thus in any case, \( g^2 \in H_1 \cap H_2 \cap H_3 = \mathbb{Z}(G) \) and so we get again \( \mathcal{O}_1(G) \leq \mathbb{Z}(G) \). For any \( x, y \in G \), \([x, y]^2 = [x^2, y] = 1\) and so \( G' \) is elementary abelian. We have obtained the groups from part (b) of our theorem and we are done.

\[\square\]

**References**

