ESTIMATION OF THE EDDY THERMAL CONDUCTIVITY FOR LAKE BOTONEGA

Danila Lozzi-Kožar[1] – Ivica Kožar[2*]

[1]Croatian Waters, Đure Šporera 3, Rijeka, Croatia
[2]Department for Computer Modeling, Faculty of Civil Engineering, University of Rijeka, Radmile Matejčić 3, Rijeka, Croatia

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Abstract:

This paper presents a part of a computer model that is suitable for limited temperature prediction and its application for Lake Botonega, which is located in Istria, Croatia. The main assumption of this study is that the heat transfer can be described by the eddy diffusivity model to formulate the model of the heating and cooling of a lake using a series of water and air temperature measurements. The coefficient of thermal diffusion, which is a function of the lake depth, is determined using the inverse model of eddy thermal diffusivity. The inverse model is linearized using the finite element approach. The model of lake thermal diffusivity consists of a conductive part and a radiative part, with the latter part being replaced with the heat flux on the boundary. The model parameters are calculated in two steps—a predictor step and a corrector step—and the coefficient of conduction is calculated instead of the diffusion.
The estimated parameters are intended for inclusion in a simple three-dimensional thermal model, which provides the lake temperature prediction that is based on previous temperature measurements, as demonstrated in the examples.

1 Introduction

This paper presents part of the computer model that is suitable for the limited temperature prediction for Lake Botonega. The lake serves as an artificial storage reservoir and is located in the central part of the Istrian peninsula of Croatia; it is the main source of water supply for the towns of Pazin, Poreč and Rovinj. It has a surface area of 2.5 km^2 and a maximum depth of 16 m. The lake is thermally mixed in the winter and is stratified from spring to autumn. During the summer months, the water becomes too hot for water supply. The main purpose of the model is the prediction of the water temperature of the lake during summer based on the long-term weather forecast.
The model is based on a series of water and air temperature measurements, which provide the only available data. The measured data consist of the depth and the corresponding water temperature (“temperature profile”), which is measured at the

* Corresponding author. Tel.: +385 51 265 993; fax: +385 51 265 998
E-mail address: ivica.kozar@gradri.uniri.hr

water intake in Lake Botonega (Fig. 1). The temperature on the surface of the lake is the temperature at a depth of 0.1 m below the surface of the lake. In some periods (especially in summer), intake 1 was above the surface of the lake and only the water temperature at the surface and intake spots 2, 3 and 4 are analyzed. Additionally, the average air temperature above the lake was recorded for particular days in 1998 and 1999.

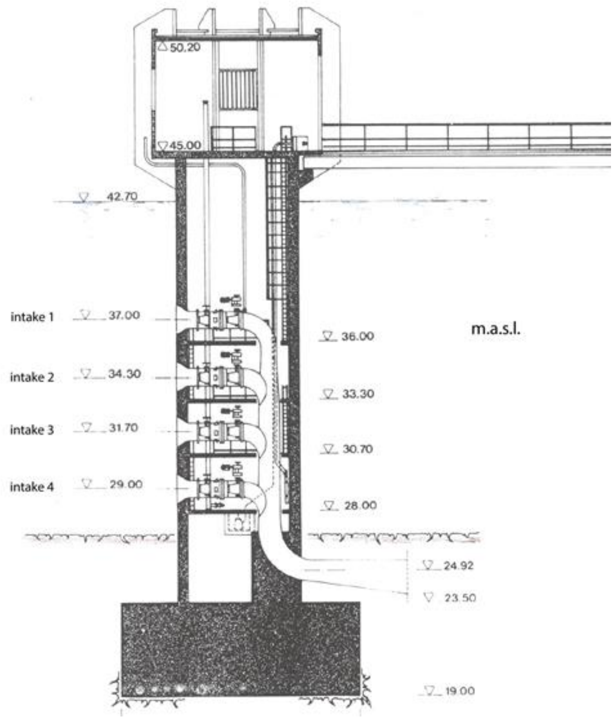

Figure 1. Water intake

Determination of thermal properties of lakes (and related quantities like concentration of various gases, etc.) has been of interest for a long time and there are many references in Geophysical and Limnology journals, e.g. [1], [2]. Generally, they are based on forward models requiring a great number of measured parameters, like [1] that is based on the Princeton dynamic lake model from 1987 based on primitive equations and used for modeling of circulation and thermal structure of Lake Michigan. [2] describes the importance of heat fluxes at the surface and bellow in formation of stratification and this paper recognizes their importance, too.

One has to mention the great difference between Lake Michigan and Lake Botonega: the former is vast and deep (about the size of Croatia) and the later is small and shallow. There has been exhaustive measurement around Lake Michigan for tens of years, while for Lake Botonega there are only temperature measurements available. However, the paper demonstrates that using inverse model it is possible to formulate a useful model based on scarce data.

The main assumption in our approach is that heat transfer can be described by the eddy thermal diffusion model, in which the thermal coefficients comprise all effects that have not been measured. The nonstationary equation of heat transfer establishes the relation between a series of water temperature measurements and a series of air temperature measurements; the heat transfer coefficients are obtained using an inverse analysis. By applying the estimated parameters in a simple three-dimensional finite element model of heating and cooling, the lake temperature prediction, which is based on previous temperature measurements, can be obtained. Beginning with water temperatures and changing air temperatures, successful predictions for a maximum period of one month have been obtained, as confirmed by the examples. All measured data are detailed in [3].

The inverse heat transfer model assumes that the vertical eddy thermal conductivity *k(z)*, which has a dimension of the thermal conductivity (W/mK), includes the influence of other factors, e.g., turbulent mixing of water. The unknown parameter is the vertical eddy thermal conductivity, whereas two lateral thermal conductivities retain their value. This facilitates the construction of a one-dimensional inverse model, in which the estimated parameters are compatible with a three-dimensional forward model and can be easily inserted into the forward model for temperature prediction.

The main advantage of our inverse model is the reduced number of measured parameters. This model is necessary due to the lack of measured quantities (only water and air temperature, which is measured over several years, are available). The nonlinear inverse problem has been successfully reduced to a linear least square problem, as demonstrated in this paper. Quantitatively, the solution in this case is always optimal regarding the number of parameters and measured quantities. Qualitatively, we do not need parameters that would not be determined (e.g., the influence of turbulent diffusion is assumed to be included in the value of vertical eddy thermal diffusivity). The accuracy of

the model is determined by the number of parameters and measured quantities (no substitute is available for quantities that are not measured but it is a physical reality that is pertinent to this problem). Data resolution and model resolution matrices illustrate the model properties.
The main problem of our inverse model is similar to the main advantage: the limited number of parameters. The traditional approach of the eddy diffusion model assumes the measurement and determination of a large number of parameters and constants (net solar radiation, albedo, and emissivity) and the calculation of their contribution to the vertical eddy thermal diffusion coefficient (see e.g., [4] and [5]). This influence is reflected in the measured water and air temperatures. A similar approach to the solution of an inverse thermal conductivity problem can be found in [6]. However, we had to make assumptions about the bottom and surface heat transfer, as in [7]. Unlike [6] and [7], in our problem there are additional mathematical difficulties related to the high nonlinearity and the operator rank deficiency that is caused by water stratification during the summer. Physically stratification is a state in which no vertical heat transfer occurs through the water; mathematically, we have singular operators for parameter estimation. The first problem is overcome by linearization obtained with the use of specially formulated novel finite elements. The second problem is solved with Tikhonov regularization, in which the parameter has been obtained using the L-curve criterion [8].
Preparatory work for the model formulation included the formation of a 3D finite element mesh of the lake's bottom. Novel type of Kriging interpolation has been employed for this rather difficult problem, as described in [9].
Some numerical methods developed in the work can be applied in various fluid modeling problems [10].
After the Introduction paper is composed as follows: chapter 2 shortly describes the forward 1D nonlinear eddy diffusion heat transfer model and chapter 3 presents its inverse formulation. There is a description of some difficulties arising with the application of the usual solution procedure based on the Levenberg-Marquardt method. The difficulties are alleviated through linearization and regularization as described in the sequel of chapter 3. In chapter 4 numerical examples illustrate the method and chapter 5 presents conclusions.

2 Formulation of the 1D eddy diffusion model for heat transfer using the finite element method

The main characteristic of Lake Botonega is its shallow depth and stratified water during the summer. Stratification is evident from water temperature measurements and can be estimated by the modified densometric Froude number [11] $F_d = \sqrt{\frac{1}{ge}}\frac{LQ}{D_m V}$ where g = gravitational constant [m/s^2], $e = 10^{-6}$ density gradient [1/m], L = reservoir length [m], Q = mean reservoir flow [m^3/s], D_m = mean reservoir depth [m], and V = reservoir volume [m^3]. For Lake Botonega, $F_d \approx 0.07 << 1/\pi$, which describes a strong stratification potential.
We assume that the main source of water temperature changes is uniformly and simultaneously derived for the entire lake area (no separate heat sources or sinks are located in the lake). The assumption of homogeneity enables us to disregard the lateral transport behavior. We can apply the vertical turbulent temperature diffusion model. General formulation of the one-dimensional eddy diffusion model for heat transfer is detailed in many sources (e.g., [12], [13], [14]). For completeness, it will be summarized in this paper. The vertical turbulent diffusion model from [11f] is adapted to expose the coefficient of conductivity instead of the diffusivity:

$$\rho c \frac{\partial T}{\partial t} = \frac{1}{A(z)}\frac{\partial}{\partial z}\left(k(z)A(z)\frac{\partial T}{\partial z}\right) \qquad (1)$$

It may be written in the form of a basic equation of transient heat transfer, as noted by [12]

$$\rho c(z)\frac{\partial T}{\partial t} - \frac{\partial}{\partial z}\left(k(z)\frac{\partial T}{\partial z}\right) = Q \qquad (2)$$

The matrix form of equation (2), which is discretized with finite elements, is

$$\mathbf{C}(\mathbf{c_p})\frac{\Delta T}{\Delta t} + \mathbf{K}(\mathbf{k})\mathbf{T} = 0 \qquad (3)$$

For simplicity, no source term is included in equation (3); only the boundary conditions determine the solution, i.e., temperature field $T(z)$

discretized by the solution vector **T**. The specific heat matrix is

$$\mathbf{C}(\mathbf{c}_\mathbf{\rho}) = \int_0^{l_e} c_\rho N(z) \frac{dN(z)}{dz} dz \tag{4}$$

and $c_\rho = c\rho$ (volumetric heat capacity).

The integration is performed over one finite element, and the total matrix is obtained by summarizing the contributions of all finite elements. $\mathbf{K}(\mathbf{k})$ is the thermal conductivity matrix by the finite element method, and *k(z)* is the function of the vertical eddy thermal conductivity

$$\mathbf{K}(\mathbf{k}) = \int_0^{l_e} N(z) \left[\frac{\partial}{\partial z} \left(k(z) \frac{\partial T}{\partial z} \right) \right] dz \tag{5}$$

Thus, the “forward” problem is defined using the finite element method: for known parameters $\mathbf{c}_\rho$ and **k**, the temperature field **T** is uniquely determined. Equations (4) and (5) describe the forward problem and are well known (e.g., [15]).

3 Formulation of the inverse problem

In the inverse problem, we attempt to determine parameters that characterize the model that is described by a forward formulation. Borrowing the notation from [8], the forward problem can be described as *G(m)=d*, where *m* represents the parameters and *d* is the outcome (result) of the forward model. In the case of a discretized model, **m** and **d** represent vectors. In our case, *m=m(z)* is discretized into **m**, which is the vector of thermal conduction coefficients, and *d=d(z)* is discretized into **d**, which is the vector of temperatures along the lake depth *z*. Parameter estimation using inverse formulation assumes that we know (from measurements or otherwise) the outcome **d** of the corresponding forward problem. For the discretized linear problem, the inverse formulation of the model is $\mathbf{m}=\mathbf{G}^{-g}\mathbf{d}$, where $\mathbf{G}^{-g}$ is the generalized inverse matrix of **G**. The formulation of the inverse problem from a forward formulation is not always straightforward. However, obtaining **d** from measurements has some advantages. In [14] the authors predict the heat transfer coefficient for boundary flow based on the eddy diffusivity concept. In the case of coefficient prediction for the forward model, an additional fluid parameter, such as the viscosity (ν) and rate of dissipation (ε), is included in the changed equation for the heat transfer coefficient. In the case of inverse modeling, knowledge of the additional parameters is not needed; the influence of the additional parameters is already included in the measurement results. Therefore, the inverse procedure, which is based on measurements, automatically considers relevant parameters.

The inverse procedure is formulated beginning with equation (2) with known results from the water temperature measurements. The heat flux at the bottom of the lake is zero (adiabatic boundary condition), and the heat flux through the air-water interface is a function of the measured surface water temperature, the mean air temperature and the estimated convection heat transfer coefficient *h*. Applying the inverse procedure, we determine the unknown vertical eddy thermal conductivity *k(z)* that is represented by the vector **k**. We obtain a solution for the problem in two steps:

1) We can assume that the problem is stationary within one time step Δt [16]; the temperature profile *T(z)* is determined from the measurements. Then, we can write

$$\frac{\partial}{\partial z}\left(k(z) \frac{\partial T}{\partial z} \right) = \rho c \frac{\partial T}{\partial t} \tag{6}$$

where $\rho c \frac{\partial T}{\partial t}$ is the accumulated heat term, e.g., the amount of heat brought into or taken out of the system in one time step. We assume that the (measured) temperature difference in time Δt is proportional to the amount of heat ΔQ brought into or taken out of the system. Now, equation (6) has the form of the heath flux equation $\partial/\partial z \left(k(z) \partial T/\partial z \right) = \Delta Q = q$.

2) We determine the heat flux *q* at the boundary from the assumed boundary conditions:

- $q_{\text{below}} = 0$ no heat transfer through the bottom (adiabatic boundary condition)
- $q_{\text{above}} = \Delta T h$ ΔT is the temperature difference between water and air. *h* is the estimated convection heat transfer coefficient (the components for the determination of the

coefficient include the net solar short wave radiation, the net long-wave radiation, and the evaporative heat flux).

3.1 Note on Levenberg-Marquardt approach to the inverse problem

The usual solution of the inverse problem represented with equation (6) is an iterative optimization of the unknown vector *k(z)* (Levenberg-Marquardt method discussed in [16]). The optimization procedure is simpler and more stable if the requested value is parameterized e.g., we assume equation

$$k(z)=\beta_1 z^4+\beta_2 z^3+\beta_3 z^2+\beta_4 z+\beta_5 \tag{7}$$

where the unknown parameters β_r (r=1...5) have to be determined to ensure that *k(z)* is optimal.
The system of equations that minimize the problem is

$$\frac{\partial S}{\partial \beta_r}=-2\sum_m\left(T_m^{mes}-T_m^{cal}\right)\frac{\partial T^{cal}}{\partial \beta_r}=0 \tag{8}$$

where m is the number of measured points, T^{mes} is the measured temperature and T^{cal} is the estimated (calculated) temperature.
When no functional connection exists between the temperature T and parameter β, we cannot directly calculate the optimal parameter β_r. Using an incremental procedure, we calculate the change of optimal parameters $\delta\beta_r$, which decrease in the process of convergence. We can write the incremental formulation for the increment $\delta\beta_r$ of parameter β_r using a Taylor series expansion as follows:

$$T_i(\beta,\beta_r+\delta\beta_r)=T_i(\beta,\beta_r)+\frac{d}{d\beta_r}T_i(\beta,\beta_r)\delta\beta_r \tag{9}$$

Now, equation (8) becomes

$$\frac{dS}{d\beta_r}=0=-2\sum_m\left[T_m-T_m(\beta,\beta_r)-X_{i,r}\delta\beta_r\right]X_{i,r} \tag{10}$$

where $X_{i,r}=dT_i/d\beta_r$ (*i=1…n*) and n is the dimension of the vector $\mathbf{T}$. The following equation

$$\delta\beta_r=\frac{\sum_m\left[T_m-T_m(\beta,\beta_r)\right]X_{i,r}}{\sum_m X_{i,r}} \tag{11}$$

is a system of “r” equations for determining the “r” of parameter β.
The computation of the sensitivity coefficients $dT^c/d\beta_r$ can only be analytically obtained. $X_{i,r}$ can be approximately resolved using the finite difference method

$$X_{i,r}\approx\frac{T(\beta,\beta_r+\delta\beta_r)-T(\beta,\beta_r)}{\delta\beta_r} \tag{12}$$

The accuracy of equation (12) can be increased if we employ the central difference formulation or a more complex formulation instead of the forward difference formulation.
The approach described by the standard inverse method was tested on the example of the measured temperature profiles for Lake Botonega and provides useless results. We have only confirmed the well-known fact that the method is sensitive to the choice of the initial parameters.
A new original inverse modelling approach is proposed and adjusted for the finite element method.

3.2 Linearized inverse problem

We begin with the matrix formulation of the transient heat transfer equation (5) using a finite element discretization and obtain the matrix equation $\mathbf{C}(\mathbf{c}_\rho)\Delta T/\Delta t+\mathbf{K}(\mathbf{k})\mathbf{T}=0$ of the forward/direct problem. In inverse modelling, new matrices are formulated, in which the parameters and unknowns of the standard (“forward”) model change places.
A novel matrix $\mathbf{H}(\Delta\mathbf{T})$ is formulated as a function of the temperature difference within the time step Δt (instead of c_ρ in the forward problem)

$$\mathbf{C}(\mathbf{c}_\rho)\Delta\mathbf{T}\cong\mathbf{H}(\Delta\mathbf{T})\mathbf{c}_\rho \tag{13}$$

where $\Delta\mathbf{T}$ is the vector of the temperature difference ΔT and matrix $\mathbf{H}(\Delta\mathbf{T})$ is not an [n×n] matrix.

The novel matrix $\mathbf{M}(\mathbf{T})$ is formulated as a function of temperature $\mathbf{T}$ (instead of parameter $\mathbf{k}$ as in the forward problem)

$$\mathbf{K}(\mathbf{k})\mathbf{T} \cong \mathbf{M}(\mathbf{T})\mathbf{k} \tag{14}$$

where $\mathbf{T}$ is the vector of temperature. Matrix $\mathbf{M}(\mathbf{T})$ is not an [n×n] matrix.

Matrices $\mathbf{H}(\Delta\mathbf{T})$ and $\mathbf{M}(\mathbf{T})$ have the dimension $[n_m+2,n_m-1]$, where n_m is the number of discretization points along the lake depth. This result is a consequence of the use of linear finite elements, where k and c_ρ are constants within each element and the number of nodes (where the temperature is defined) exceeds coefficients k and c_ρ.

Now, we have a new matrix equation that is suitable for the inverse model

$$\mathbf{H}(\Delta\mathbf{T})\mathbf{c}_\mathbf{\rho}\frac{1}{\Delta t} + \mathbf{M}(\mathbf{T})\mathbf{k} = 0 \tag{15}$$

From equations (15) and (13), vector $\mathbf{k}(z)$ can be calculated if we know $\mathbf{c}_\mathbf{\rho}$. In this step (predictor), we assume $\mathbf{c}_\mathbf{\rho}$ to represent the physical value for water and

$$\mathbf{M}(\mathbf{T})\mathbf{k} = -\frac{1}{\Delta t}\mathbf{C}(\mathbf{c}_\mathbf{\rho})\Delta\mathbf{T} \tag{16}$$

Thus,

$$\mathbf{k} = -\mathbf{M}(\mathbf{T})^{-1}\frac{1}{\Delta t}\mathbf{C}(\mathbf{c}_\mathbf{\rho})\Delta\mathbf{T} \tag{17}$$

This inverse system has more equations than unknowns, and an optimal solution is obtained according to the method of least squares. Mathematically, the generalized inverse matrix is employed in the solution, which is equivalent to the approximation with the method of least squares.

The suitability of the solution is assessed through resolution matrices. The data resolution matrix reveals the possibility of the model to reconstruct the initial data. The model resolution matrix illustrates the quality of the model parameters that are deducted from the available data. In the linear least squares model, parameters are always well resolved.

3.2.1 Boundary conditions

In equation (15), the coefficient $\mathbf{k}$ is the main variable; we assume $\mathbf{c}_\mathbf{\rho}$ to be known in the first approximation. The calculation of the vector of the unknown heat transfer coefficient $\mathbf{k}$ using equation (17) can be performed with or without the prescribed boundary conditions at the bottom and the surface of the lake. The introduction of the boundary conditions for the heat flux q requires some assumptions. From Fourier's law of heat conduction $q = k \cdot dT / dz$, we will calculate k at both boundaries (at the bottom of the lake and toward the air)

$$k = \frac{q}{\frac{dT}{dz}} = \frac{q}{\frac{T_b - T}{\Delta z}} \tag{18a}$$

respectively

$$k = \frac{q}{\frac{dT}{dz}} = \frac{q}{\frac{T_a - T}{\Delta z}} \tag{18b}$$

where T_b and T_a are temperatures at the bottom and on the surface of the lake.

The heat flux at the bottom of the lake is assumed to be zero, where $q_b = 0$; thus, the k at the bottom is also zero. On the surface of the lake, we have $q_a = (T_{air} - T_{water})h$, where h is a coefficient of the convective diffusion with the value taken from [17]. Measuring h would improve the model accuracy, which would require additional measuring devices (such as solarimeter or similar) and is planned as a future extension of the model.

The relevance of the boundary conditions is assessed by a comparison of the inverse model matrix eigenvalues with and without boundary conditions in one of the examples.

3.2.2 Tikhonov regularization

The matrix $\mathbf{M}(\mathbf{T})$ is the $[n_m+2,n_m-1]$ matrix, which cannot be inverted; thus, the generalized inversion procedure is necessary. If two adjacent points have the same measured temperature (as occurs in a stratified fluid), the matrix $\mathbf{M}(\mathbf{T})$ becomes singular (rank is [*n_m-1-number of equal consecutive measured temperatures*]). Thus, the matrix $\mathbf{M}(\mathbf{T})$ is also an indicator of thermal stratification. A singular matrix can be inverted using a singular value decomposition (SVD) procedure; however, significantly better results are obtained if a regularization procedure is applied. We have adopted Tikhonov's regularization procedure by applying

$$\mathbf{M}(\mathbf{T})^{-1} = (\mathbf{M}^{\mathrm{T}}\mathbf{M} + \alpha\mathbf{I})^{-1}\mathbf{M} \qquad (19)$$

In equation (19), the parameter α for Tikhonov's regularization procedure is obtained using the L-curve criterion; however, other possible solutions (refer to [8]). The L-curve helps to obtain the optimal value of α as a trade-off between the solution norm $\|\mathbf{m}\|_2$ and the residual norm $\|\mathbf{M}(\mathbf{T})\mathbf{m} - \Delta\mathbf{Q}\|_2$. These norms are a function of α; the norm that corresponds to the sharp corner of L-curve is selected. In one of the examples, the corresponding L-curve is depicted.

3.2.3 Time scale adjustment

The water temperature prediction is obtained by introduction of the calculated **k** from equation (17) into the matrix form of the equation (2). However, the time increments *Δt* are not identical in both equations: in equation (17), the increment is reflected by the time difference between the measurements, whereas the increment consists of the time step in the prediction/simulation in equation (2). This difference in time scales is resolved with modification of the heat capacity by assuming that variable $\mathbf{c}_\rho$ is not a physical constant but is a model parameter. Physical justification for the modification of the variable $\mathbf{c}_\rho$ is that we describe heat exchange in the lateral direction (lateral heat flux between areas of different temperature).

Equation (15) can be re-written in the incremental form

$$\frac{1}{\Delta t}\mathbf{H}\left(\mathbf{T}^{\mathbf{1}}\right)\mathbf{c}_{\boldsymbol{\rho}}^{\mathbf{1}} + \mathbf{K}(\mathbf{k})\mathbf{T}^{\mathbf{1}} = \frac{1}{\Delta t}\mathbf{C}\left(\mathbf{c}_{\boldsymbol{\rho}}^{\mathbf{0}}\right)\mathbf{T}^{\mathbf{0}} \qquad (20)$$

where superscript ^{0} denotes the backward value (backward time step), superscript ^{1} denotes the current value (current time step) and *Δt* is the time step in the prediction/simulation. The corrected vector $\mathbf{c}_{\boldsymbol{\rho}}^{\mathbf{1}}$ follows from equation (20)

$$\mathbf{c}_{\boldsymbol{\rho}}^{\mathbf{1}} = \mathbf{H}\left(\mathbf{T}^{\mathbf{1}}\right)^{-1}\left[\mathbf{C}\left(\mathbf{c}_{\boldsymbol{\rho}}^{\mathbf{0}}\right)\mathbf{T}^{\mathbf{0}} - \Delta t\mathbf{K}(\mathbf{k})\mathbf{T}^{\mathbf{1}}\right] \qquad (21)$$

Matrix $\mathbf{H}\left(\mathbf{T}^{\mathbf{1}}\right)$ is not an [n×n] matrix, and the generalized inverse $\mathbf{H}\left(\mathbf{T}^{\mathbf{1}}\right)^{-1}$ is

$$\mathbf{H}\left(\mathbf{T}^{\mathbf{1}}\right)^{-1} = \left(\mathbf{H}^{\mathbf{T}}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathbf{T}} \qquad (22)$$

Tikhonov's or another regularization procedure is not necessary because the matrix is always well conditioned. Due to the character of the problem, we have to apply a generalized inverse; this solution is not exact but is only optimal, i.e., measured data cannot be fully reconstructed but a reasonable solution can be obtained. The data and model resolution matrices can be constructed to assess the quality of the parameter estimation.

The forward formulation is used to reconstruct the measured temperatures. In the reconstruction procedure, the calculated vectors **k** and $\mathbf{c}_\rho$ have to be included in the transient equation $\mathbf{C}\left(\mathbf{c}_{\boldsymbol{\rho}}\right)\Delta T/\Delta t + \mathbf{K}(\mathbf{k})\mathbf{T} = 0$ based on the finite element discretization.

4 Examples

In the sequel, some examples illustrate (and validate) the new procedure. For testing purpose, we begin with the initial condition of the measured temperature profile in the previous period (the previous month). Through several time steps, a new state is obtained. The obtained temperature should correspond with the measured temperature for the following month.

The physical parameters of water used in the calculations are as follows: density ρ=1000 kg/m^3, specific heat c=4200 J/kgK, estimated convection heat transfer coefficient h=100 W/m^2, and thermal conductivity k=0.58 W/mK.

The calculation begins with the initial condition of the measured temperature profile in the previous month. The inverse model is applied, and the equivalent vertical eddy thermal conductivity *k(z)* and the equivalent heat flux is determined. The calculated coefficients are inserted into the forward model in several time steps (in our example 5), and a prediction of the new temperature profile for the following month is obtained. In our example, the temperature profiles for two consecutive years are known; thus, we can compare the predicted values with the measured values.

4.1 Winter month example - cooling

In the first example, we observe the prediction of lake temperature during winter cooling using the measured water temperatures from mid-December to predict the water temperatures at the beginning of February, assuming that the air temperature above the lake is known. Because we know the water temperature measurement for February, we can simply assess the accuracy of the temperature prediction. We can compare the results for the nodal positions of the finite element mesh, which do not coincide with the points of measurement; thus, interpolation was needed. In Fig. 2, measured water profiles with the necessary interpolation are employed to obtain measurements for the finite element nodal positions.

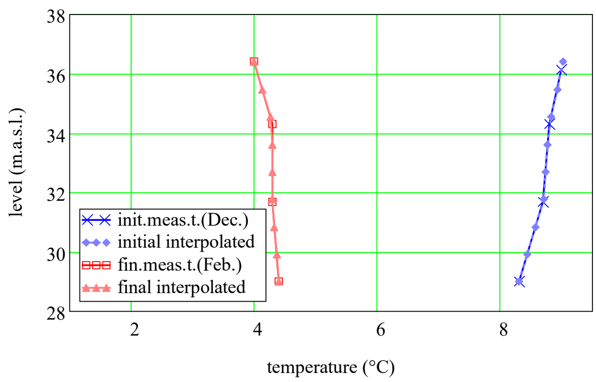

Figure 2. Measured and interpolated temperatures for 8.12.1997. and 10.02.1998.

Applying the described procedures, we obtain the resulting heat flux and equivalent eddy thermal conductivity, as presented in Fig. 3.

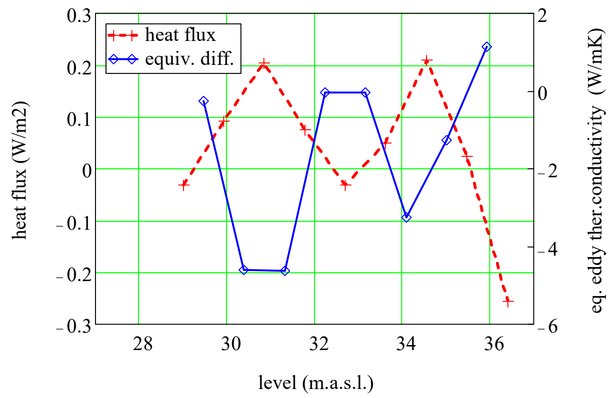

Figure 3. Calculated heat flux and equivalent eddy thermal conductivity along the lake depth (meters above sea level).

Table 1. Comparison of eigenvalues for winter time example

$SVDM_1 \begin{pmatrix} 1.01 \\ 1.001 \\ 0.203 \\ 0.062 \\ 0.047 \\ 0.035 \\ 0 \\ 0 \end{pmatrix}$	$SVDMr_1 \begin{pmatrix} 0.232 \\ 0.095 \\ 0.067 \\ 0.05 \\ 0.036 \\ 0.028 \\ 0 \\ 0 \end{pmatrix}$

The inspection of the singular value decomposition of matrix **M** with (SVDM1) and without (SVDMr1) the boundary conditions illustrates the importance of boundary conditions.

The SVD matrix **M** does not have full rank; this example requires the application of Tikhonov regularization as previously described. The parameter for the regularization is determined from the comparison of the residual and solution norms, as depicted in Fig. 4. The solution norm $\|\mathbf{m}\|_2 = \left(\mathbf{M}^{\mathrm{T}}\mathbf{M} + \alpha\mathbf{I}\right)^{-1}\mathbf{M}$ residual norm $\|\mathbf{M(T)m} - \mathbf{\Delta Q}\|_2$ are functions of the Tikhonov parameter α. We have selected the parameter α=0.001, which corresponds to the dark dot in Fig. 4.

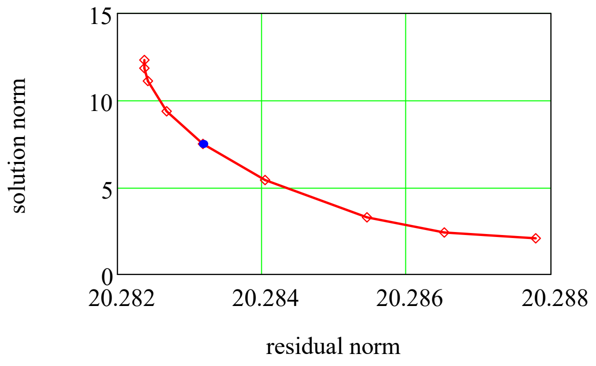

Figure 4. Comparison of residual and solution norms for determination of the Tikhonov regularization parameter.

The final temperature prediction is provided in Fig. 5. The prediction pertains to approximately sixty days and is satisfactory.

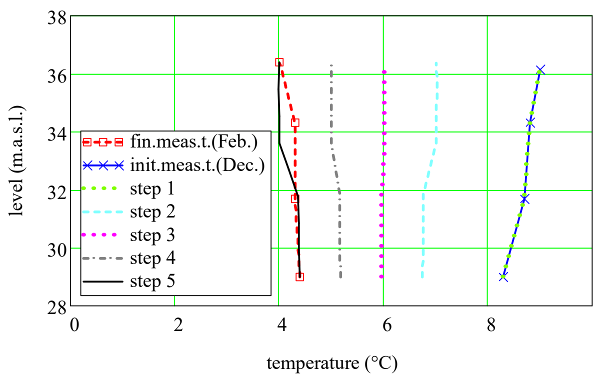

Figure 5. Temperature prediction for the beginning of February from mid-December.

Figure 5 shows our most demanding example of temperature prediction due to a lack of measurements in January 1998 and a time span of two months. The calculated results pertain to nine finite element nodal points for five time steps (12 days for each time step). In addition, a thin stratification produces a singular inverse matrix operator. In this case, the influence of the Tikhonov parameter is significant.

4.2 Summer month example - heating

In this example, we observe the prediction of the lake temperature during summer heating using temperatures from June and July. Figure 6 shows the measured water profiles with the necessary interpolation to match the finite element nodal positions.

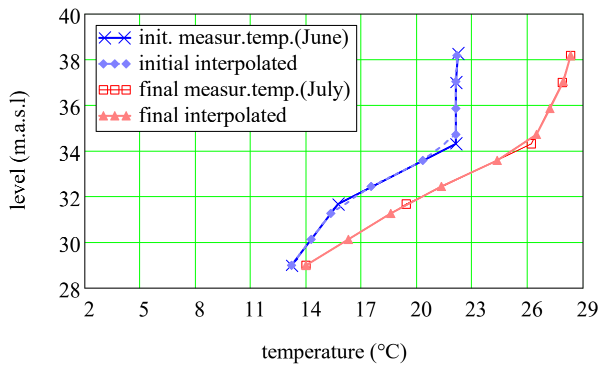

Figure 6. Measured and interpolated temperatures for 18.06.1997. and 28.07.1998.

Applying the described procedures, we obtain the resulting heat flux and equivalent eddy thermal conductivity, as presented in Fig. 7.

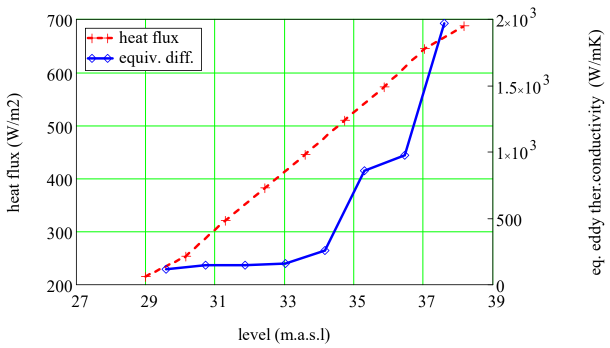

Figure 7. Calculated heat flux and equivalent eddy thermal conductivity along the lake depth (meters above sea level).

The inspection of singular value decomposition of matrix **M** with (SVDM1) and without (SVDMr1) introduction of boundary conditions indicates that the boundary conditions are not significant for the well behaved inverse operator (matrix **M** is full rank).
In this case, Tikhonov regularization is not needed. We will examine the data $\mathbf{R_d}$ and model $\mathbf{R_m}$ resolution matrices, in which the numerical values are graphically presented in Fig. 8 and a dominant diagonal is immediately identified.

Table 2. Comparison of eigenvalues for summer time example

$SVDM_l \begin{pmatrix} 4.558 \\ 3.565 \\ 2.747 \\ 1.921 \\ 1.131 \\ 0.999 \\ 0.837 \\ 0.378 \end{pmatrix}$	$SVDMr_l \begin{pmatrix} 4.56 \\ 3.635 \\ 2.942 \\ 2.101 \\ 1.232 \\ 0.995 \\ 0.497 \\ 0.101 \end{pmatrix}$

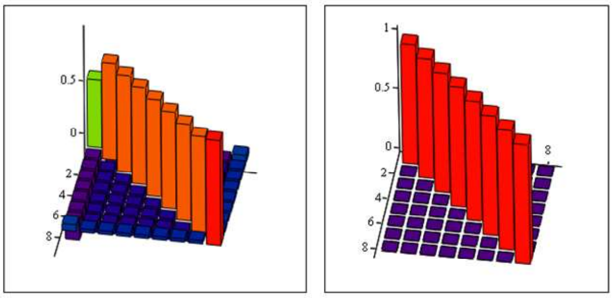

Figure 8. Data resolution matrix and model resolution matrix.

In this example, we have introduced the resolution matrices as a measure of the quality of the model (refer to [18]). Resolution matrices would not be appropriate in the previous example because its behavior is dominated by the Tikhonov regularization. The data resolution matrix is $\mathbf{R_d}=\mathbf{M(T)}^{-g}\ \mathbf{M(T)}$, and the model resolution matrix is $\mathbf{R_m}=\mathbf{M(T)}\ \mathbf{M(T)}^{-g}$, where the superscript $^{-g}$ signifies the generalized inverse and $\mathbf{R_d}$ is a measure of the data reconstruction capability of the model. When $\mathbf{R_d}$=I (identity matrix), the measured data can be fully reconstructed from the model. $\mathbf{R_m}$ is a measure of how the model parameters are determined by the data; when $\mathbf{R_m}$=I, the model parameters are uniquely determined by the data.
As shown in Fig. 8, the data parameters can be satisfactorily resolved while the model parameters are completely resolved. The consequence of this resolution is visible in the prediction results depicted in Fig. 9.

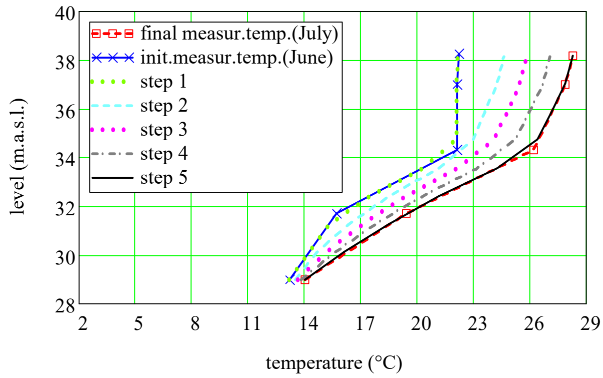

Figure 9. Temperature prediction for the end of July beginning in mid-Jun.

4.3 Overview of other temperature predictions

Figure 10 provides an overview of the temperature predictions for several months. Some of the predictions pertain to heating and some of the predictions pertain to cooling; an interesting case of cooling at the surface is observed, whereas simultaneously heating occurs at the bottom.
Figure 10.a. shows a simple case of winter heating and an excellent prediction.

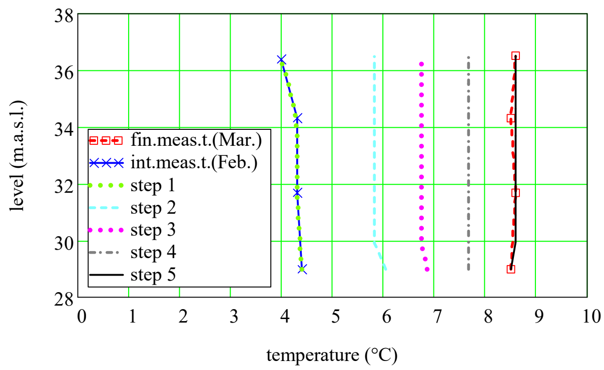

Figure 10.a. Measured and interpolated temperatures for 10.02.1998 and 08.03.1998.

Figure 10.b. shows the beginning of the formation of the surface temperature layer with reduced thermal communication through the surface of the lake.

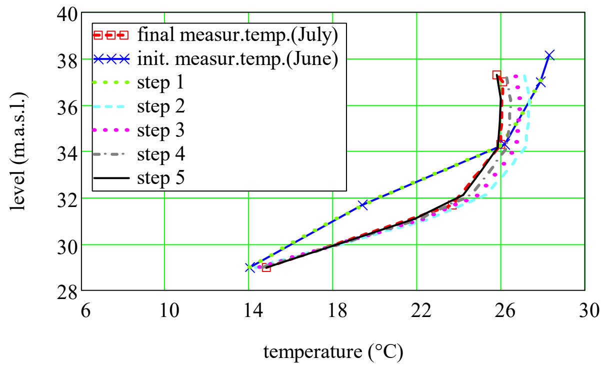

Figure 10.b. Measured and interpolated temperatures for 28.07.1998. and 25.08.1998.

Cooling is presented in Fig. 10.c. Cooling occurs on the surface and heating occurs on the bottom; at one moment, the temperature of the lake is almost uniform.

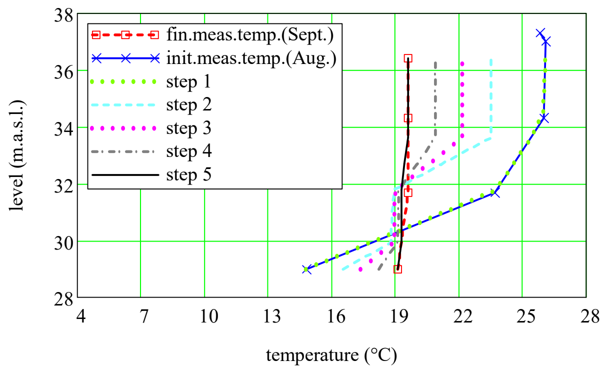

Figure 10.c. Measured and interpolated temperatures for 25.08.1998 and 24.09.1998.

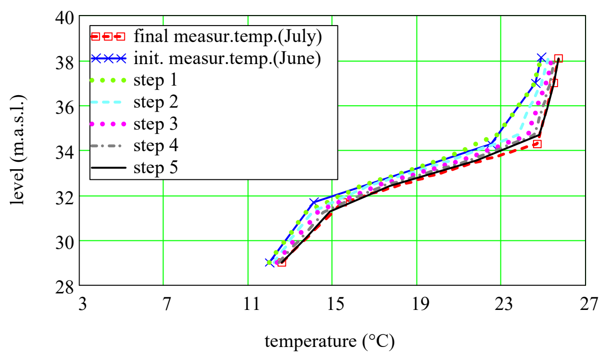

Figure 10.d. Measured and interpolated temperatures for 02.06.1999 and 09.07.1999.

Figure 10.d. presents temperature changes due to thermal stratification. This case is the most difficult case because the matrix **M** in the inverse model is singular.

5 Discussion

The coefficient of thermal conduction is assumed depth dependent and is determined by applying the inverse procedure on the model connecting measured temperatures of air and water. The inverse formulation is based on the eddy diffusivity concept. The thermal conductivity is selected as an additional parameter with the thermal diffusivity due to the two-stage procedure of coefficient computation. However, the concept is completely equivalent: $k(z)$ can be divided by $c(z)$ to proclaim the new variable for diffusivity.

The crucial step that enables a successful model is the linearization that has been performed through finite elements. This novel approach requires additional effort in the formulation of special finite elements but is generally applicable and robust.

This simple lake model is based on two assumptions that have not been quantified (due to the substantial amount of work, the study would entail a separate research project). The first assumption is pertinent to the fact that Lake Botonega is very shallow; thus, the entire surface can be considered to be homogeneous and the vertical heat flux can be separately treated. Whether these predictions can be obtained for a significantly deeper lake is unknown. The second assumption is the influence of solar radiation, which is expressed by the value of the estimated convection heat transfer coefficient h, which is assumed. The authors believe that the inclusion of this information would noticeably improve the model. This belief is based on the significance of the boundary conditions in the inverse model; the surface boundary conditions are highly dependent on the coefficient h (also, see [2]).

6 Conclusion

A simple thermal model of heating and cooling of a lake can be constructed using only measured temperatures of water and air. Lake thermal diffusivity parameters are assumed depth dependent and can be determined from an inverse model using temperature measurements. The successful inverse model should consider the following facts: 1) the inverse model should be linearized, 2) the inclusion of boundary conditions significantly improves the inverse model, and 3) the determined heat transfer parameters should be adjusted to the time scale that is employed in the temperature predictions. By obeying these guidelines, successful temperature predictions are obtained. The examples have demonstrated accurate predictions for a maximum time period of one month.

Notations

$k(z)$	vertical eddy thermal conductivity (W/mK)
$A(z)$	lake area as a function of depth (m^2)
q	heat flux (W/m^2)
T	temperature (°C)
z	vertical coordinate (m)
$N(z)$	element shape function
D	derivative matrix
k	vector of eddy thermal conductivity (W/mK)
ρ	density (kg/m^3)
c	specific heat capacity (J/kgK)
t	time (s)
Q	rate of heat generation (W/m^3)
c_ρ	heat capacity
$\mathbf{c}_\rho$	heat capacity vector
C	specific heat capacity matrix (by the finite element method)
K	thermal conductivity matrix (by the finite element method)
h	estimated convection heat transfer coefficient (W/m^2K)
Δt	time step
ΔT	temperature difference (°C)
β_r	unknown parameter
$\delta\beta_r$	increment of parameter β_r
m	number of measured points
T^{mes}	measured temperature (°C)
T^{cal}	model estimated temperature (°C)
H	matrix as a function of the temperature difference in the inverse model
ΔT	vector of temperature increments (°C)
M	matrix as a function of temperature in the inverse model
T	vector of temperatures (°C)
q_b	heat flux at the bottom of the lake (W/m^2)
q_a	heat flux on the surface of the lake (W/m^2)
T_{air}	air temperature (°C)
T_{water}	water temperature (°C)
m.a.s.l.	meters above sea level

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