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Existence of non-oscillatory solutions of a kind of first-order neutral differential equation

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Abstract. This paper deals with the existence of non-oscillatory solutions to a kind of first-order neutral equations having both delay and advance terms. The new results are established using the Banach contraction principle.

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Key words: Neutral equations, Banach contraction principle, Non-oscillatory solution

1. Introduction

In this paper, we study the first-order neutral differential equation

$$\begin{aligned}\frac{d}{dt}\big[x(t)+P_1(t)x(t-\tau_1)+P_2(t)x(t+\tau_2)\big]\\ +Q_1(t)g_1(x(t-\sigma_1))-Q_2(t)g_2(x(t+\sigma_2))-f(t)=0,\end{aligned}\tag{1}$$

where $P_i \in C([t_0,\infty),\mathbb{R})$, $Q_i \in C([t_0,\infty),[0,\infty))$, $i=1,2$. $\tau_i>0$ and $\sigma_i \geq 0$, $i=1,2$. $f \in C([t_0,\infty),\mathbb{R})$ and $g_i \in C(\mathbb{R},\mathbb{R})$. We assume that $g_i, i=1,2$ satisfy the local Lipschitz condition and $xg_i(x)>0$, $i=1,2$, for $x \neq 0$.

The problem of oscillation of solutions of neutral functional differential equations is of both theoretical and practical interest. Recently, there has been an interest in establishing the oscillatory and the non-oscillatory behavior of first, second, third and higher order neutral functional differential equations. In 2002, Zhou and Zhang [31] extended the results of Kulenović and Hadžiomerspahić in [18] to a higher order linear neutral delay differential equation of the form

$$\frac{d^n}{dt^n}[x(t)+cx(t-\tau)]+(-1)^{n+1}[P(t)x(t-\sigma)-Q(t)x(t-\sigma)]=0.$$

In 2005, the existence of non-oscillatory solutions of first-order linear neutral delay differential equations of the form

$$\frac{d}{dt}[x(t)+P(t)x(t-\tau)]+Q_1(t)x(t-\sigma_1)-Q_2(t)x(t-\sigma_2)=0,$$

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was investigated by Zhang et al. [30] and in the same year, Yu and Wang [29] studied non-oscillatory solutions of second-order nonlinear neutral delay equations of the form

$$\big[r(t)[x(t)+P(t)x(t-\tau)]'\big]' + Q_1(t)f(x(t-\sigma_1)) - Q_2(t)g(x(t-\sigma_2)) = 0.$$

For more related works, we refer the reader to papers [1, 4, 5, 7, 8, 9, 10, 12, 13, 16, 17, 20, 21, 22, 23, 24, 25, 26, 28, 32] and books [2, 3, 6, 14, 15, 19, 27].

However, to the best of our knowledge, there are no results of the existence of non-oscillatory solutions for the first-order neutral differential equations having both delay and advance terms. Motivated by the works mentioned above, in this paper, we study the existence of non-oscillatory solutions for (1).

Let $m = \max\{\tau_1, \sigma_1, \sigma_2\}$. By a solution of (1) we mean a function $x \in C([t_1 - m, \infty), \mathbb{R})$, for some $t_1 \geq t_0$, such that $x(t) + P_1(t)x(t-\tau_1) + P_2(t)x(t+\tau_2)$ is continuously differentiable on $[t_1, \infty)$ and (1) is satisfied for $t \geq t_1$.

As customary, a solution of (1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise the solution is called non-oscillatory.

The following theorem will be used to prove the main results in the next section.

Theorem 1 (Banach's Contraction Mapping Principle, see [11])**.** *A contraction mapping on a complete metric space has exactly one fixed point.*

2. Main results

To show that an operator S satisfies the conditions for the contraction mapping principle, we consider different cases for the ranges of the coefficients $P_1(t)$ and $P_2(t)$.

Theorem 2. *Assume that* $0 \leq P_1(t) \leq p_1 < 1$, $0 \leq P_2(t) \leq p_2 < 1 - p_1$ *and*

$$\int_{t_0}^{\infty} Q_i(s)ds < \infty, i = 1, 2, \quad \textit{and} \quad \int_{t_0}^{\infty} |f(s)|ds < \infty; \tag{2}$$

then (1) *has a bounded non-oscillatory solution.*

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the supremum norm. Set

$$\Omega = \{x \in \Lambda : M_1 \leq x(t) \leq M_2,\ t \geq t_0\}.$$

It is clear that Ω is a bounded, closed and convex subset of Λ.

Let L_i, $i = 1, 2$ denote the Lipschitz constants of functions g_i, $i = 1, 2$ on the set Ω, and $L = \max\{L_1, L_2\}$, $\beta_i = \max_{x \in \Omega} g_i(x), i = 1, 2, \ldots$ Because of (2), we can choose $t_1 > t_0$,

$$t_1 \geq t_0 + \max\{\tau_1, \sigma_1, \sigma_2\} \tag{3}$$

sufficiently large such that

$$\int_t^{\infty} \Big[Q_1(s)\beta_1 + |f(s)|\Big]ds \leq M_2 - \alpha, \quad t \geq t_1, \tag{4}$$

$$\int_t^\infty \Big[Q_2(s)\beta_2 + |f(s)|\Big]ds \le \alpha - (p_1+p_2)M_2 - M_1, \quad t \ge t_1, \tag{5}$$

and

$$\int_t^\infty \Big[Q_1(s) + Q_2(s)\Big]ds \le \frac{1-(p_1+p_2)}{L}, \quad t \ge t_1, \tag{6}$$

where M_1 and M_2 are positive constants such that

$$(p_1+p_2)M_2 + M_1 < M_2 \quad \text{and} \quad \alpha \in \big((p_1+p_2)M_2 + M_1, M_2\big).$$

Consider the operator $S : \Omega \to \Lambda$ defined by

$$(Sx)(t) = \begin{cases} \alpha - P_1(t)x(t-\tau_1) - P_2(t)x(t+\tau_2) \\ + \displaystyle\int_t^\infty \Big[Q_1(s)g_1(x(s-\sigma_1)) - Q_2(s)g_2(x(s+\sigma_2)) - f(s)\Big]ds, \quad t \ge t_1, \\ (Sx)(t_1), \quad t_0 \le t \le t_1. \end{cases}$$

Clearly, Sx is continuous. For $t \ge t_1$ and $x \in \Omega$, it follows from (4) and (5) that

$$\begin{aligned} (Sx)(t) \le& \alpha + \int_t^\infty \Big[Q_1(s)g_1(x(s-\sigma_1)) - f(s)\Big]ds \\ \le& \alpha + \int_t^\infty \Big[Q_1(s)\beta_1 + |f(s)|\Big]ds \\ \le& M_2, \end{aligned}$$

and

$$\begin{aligned} (Sx)(t) \ge& \alpha - P_1(t)x(t-\tau_1) - P_2(t)x(t+\tau_2) \\ &- \int_t^\infty \Big[Q_2(s)g_2(x(s+\sigma_2)) + f(s)\Big]ds \\ \ge& \alpha - p_1M_2 - p_2M_2 - M_1 \\ \ge& M_1. \end{aligned}$$

This means that $S\Omega \subset \Omega$. Since Ω is a bounded, closed, convex subset of Λ, in order to apply the Banach contraction principle we have to show that S is a contraction mapping on Ω. For $x_1, x_2 \in \Omega$ and $t \ge t_1$,

$$\begin{aligned} &\big|(Sx_1)(t) - (Sx_2)(t)\big| \\ &\qquad \le P_1(t)|x_1(t-\tau_1) - x_2(t-\tau_1)| + P_2(t)|x_1(t+\tau_2) - x_2(t+\tau_2)| \\ &\qquad + \int_t^\infty Q_1(s)|g_1(x_1(s-\sigma_1)) - g_1(x_2(s-\sigma_1))|ds \\ &\qquad + \int_t^\infty Q_2(s)|g_2(x_1(s+\sigma_2)) - g_2(x_2(s+\sigma_2))|ds \end{aligned}$$

or by (6)

$$\begin{aligned}\big|(Sx_1)(t)-(Sx_2)(t)\big| &\le \|x_1-x_2\|\left(p_1+p_2+L\int_t^\infty [Q_1(s)+Q_2(s)]\,ds\right)\\ &\le \|x_1-x_2\|\left[p_1+p_2+L\cdot\frac{1-(p_1+p_2)}{L}\right]\\ &= \|x_1-x_2\|,\end{aligned}$$

which implies that

$$\|Sx_1-Sx_2\|\le\|x_1-x_2\|,$$

which shows that S is a contraction mapping on Ω. Thus, S has a unique fixed point which is a positive and bounded solution of (1). This completes the proof. □

Theorem 3. *Assume that* $0\le P_1(t)\le p_1<1$, $p_1-1<p_2\le P_2(t)\le 0$ *and* (2) *hold; then* (1) *has a bounded non-oscillatory solution.*

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0,\infty)$ with the supremum norm. Set

$$\Omega=\{x\in\Lambda: N_1\le x(t)\le N_2,\ t\ge t_0\}.$$

It is clear that Ω is a bounded, closed and convex subset of Λ.

Let L_i, $i=1,2$ denote the Lipschitz constants of functions g_i, $i=1,2$ on the set Ω, and $L=\max\{L_1,L_2\}$, $\beta_i=\max\limits_{x\in\Omega} g_i(x), i=1,2,\ldots$ Because of (2), we can choose a $t_1>t_0$,

$$t_1\ge t_0+\max\{\tau_1,\sigma_1,\sigma_2\}$$

sufficiently large such that

$$\int_t^\infty \Big[Q_1(s)\beta_1+|f(s)|\Big]ds\le(1+p_2)N_2-\alpha,\quad t\ge t_1, \tag{7}$$

$$\int_t^\infty \Big[Q_2(s)\beta_2+|f(s)|\Big]ds\le\alpha-p_1N_2-N_1,\quad t\ge t_1, \tag{8}$$

and

$$\int_t^\infty \Big[Q_1(s)+Q_2(s)\Big]ds\le\frac{1-(p_1+p_2)}{L},\quad t\ge t_1, \tag{9}$$

where N_1 and N_2 are positive constants such that

$$N_1+p_1N_2<(1+p_2)N_2\quad\text{and}\quad \alpha\in(N_1+p_1N_2,(1+p_2)N_2).$$

Consider the operator $S:\Omega\to\Lambda$ defined by

$$(Sx)(t)=\begin{cases}\alpha-P_1(t)x(t-\tau_1)-P_2(t)x(t+\tau_2)\\ +\displaystyle\int_t^\infty\Big[Q_1(s)g_1(x(s-\sigma_1))-Q_2(s)g_2(x(s+\sigma_2))-f(s)\Big]ds, t\ge t_1,\\ (Sx)(t_1),\quad t_0\le t\le t_1.\end{cases}$$

Obviously, Sx is continuous. For $t \geq t_1$ and $x \in \Omega$, it follows from (7) and (8) that

$$\begin{aligned}(Sx)(t) \leq& \alpha - p_2 N_2 + \int_t^{\infty} \Big[Q_1(s)\beta_1 + |f(s)|\Big] ds \\ \leq& N_2,\end{aligned}$$

and

$$\begin{aligned}(Sx)(t) \geq& \alpha - p_1 N_2 - \int_t^{\infty} \Big[Q_2(s)\beta_2 + |f(s)|\Big] ds \\ \geq& N_1.\end{aligned}$$

This proves that $S\Omega \subset \Omega$. Since Ω is a bounded, closed, convex subset of Λ, in order to apply the Banach contraction principle we have to show that S is a contraction mapping on Ω. For $x_1, x_2 \in \Omega$ and $t \geq t_1$, by using (9), we can obtain

$$\begin{aligned}\big|(Sx_1)(t) - (Sx_2)(t)\big| &\leq \|x_1 - x_2\| \left(p_1 + p_2 + L \int_t^{\infty} [Q_1(s) + Q_2(s)]\, ds\right) \\ &\leq \|x_1 - x_2\| \left[p_1 + p_2 + L \cdot \frac{1 - (p_1 + p_2)}{L}\right] \\ &= \|x_1 - x_2\|.\end{aligned}$$

This implies

$$\|Sx_1 - Sx_2\| \leq \|x_1 - x_2\|,$$

which shows that S is a contraction mapping on Ω. Thus, S has a unique fixed point which is a positive and bounded solution of (1). This completes the proof. □

Theorem 4. *Assume that* $1 < p_1 \leq P_1(t) \leq p_{1_0} < \infty$, $0 \leq P_2(t) \leq p_2 < p_1 - 1$ *and* (2) *hold; then* (1) *has a bounded non-oscillatory solution.*

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the supremum norm. Set

$$\Omega = \{x \in \Lambda : M_3 \leq x(t) \leq M_4,\ t \geq t_0\}.$$

It is clear that Ω is a bounded, closed and convex subset of Λ.

Let L_i, $i = 1, 2$ denote the Lipschitz constants of functions g_i, $i = 1, 2$ on the set Ω, and $L = \max\{L_1, L_2\}$, $\beta_i = \max_{x \in \Omega} g_i(x)$, $i = 1, 2, \ldots$ In view of (2), we can choose $t_1 > t_0$,

$$t_1 + \tau_1 \geq t_0 + \max\{\sigma_1, \sigma_1\} \tag{10}$$

sufficiently large such that

$$\int_t^{\infty} \Big[Q_1(s)\beta_1 + |f(s)|\Big] ds \leq p_1 M_4 - \alpha, \quad t \geq t_1, \tag{11}$$

$$\int_t^{\infty} \Big[Q_2(s)\beta_2 + |f(s)|\Big] ds \leq \alpha - p_{1_0} M_3 - (1 + p_2) M_4, \quad t \geq t_1, \tag{12}$$

and

$$\int_t^\infty \Big[Q_1(s)+Q_2(s)\Big]ds \le \frac{(p_1+p_2)-1}{L}, \quad t \ge t_1, \tag{13}$$

where M_3 and M_4 are positive constants such that

$$p_{1_0}M_3+(1+p_2)M_4 < p_1M_4 \quad \text{and} \quad \alpha \in \big(p_{1_0}M_3+(1+p_2)M_4, p_1M_4\big).$$

Define a mapping $S:\Omega \to \Lambda$ as follows:

$$(Sx)(t)=\begin{cases} \dfrac{1}{P_1(t+\tau_1)}\Big\{\alpha - x(t+\tau_1)-P_2(t+\tau_1)x(t+\tau_1+\tau_2) \\ +\displaystyle\int_{t+\tau_1}^\infty \Big[Q_1(s)g_1(x(s-\sigma_1))-Q_2(s)g_2(x(s+\sigma_2))-f(s)\Big]ds\Big\}, \quad t\ge t_1, \\ (Sx)(t_1), \quad t_0 \le t \le t_1. \end{cases}$$

Clearly, Sx is continuous. For $t \ge t_1$ and $x \in \Omega$, from (11) and (12) it follows that

$$\begin{aligned}(Sx)(t) \le & \frac{1}{P_1(t+\tau_1)}\left(\alpha + \int_t^\infty \Big[Q_2(s)\beta_2+|f(s)|\Big]ds\right)\\ \le & \frac{1}{p_1}\left(\alpha + \int_t^\infty \Big[Q_2(s)\beta_2+|f(s)|\Big]ds\right)\\ \le & M_4,\end{aligned}$$

and

$$\begin{aligned}(Sx)(t) \ge & \frac{1}{P_1(t+\tau_1)}\left(\alpha-(1+p_2)M_4 - \int_t^\infty \Big[Q_2(s)\beta_2+|f(s)|\Big]ds\right)\\ \ge & \frac{1}{p_{1_0}}\left(\alpha-(1+p_2)M_4 - \int_t^\infty \Big[Q_2(s)\beta_2+|f(s)|\Big]ds\right)\\ \ge & M_3.\end{aligned}$$

This means that $S\Omega \subset \Omega$. Since Ω is a bounded, closed, convex subset of Λ, in order to apply the Banach contraction principle we have to show that S is a contraction mapping on Ω. For $x_1, x_2 \in \Omega$ and $t \ge t_1$, by using (13), we can obtain

$$\begin{aligned}\big|(Sx_1)(t)-(Sx_2)(t)\big| &\le \frac{1}{p_1}\|x_1-x_2\|\left(1+p_2+L\int_t^\infty (Q_1(s)+Q_2(s))\,ds\right)\\ &\le \|x_1-x_2\|.\end{aligned}$$

This implies

$$\|Sx_1-Sx_2\| \le \|x_1-x_2\|,$$

which shows that S is a contraction mapping on Ω. Thus, S has a unique fixed point which is a positive and bounded solution of (1). This completes the proof. □

Theorem 5. *Assume that* $1<p_1\le P_1(t)\le p_{1_0}<\infty$, $1-p_1<p_2\le P_2(t)\le 0$ *and* (2) *hold; then* (1) *has a bounded non-oscillatory solution.*

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0,\infty)$ with the supremum norm. Set

$$\Omega = \{x \in \Lambda : N_3 \le x(t) \le N_4,\ t \ge t_0\}.$$

It is clear that Ω is a bounded, closed and convex subset of Λ.

Let L_i, $i=1,2$ denote the Lipschitz constants of functions g_i, $i=1,2$ on the set Ω, and $L=\max\{L_1,L_2\}$, $\beta_i = \max\limits_{x\in\Omega} g_i(x), i=1,2,\ldots$ In view of (2), we can choose $t_1 > t_0$,

$$t_1+\tau_1 \ge t_0+\max\{\sigma_1,\sigma_1\}$$

sufficiently large such that

$$\int_t^\infty \Big[Q_1(s)\beta_1+|f(s)|\Big]ds \le (p_1+p_2)N_4-\alpha, \quad t\ge t_1, \tag{14}$$

$$\int_t^\infty \Big[Q_2(s)\beta_2+|f(s)|\Big]ds \le \alpha-p_{1_0}N_3-N_4, \quad t\ge t_1, \tag{15}$$

and

$$\int_t^\infty \Big[Q_1(s)+Q_2(s)\Big]ds \le \frac{(p_1+p_2)-1}{L}, \quad t\ge t_1, \tag{16}$$

where N_3 and N_4 are positive constants such that

$$p_{1_0}N_3+N_4<(p_1+p_2)N_4 \quad \text{and} \quad \alpha \in \big(p_{1_0}N_3+N_4,(p_1+p_2)N_4\big).$$

Define a mapping $S:\Omega\to\Lambda$ as follows:

$$(Sx)(t) = \begin{cases} \dfrac{1}{P_1(t+\tau_1)}\Big\{\alpha - x(t+\tau_1)-P_2(t+\tau_1)x(t+\tau_1+\tau_2) \\ +\displaystyle\int_{t+\tau_1}^\infty \Big[Q_1(s)g_1(x(s-\sigma_1))-Q_2(s)g_2(x(s+\sigma_2))-f(s)\Big]ds\Big\}, \quad t\ge t_1, \\ (Sx)(t_1), \quad t_0\le t\le t_1. \end{cases}$$

Clearly, Sx is continuous. For $t\ge t_1$ and $x\in\Omega$, from (14) and (15) it follows that

$$\begin{aligned}(Sx)(t) &\le \frac{1}{P_1(t+\tau_1)}\left(\alpha-p_2N_4+\int_t^\infty \Big[Q_1(s)\beta_1+|f(s)|\Big]ds\right)\\ &\le \frac{1}{p_1}\left(\alpha-p_2N_4+\int_t^\infty \Big[Q_1(s)\beta_1+|f(s)|\Big]ds\right)\\ &\le N_4,\end{aligned}$$

and

$$\begin{aligned}(Sx)(t) &\ge \frac{1}{P_1(t+\tau_1)}\left(\alpha-N_4-\int_t^\infty \Big[Q_2(s)\beta_2+|f(s)|\Big]ds\right)\\ &\ge \frac{1}{p_{1_0}}\left(\alpha-N_4-\int_t^\infty \Big[Q_2(s)\beta_2+|f(s)|\Big]ds\right)\\ &\ge N_3.\end{aligned}$$

This proves that $S\Omega \subset \Omega$. To apply the Banach contraction principle it remains to show that S is a contraction mapping on Ω. For $x_1, x_2 \in \Omega$ and $t \geq t_1$, by using (16), we can have

$$\begin{aligned}\big|(Sx_1)(t) - (Sx_2)(t)\big| &\leq \frac{1}{p_1}\|x_1 - x_2\| \left(1 - p_2 + L\int_t^\infty \Big[Q_1(s) + Q_2(s)\Big]ds\right)\\ &\leq \|x_1 - x_2\|.\end{aligned}$$

This implies

$$\|Sx_1 - Sx_2\| \leq \|x_1 - x_2\|,$$

which shows that S is a contraction mapping on Ω. Thus, S has a unique fixed point which is a positive and bounded solution of (1). This completes the proof. □

Theorem 6. *Assume that* $-1 < p_1 \leq P_1(t) \leq 0$, $0 \leq P_2(t) \leq p_2 < 1 + p_1$ *and* (2) *hold; then* (1) *has a bounded non-oscillatory solution.*

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the supremum norm. Set

$$\Omega = \{x \in \Lambda : M_5 \leq x(t) \leq M_6,\ t \geq t_0\}.$$

It is clear that Ω is a bounded, closed and convex subset of Λ.

Let L_i, $i = 1, 2$ denote the Lipschitz constants of functions g_i, $i = 1, 2$ on the set Ω, and $L = \max\{L_1, L_2\}$, $\beta_i = \max\limits_{x\in\Omega} g_i(x), i = 1, 2, \ldots$ Because of (2), we can choose $t_1 > t_0$,

$$t_1 \geq t_0 + \max\{\tau_1, \sigma_1, \sigma_2\}$$

sufficiently large such that

$$\int_t^\infty \Big[Q_1(s)\beta_1 + |f(s)|\Big]ds \leq (1 + p_1)M_6 - \alpha, \quad t \geq t_1, \tag{17}$$

$$\int_t^\infty \Big[Q_2(s)\beta_2 + |f(s)|\Big]ds \leq \alpha - p_2 M_6 - M_5, \quad t \geq t_1\,, \tag{18}$$

and

$$\int_t^\infty \Big[Q_1(s) + Q_2(s)\Big]ds \leq \frac{1 + p_1 - p_2}{L}, \quad t \geq t_1, \tag{19}$$

where M_5 and M_6 are positive constants such that

$$M_5 + p_2 M_6 < (1 + p_1)M_6 \quad \text{and} \quad \alpha \in (M_5 + p_2 M_6, (1 + p_1)M_6)\,.$$

Define an operator $S : \Omega \to \Lambda$ as follows:

$$(Sx)(t) = \begin{cases} \alpha - P_1(t)x(t - \tau_1) - P_2(t)x(t + \tau_2) \\ + \displaystyle\int_t^\infty \Big[Q_1(s)g_1(x(s - \sigma_1)) - Q_2(s)g_2(x(s + \sigma_2)) - f(s)\Big]ds, \quad t \geq t_1, \\ (Sx)(t_1), \quad t_0 \leq t \leq t_1. \end{cases}$$

Obviously, Sx is continuous. For $t \geq t_1$ and $x \in \Omega$, from (17) and (18) it follows that

$$\begin{aligned}(Sx)(t) \leq& \alpha - p_1 M_6 + \int_t^\infty \Big[Q_1(s)\beta_1 + |f(s)|\Big] ds \\ \leq& M_6,\end{aligned}$$

and

$$\begin{aligned}(Sx)(t) \geq& \alpha - p_2 M_6 - \int_t^\infty \Big[Q_2(s)\beta_2 + |f(s)|\Big] ds \\ \geq& M_5.\end{aligned}$$

This proves that $S\Omega \subset \Omega$. To apply the Banach contraction principle it remains to show that S is a contraction mapping on Ω. For $x_1, x_2 \in \Omega$ and $t \geq t_1$, by using (19), we can get

$$\begin{aligned}\big|(Sx_1)(t) - (Sx_2)(t)\big| &\leq \|x_1 - x_2\| \left(-p_1 + p_2 + L\int_t^\infty \Big[Q_1(s) + Q_2(s)\Big] ds\right) \\ &\leq \|x_1 - x_2\|,\end{aligned}$$

which implies that

$$\|Sx_1 - Sx_2\| \leq \|x_1 - x_2\|,$$

where the supremum norm is used. Thus, S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1). This completes the proof. □

Theorem 7. *Assume that $-1 < p_1 \leq P_1(t) \leq 0$, $-1 - p_1 < p_2 \leq P_2(t) \leq 0$ and (2) hold; then (1) has a bounded non-oscillatory solution.*

Proof. Let Λ be the set of continuous and bounded functions on $[t_0, \infty)$ with the supremum norm. Set

$$\Omega = \{x \in \Lambda : N_5 \leq x(t) \leq N_6,\ t \geq t_0\}.$$

It is clear that Ω is a bounded, closed and convex subset of Λ.

Let L_i, $i = 1, 2$ denote the Lipschitz constants of functions g_i, $i = 1, 2$ on the set Ω, and $L = \max\{L_1, L_2\}$, $\beta_i = \max_{x \in \Omega} g_i(x), i = 1, 2, \ldots$ Because of (2), we can choose $t_1 > t_0$,

$$t_1 \geq t_0 + \max\{\tau_1, \sigma_1, \sigma_2\}$$

sufficiently large such that

$$\int_t^\infty \Big[Q_1(s)\beta_1 + |f(s)|\Big] ds \leq (1 + p_1 + p_2)N_6 - \alpha, \quad t \geq t_1, \tag{20}$$

$$\int_t^\infty \Big[Q_2(s)\beta_2 + |f(s)|\Big] ds \leq \alpha - N_5, \quad t \geq t_1, \tag{21}$$

and

$$\int_t^\infty \Big[Q_1(s)+Q_2(s)\Big]ds \le \frac{1+p_1+p_2}{L}, \quad t \ge t_1, \tag{22}$$

where N_5 and N_6 are positive constants such that

$$N_5 < (1+p_1+p_2)N_6 \quad \text{and} \quad \alpha \in (N_5, (1+p_1+p_2)N_6).$$

Define an operator $S : \Omega \to \Lambda$ as follows:

$$(Sx)(t) = \begin{cases} \alpha - P_1(t)x(t-\tau_1) - P_2(t)x(t+\tau_2) \\ + \int_t^\infty \Big[Q_1(s)g_1(x(s-\sigma_1)) - Q_2(s)g_2(x(s+\sigma_2)) - f(s)\Big]ds, t \ge t_1, \\ (Sx)(t_1), t_0 \le t \le t_1. \end{cases}$$

Obviously, Sx is continuous. For $t \ge t_1$ and $x \in \Omega$, from (20) and (21) it follows that

$$\begin{aligned}(Sx)(t) \le& \alpha - p_1N_6 - p_2N_6 + \int_t^\infty \Big[Q_1(s)\beta_1 + |f(s)|\Big]ds \\ \le& N_6,\end{aligned}$$

and

$$\begin{aligned}(Sx)(t) \ge& \alpha - \int_t^\infty \Big[Q_2(s)\beta_2 + |f(s)|\Big]ds \\ \ge& N_5.\end{aligned}$$

This proves that $S\Omega \subset \Omega$. To apply the Banach contraction principle it remains to show that S is a contraction mapping on Ω. Thus, for $x_1, x_2 \in \Omega$ and $t \ge t_1$, by using (22), we can obtain

$$\begin{aligned}|(Sx_1)(t) - (Sx_2)(t)| &\le \|x_1 - x_2\| \left(-p_1 - p_2 + L\int_t^\infty \Big[Q_1(s)+Q_2(s)\Big]ds\right) \\ &\le \|x_1 - x_2\|,\end{aligned}$$

which implies that

$$\|Sx_1 - Sx_2\| \le \|x_1 - x_2\|,$$

where the supremum norm is used. Thus, S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1). This completes the proof. □

Theorem 8. *Assume that* $-\infty < p_{1_0} \le P_1(t) \le p_1 < -1$, $0 \le P_2(t) \le p_2 < -p_1 - 1$ *and* (2) *hold; then* (1) *has a bounded non-oscillatory solution.*

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the supremum norm. Set

$$\Omega = \{x \in \Lambda : M_7 \le x(t) \le M_8,\ t \ge t_0\}.$$

It is clear that Ω is a bounded, closed and convex subset of Λ.

Let L_i, $i = 1, 2$ denote the Lipschitz constants of functions g_i, $i = 1, 2$ on the set Ω, and $L = \max\{L_1, L_2\}$, $\beta_i = \max\limits_{x\in\Omega} g_i(x), i = 1, 2, \dots$ In view of (2), we can choose $t_1 > t_0$,

$$t_1 + \tau_1 \geq t_0 + \max\{\sigma_1, \sigma_1\}$$

sufficiently large such that

$$\int_t^\infty \Big[Q_1(s)\beta_1 + |f(s)|\Big] ds \leq p_{1_0} M_7 + \alpha, \quad t \geq t_1, \tag{23}$$

$$\int_t^\infty \Big[Q_2(s)\beta_2 + |f(s)|\Big](s) ds \leq (-p_1 - 1 - p_2) M_8 - \alpha, \quad t \geq t_1, \tag{24}$$

and

$$\int_t^\infty \Big[Q_1(s) + Q_2(s)\Big] ds \leq \frac{-1 - p_1 - p_2}{L}, \quad t \geq t_1, \tag{25}$$

where M_7 and M_8 are positive constants such that

$$-p_{1_0} M_7 < (-p_1 - 1 - p_2) M_8 \quad \text{and} \quad \alpha \in (-p_{1_0} M_7, (-p_1 - 1 - p_2) M_8)\,.$$

Define a mapping $S : \Omega \to \Lambda$ as follows:

$$(Sx)(t) = \begin{cases} \dfrac{-1}{P_1(t+\tau_1)} \Big\{ \alpha + x(t+\tau_1) + P_2(t+\tau_1) x(t+\tau_1+\tau_2) \\ - \displaystyle\int_{t+\tau_1}^\infty \Big[Q_1(s) g_1(x(s-\sigma_1)) - Q_2(s) g_2(x(s+\sigma_2)) - f(s)\Big] ds \Big\}, t \geq t_1 \\ (Sx)(t_1), t_0 \leq t \leq t_1. \end{cases}$$

Clearly, Sx is continuous. For $t \geq t_1$ and $x \in \Omega$, it follows from (23) and (24) that

$$\begin{aligned} (Sx)(t) \leq& \frac{-1}{p_1} \left(\alpha + M_8 + p_2 M_8 + \int_t^\infty \Big[Q_1(s)\beta_1 + |f(s)|\Big] ds \right) \\ \leq& M_8, \end{aligned}$$

and

$$\begin{aligned} (Sx)(t) \geq& \frac{-1}{p_{1_0}} \left(\alpha - \int_t^\infty \Big[Q_2(s)\beta_2 + |f(s)|\Big] ds \right) \\ \geq& M_7. \end{aligned}$$

This implies that $S\Omega \subset \Omega$. To apply the Banach contraction principle it remains to show that S is a contraction mapping on Ω. Thus, for $x_1, x_2 \in \Omega$ and $t \geq t_1$, by using (25), we get

$$\begin{aligned} |(Sx_1)(t) - (Sx_2)(t)| &\leq \frac{-1}{p_1} \|x_1 - x_2\| \left(1 + p_2 + L \int_t^\infty \Big[Q_1(s) + Q_2(s)\Big] ds \right) \\ &\leq \|x_1 - x_2\|. \end{aligned}$$

This implies

$$\|Sx_1 - Sx_2\| \le \|x_1 - x_2\|,$$

where the supremum norm is used. Thus, S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1). This completes the proof. $\square$

Theorem 9. *Assume that* $-\infty < p_{1_0} \le P_1(t) \le p_1 < -1$, $p_1 + 1 < p_2 \le P_2(t) \le 0$ *and* (2) *hold; then* (1) *has a bounded non-oscillatory solution.*

Proof. Let Λ be the set of continuous and bounded functions on $[t_0, \infty)$ with the supremum norm. Set

$$\Omega = \{x \in \Lambda : N_7 \le x(t) \le N_8,\ t \ge t_0\}.$$

It is clear that Ω is a bounded, closed and convex subset of Λ.

Let L_i, $i = 1, 2$ denote the Lipschitz constants of functions g_i, $i = 1, 2$ on the set Ω, and $L = \max\{L_1, L_2\}$, $\beta_i = \max\limits_{x\in\Omega} g_i(x), i = 1, 2, \ldots$ In view of (2), we can choose $t_1 > t_0$,

$$t_1 + \tau_1 \ge t_0 + \max\{\sigma_1, \sigma_1\}$$

sufficiently large such that

$$\int_t^\infty \Big[Q_1(s)\beta_1 + |f(s)|\Big] ds \le p_{1_0} N_7 + p_2 N_8 + \alpha, \quad t \ge t_1, \tag{26}$$

$$\int_t^\infty \Big[Q_2(s)\beta_2 + |f(s)|\Big] ds \le (-p_1 - 1)N_8 - \alpha, \quad t \ge t_1, \tag{27}$$

and

$$\int_t^\infty \Big[Q_1(s) + Q_2(s)\Big] ds \le \frac{p_2 - p_1 - 1}{L}, \quad t \ge t_1, \tag{28}$$

where N_7 and N_8 are positive constants such that

$$-p_{1_0} N_7 - p_2 N_8 < (-p_1 - 1)N_8 \quad \text{and} \quad \alpha \in \big(-p_{1_0} N_7 - p_2 N_8, (-p_1 - 1)N_8\big).$$

Define a mapping $S : \Omega \to \Lambda$ as follows:

$$(Sx)(t) = \begin{cases} \dfrac{-1}{P_1(t+\tau_1)}\Big\{\alpha + x(t+\tau_1) + P_2(t+\tau_1)x(t+\tau_1+\tau_2) \\ -\displaystyle\int_{t+\tau_1}^\infty \Big[Q_1(s)g_1(x(s-\sigma_1)) - Q_2(s)g_2(x(s+\sigma_2)) - f(s)\Big] ds\Big\}, \quad t \ge t_1, \\ (Sx)(t_1), \quad t_0 \le t \le t_1. \end{cases}$$

Clearly, Sx is continuous. For $t \ge t_1$ and $x \in \Omega$, from (26) and (27) it follows that

$$\begin{aligned} (Sx)(t) \le& \frac{-1}{p_1}\left(\alpha + N_8 + \int_t^\infty \Big[Q_1(s)\beta_1 + |f(s)|\Big] ds\right) \\ \le& N_8 \end{aligned}$$

and

$$\begin{aligned}(Sx)(t) \geq & \frac{-1}{p_{1_0}}\left(\alpha + p_2 N_8 - \int_t^{\infty}\Big[Q_2(s)\beta_2 + |f(s)|\Big]ds\right)\\ \geq & N_7.\end{aligned}$$

These prove that $S\Omega \subset \Omega$. To apply the Banach contraction principle it remains to show that S is a contraction mapping on Ω. Thus, for $x_1, x_2 \in \Omega$, $t \geq t_1$, by using (28), we can obtain

$$\begin{aligned}|(Sx_1)(t) - (Sx_2)(t)| &\leq \frac{-1}{p_1}\|x_1 - x_2\|\left(1 - p_2 + L\int_t^{\infty}\Big[Q_1(s) + Q_2(s)\Big]ds\right)\\ &\leq \|x_1 - x_2\|.\end{aligned}$$

This implies

$$\|Sx_1 - Sx_2\| \leq \|x_1 - x_2\|,$$

where the supremum norm is used. Hence, S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1). This completes the proof. □

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References

[1] R. Agarwal, S. Grace, *Oscillation theorems for certain neutral functional differential equations*, Comput. Math. Appl. **38**(1999), 1–11.

[2] R. Agarwal, S. Grace, D. O'Regan, *Oscillation Theory for Difference and Functional Differential Equations*, Kluwer Academic Publishers, 2000.

[3] R. Agarwal, M. Bohner, W. Li, *Nonoscillation and Oscillation: Theorey for Functional Differential Equations*, Marcel Dekker Inc., New York, 2004.

[4] R. Agarwal, M. Bohner, T. Li, *Oscillation of second-order nonlinear neutral dynamic equations*, Dynam. Systems Appl. **22**(2013), 535–542.

[5] R. Agarwal, C. Zhang, T. Li, *Some remarks on oscillation of second order neutral differential equations*, Appl. Math. Comput. **274**(2016), 178–181.

[6] D. Bainov, D. Mishev, *Oscillation Theory for Neutral Differential Equations with Delay*, Adam Hilger, 1991.

[7] B. Baculikova, T. Li, J. Dzurina, *Oscillation theorems for second order neutral differential equations*, Electron. J. Qual. Theory Differ. Equ. **74**(2011), 1–13.

[8] T. Candan, R. Dahiya, *Oscillation theorems for nth-order neutral functional differential equations*, Math. Comput. Modelling. **43**(2006), 357–367.

[9] T. Candan, R. Dahiya, *Existence of nonoscillatory solutions of first and second order neutral differential equations with distributed deviating arguments*, J. Franklin Inst. **347**(2010), 1309–1316.

[10] T. Candan, *Existence of nonoscillatory solutions of first-order nonlinear neutral differential equations*, Appl. Math. Lett. **26**(2013), 1182–1186.

[11] T. Candan, *Existence of non-oscillatory solutions to first-order neutral differential equations*, Electron. J. Diff. Equ. **39**(2016), 1–11.

[12] M. Chen, J. Yu, Z. Wang, *Nonoscillatory solutions of neutral delay differential equations*, Bull. Austral. Math. Soc. **48**(1993), 475–483.

[13] B. Dorociaková, A. Najmanová, R. Olach, *Existence of nonoscillatory solutions of first-order neutral differential equations*, Abstr. Appl. Anal. **2011**(2011), Article ID 346745, 9pp.

[14] L. Erbe, Q. Kong, B. Zhang, *Oscillation Theory for Functional Differential Equations*, Marcel Dekker Inc., New York, 1995.

[15] I. Györi, G. Ladas, *Oscillation Theory of Delay Differential Equations With Applications*, Clarendon Press, Oxford, 1991.

[16] Z. Han, T. Li, C. Zhang, S. Sun, *Oscillatory behavior of solutions of certain third-order mixed neutral functional differential equations*, Bull. Malays. Math. Sci. Soc. **35**(2012), 611–620.

[17] Y. Jiang, Y. Fu, H. Wang, T. Li, *Oscillation of forced second-order neutral delay differential equations*, Adv. Difference Equ. **2015**(2015), 1.

[18] M. Kulenović, S. Hadžiomerspahić, *Existence of nonoscillatory solution of second-order linear neutral delay equation*, J. Math. Anal. Appl. **228**(1998), 436–448.

[19] G. Ladde, V. Lakshmikantham, B. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments*, Marcel Dekker Inc., New York, 1987.

[20] T. Li, C. Zhang, G. Xing, *Oscillation of third-order neutral delay differential equations*, Abstr. Appl. Anal. **2012**(2012), Article ID 569201, 11 pp.

[21] T. Li, E. Thandapani, *Oscillation of solutions to second-order neutral differential equations*, Electron. J. Diff. Equ. **67**(2014), 1–7.

[22] T. Li, Y. Rogovchenko, *Oscillation theorems for second-order nonlinear neutral delay differential equations*, Abstr. Appl. Anal. **2014**(2014), Articel ID 594190, 5 pp.

[23] T. Li, Y. Rogovchenko, *Oscillation of second-order neutral differential equations*, Math. Nachr. **288**(2015), 1150–1162.

[24] T. Li, Yu. V. Rogovchenko, *Oscillation criteria for even-order neutral differential equations*, Appl. Math. Lett. **61**(2016), 35–41.

[25] R. Olah, *Note on the oscillation of differential equation with advanced argument*, Math. Slav. **33**(1983), 241–248.

[26] Y. Tian, Y. Cai, T. Li, *Existence of nonoscillatory solutions to second-order nonlinear neutral difference equations*, J. Nonlinear Sci. Appl. **8**(2015), 884–892.

[27] A. Tantawy, R. Dahiya, *Oscillation of even order neutral delay differential equations*, in: *Differential Equations Stability and Control*, (S. Elaydi, Ed.), Lecture Notes in Pure and Applied Mathematics, Marcel Dekker Inc., New York, 1991, 503–509.

[28] J. Yu, Y. Wang, *Nonoscillation of a neutral delay differential equation*, Rad. Mat. **8**(1992-1996), 127–133.

[29] Y. Yu, H. Wang, *Nonoscillatory solutions of second-order nonlinear neutral delay equations*, J. Math. Anal. Appl. **311**(2005), 445–456.

[30] W. Zhang, W. Feng, J. Yan, J. Song, *Existence of nonoscillatory solutions of first-order linear neutral delay differential equations*, Comput. Math. Appl. **49**(2005), 1021–1027.

[31] Y. Zhou, B. Zhang, *Existence of nonoscillatory solutions of higher-order neutral differential equations with positive and negative coefficients*, Appl. Math. Lett. **15**(2002), 867–874.

[32] C. Zhang, R. Agarwal, M. Bohner, T. Li, *Oscillation of second-order nonlinear neutral dynamic equations with noncanonical operators*, Bull. Malays. Math. Sci. Soc. **38**(2015), 761–778.