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The generalized multiplier space and its Köthe-Toeplitz and null duals

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Abstract. The purpose of the present study is to generalize the multiplier space for introducing the concepts of αB-, βB-, γB-duals and NB-duals, where $B = (b_{n,k})$ is an infinite matrix with real entries. Moreover, these duals are computed for the sequence spaces X and $X(\Delta)$, where $X \in \{l_p, c, c_0\}$ and $1 \leq p \leq \infty$.

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1. Introduction

Let ω denote the space of all real-valued sequences. Any vector subspace of ω is called a sequence space. For $1 \leq p < \infty$, denote by l_p the space of all real sequences $x = (x_n) \in \omega$ such that

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \infty.$$

For $p = \infty$, $\left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$ is interpreted as $\sup_{n \geq 1} |x_n|$. We write c and c_0 for the spaces of all convergent and null sequences, respectively. Also, bs and cs are used for the spaces of all bounded and convergent series, respectively. Kizmaz [8, 9] defined the forward and backward difference sequence spaces. In this paper, we focus on the backward difference space

$$X(\Delta) = \{x = (x_k) : \ \Delta x \in X\},$$

for $X \in \{l_\infty, c, c_0\}$, where $\Delta x = (x_k - x_{k-1})_{k=1}^{\infty}$, $x_0 = 0$. Observe that $X(\Delta)$ is a Banach space with the norm

$$\|x\|_\Delta = \sup_{k \geq 1} |x_k - x_{k-1}|.$$

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In summability theory, the β-dual of a sequence space is very important in connection with inclusion theorems. The idea of dual sequence space was introduced by Köthe and Toeplitz [10], and it is generalized to the vector-valued sequence spaces by Maddox [11]. For the sequence spaces X and Y, the set $M(X,Y)$ defined by

$$M(X,Y)=\{z=(z_k)\in\omega \;:\; (z_kx_k)_{k=1}^{\infty}\in Y \quad \forall x=(x_k)\in X\}$$

is called the multiplier space of X and Y. With the above notation, the α-, β- γ and N-duals of a sequence space X, which are respectively denoted by X^α, X^β, X^γ and X^N, are defined by

$$X^\alpha=M(X,l_1),\;\; X^\beta=M(X,cs),\;\; X^\gamma=M(X,bs),\;\; X^N=M(X,c_0).$$

For a sequence space X, the matrix domain $X(A)$ of an infinite matrix A is defined by

$$X(A)=\{x=(x_n)\in\omega \;:\; Ax\in X\}, \tag{1}$$

which is a sequence space. The new sequence space $X(A)$ generated by the limitation matrix A from a sequence space X can be the expansion or the contraction and the overlap of the original space X.

In the past, several authors studied Köthe-Toeplitz duals of sequence spaces that are the matrix domains in classical spaces l_p, l_∞, c and c_0. For instance, some matrix domains of the difference operator were studied in [4]. The domain of the backward difference matrix in the space l_p was investigated for $1\le p\le\infty$ by Başar and Altay in [3] and was studied for $0<p<1$ by Altay and Başar in [1]. Recently the Köthe-Toeplitz duals were computed for some new sequence spaces by Erfanmanesh and Foroutannia [5], [6] and Foroutannia [7]. For more details on the domain of triangle matrices in some sequence spaces, the reader may refer to Chapter 4 of [2].

In the present study, the concept of the multiplier space is generalized and the αB-, βB-, γB- and NB-duals are determined for the classical sequence spaces l_p, c and c_0, where $1\le p\le\infty$. Moreover, the $\dagger B$-dual are investigated for the difference sequence spaces $X(\Delta)$, where $X\in\{l_\infty,c,c_0\}$ and $\dagger\in\{\alpha,\beta,N\}$.

2. The αB-, βB-, γB- and NB-duals of sequence spaces

In this section, we generalize the concept of multiplier space to introduce new generalizations of Köthe-Toeplitz duals and null duals of sequence spaces. Furthermore, we obtain these duals for the sequence spaces l_p, c and c_0, where $1\le p\le\infty$.

Let $A=(a_{n,k})$ and $B=(b_{n,k})$ be two infinite matrices of real numbers and X and Y two sequence spaces. We write $A_n=(a_{n,k})_{k=1}^{\infty}$ for the sequence in the n-th row of A. We say that A defines a matrix mapping from X into Y, and denote it by $A:X\to Y$, if and only if $A_n\in X^\beta$ for all n and $Ax\in Y$ for all $x\in X$. If we conside the matrix AB^t, where B^t is the transpose of matrix B, then the matrix AB^t defines a matrix mapping from X into Y, if and only if $(AB^t)_n\in X^\beta$ for all n and $(AB)x\in Y$ for all $x\in X$. Note that the condition $(AB^t)_n\in X^\beta$ implies that

$$\sum_{k=1}^{\infty}\left(x_k\sum_{i=1}^{\infty}a_{n,i}b_{i,k}\right)<\infty.$$

Based on this fact, we generalize the multiplier space $M(X,Y)$.

Definition 1. *Suppose that $B=(b_{n,k})$ is an infinite matrix with real entries. For the sequence spaces X and Y, the set $M_B(X,Y)$ defined by*

$$M_B(X,Y)=\left\{z\in\omega\,:\,\sum_{k=1}^{\infty}b_{n,k}z_k<\infty,\ \forall n \text{ and } \left(x_n\sum_{k=1}^{\infty}b_{n,k}z_k\right)_{n=1}^{\infty}\in Y,\ \forall x\in X\right\}$$

is called the generalized multiplier space of X and Y.

The αB-, βB-, γB- and NB-duals of a sequence space X, which are denoted by $X^{\alpha B}$, $X^{\beta B}$, $X^{\gamma B}$ and X^{NB}, respectively, are defined by

$$X^{\alpha B}=M_B(X,l_1),\quad X^{\beta B}=M_B(X,cs),\quad X^{\gamma B}=M_B(X,bs),\quad X^{NB}=M_B(X,c_0).$$

It should be noted that in the special case $B=I$, we have $M_B(X,Y)=M(X,Y)$. So

$$X^{\alpha B}=X^{\alpha},\quad X^{\beta B}=X^{\beta},\quad X^{\gamma B}=X^{\gamma},\quad X^{NB}=X^{N}.$$

Theorem 1. *If $B=(b_{n,k})$ is an invertible matrix, then $M_B(X,Y)\simeq M(X,Y)$.*

Proof. With the map $T:M_B(X,Y)\longrightarrow M(X,Y)$, which is defined by

$$Tz=\left(\sum_{k=1}^{\infty}b_{n,k}z_k\right)_{n=1}^{\infty},$$

the proof is obvious. □

We determine the generalized multiplier space for some sequence spaces. In order to do this, we state the following lemma which is essential in the study.

Lemma 1. *If $X,Y,Z\subset\omega$, then*

(*i*) *$X\subset Z$ implies $M_B(Z,Y)\subset M_B(X,Y)$,*

(*ii*) *$Y\subset Z$ implies $M_B(X,Y)\subset M_B(X,Z)$.*

Proof. The proof is elementary and so omitted. □

Remark 1. *If $B=I$, we have Lemma 1.25 from [12].*

Corollary 1. *Suppose that $X,Y\subset\omega$ and † denotes either of the symbols α, β, γ or N. Then*

(*i*) *$X^{\alpha B}\subset X^{\beta B}\subset X^{\gamma B}\subset\omega$; in particular, $X^{\dagger B}$ is a sequence space.*

(*ii*) *$X\subset Z$ implies $Z^{\dagger B}\subset X^{\dagger B}$.*

Remark 2. *If $B=I$, we have Corollary 1.26 from [12].*

With the notation of (1), we can define the spaces $X(B)$ for $X \in \{l_p, c, c_0\}$ and $1 \le p \le \infty$, as follows:

$$X(B) = \left\{ x = (x_n) \in \omega \ : \ \left(\sum_{k=1}^{\infty} b_{n,k} x_k \right)_{n=1}^{\infty} \in X \right\}.$$

Theorem 2. *We have the following statements.*

(*i*) $M_B(c_0, X) = l_\infty(B)$, *where* $X \in \{l_\infty, c, c_0\}$,

(*ii*) $M_B(l_\infty, X) = c_0(B)$, *where* $X \in \{c, c_0\}$,

(*iii*) $M_B(c, X) = c(B)$, *where* $X \in \{c, c_0\}$.

Proof. (i): Since $c_0 \subset c \subset l_\infty$, by applying Lemma 1(*ii*), we have

$$M_B(c_0, c_0) \subset M_B(c_0, c) \subset M_B(c_0, l_\infty).$$

So it is sufficient to verify $l_\infty(B) \subset M_B(c_0, c_0)$ and $M_B(c_0, l_\infty) \subset l_\infty(B)$. Suppose that $z \in l_\infty(B)$ and $x \in c_0$. We have

$$\lim_{n\to\infty} \left(x_n \sum_{k=1}^{\infty} b_{n,k} z_k \right) = 0.$$

This means that $z \in M_B(c_0, c_0)$. Thus $l_\infty(B) \subset M_B(c_0, c_0)$.

Now we assume $z \notin l_\infty(B)$. Then there is a subsequence $\left(\sum_{k=1}^{\infty} b_{n_j,k} z_k\right)_{j=1}^{\infty}$ of the sequence $(\sum_{k=1}^{\infty} b_{n,k} z_k)_{n=1}^{\infty}$ such that

$$\left| \sum_{k=1}^{\infty} b_{n_j,k} z_k \right| > j^2,$$

for $j = 1, 2, \cdots$. If the sequence $x = (x_i)$ is defined by

$$x_i = \begin{cases} \frac{(-1)^j j}{\sum_{k=1}^{\infty} b_{i,k} z_k}, & \text{if } i = n_j \\ 0, & \text{otherwise} \end{cases},$$

for $i = 1, 2, \cdots$, we have $x \in c_0$ and $x_{n_j} \sum_{k=1}^{\infty} b_{n_j,k} z_k = (-1)^j j$, for all j. Hence

$$\left(x_n \sum_{k=1}^{\infty} b_{n,k} z_k \right)_{n=1}^{\infty} \notin l_\infty.$$

This shows $M_B(c_0, l_\infty) \subset l_\infty(B)$.

(ii): We have

$$M_B(l_\infty, c_0) \subset M_B(l_\infty, c),$$

by applying Lemma 1(ii). It is sufficient to prove $c_0(B) \subset M_B(l_\infty, c_0)$ and $M_B(l_\infty, c) \subset c_0(B)$. Suppose that $z \in c_0(B)$. We have

$$\lim_{n\to\infty} \left(x_n \sum_{k=1}^{\infty} b_{n,k} z_k \right) = 0,$$

for all $x \in l_\infty$, that is, $z \in M_B(l_\infty, c_0)$. Thus $c_0(B) \subset M_B(l_\infty, c_0)$.

Now we assume $z \notin c_0(B)$. Then there are a real number as $b > 0$ and a subsequence $\left(\sum_{k=1}^{\infty} b_{n_j,k} z_k\right)_{j=1}^{\infty}$ of the sequence $\left(\sum_{k=1}^{\infty} b_{n,k} z_k\right)_{n=1}^{\infty}$ such that

$$\left| \sum_{k=1}^{\infty} b_{n_j,k} z_k \right| > b,$$

for all $j = 1, 2, \cdots$. If the sequence $x = (x_i)$ is defined by

$$x_i = \begin{cases} \frac{(-1)^j}{\sum_{k=1}^{\infty} b_{i,k} z_k}, & \text{if } i = n_j \\ 0, & \text{otherwise} \end{cases},$$

for all $i \in \mathbb{N}$, then we have $x \in l_\infty$ and

$$\left(x_n \sum_{k=1}^{\infty} b_{n,k} z_k \right)_{n=1}^{\infty} \notin c,$$

which implies $z \notin M_B(l_\infty, c)$. This shows that $M_B(l_\infty, c) \subset c_0(B)$.

(iii): Suppose that $z \in c(B)$. We deduce that $\lim_{n\to\infty} \left(x_n \sum_{k=1}^{\infty} b_{n,k} z_k\right)$ exists for all $x \in c_0$. So $z \in M_B(c, c_0)$ and $c(B) \subset M_B(c, c_0)$.

Conversely, we assume $z \in M_B(c, c)$. Let $x = (1, 1, \cdots)$. It is obvious that $x \in c$ and

$$\left(\sum_{k=1}^{\infty} b_{n,k} z_k \right)_{k=1}^{\infty} = \left(x_n \sum_{k=1}^{\infty} b_{n,k} z_k \right)_{k=1}^{\infty} \in c.$$

So $z \in c(B)$. This shows $M_B(c, c) \subset c(B)$. □

Remark 3. *If $B = I$, we have Example 1.28 from [12].*

Corollary 2. *We have $c_0^{NB} = l_\infty(B)$, $l_\infty^{NB} = c_0(B)$ and $c^{NB} = c_0(B)$.*

Below we recall the concept of normal and similarly to the Köthe-Toeplitz duals, we show that $X^{\alpha B} = X^{\beta B} = X^{\gamma B}$ when X is a normal set.

Definition 2. *A subset X of ω is said to be normal if $y \in X$ and $|x_n| \leq |y_n|$, for $n = 1, 2, \cdots$, together imply $x \in X$.*

Example 1. *The sequence spaces c_0 and l_∞ are normal, but c is not normal.*

Theorem 3. *Let X be a normal subset of ω. We have*

$$X^{\alpha B} = X^{\beta B} = X^{\gamma B}.$$

Proof. Obviously, $X^{\alpha B} \subset X^{\beta B} \subset X^{\gamma B}$, by Corollary 1(i). To prove the statement, it is sufficient to verify $X^{\gamma B} \subset X^{\alpha B}$. Let $z \in X^{\gamma B}$ and $x \in X$ be given. We define the sequence y such that

$$y_n = \left(sgn \sum_{k=1}^{\infty} b_{n,k} z_k \right) |x_n|,$$

for $n = 1, 2, \cdots$. It is clear $|y_n| \leq |x_n|$, for all n. Consequently, $y \in X$ since X is normal. So

$$\sup_n \left| \sum_{k=1}^{n} \left(y_n \sum_{k=1}^{\infty} b_{n,k} z_k \right) \right| < \infty.$$

Furthermore, by the definition of the sequence y,

$$\sum_{n=1}^{\infty} \left| x_n \sum_{k=1}^{\infty} b_{n,k} z_k \right| < \infty.$$

Since $x \in X$ was arbitrary, $z \in X^{\alpha B}$. This finishes the proof of the theorem. $\square$

Remark 4. *If $B = I$ and X is a normal subset of ω, we have*

$$X^{\alpha} = X^{\beta} = X^{\gamma},$$

hence Remark 1.27 from [12].

Now, we investigate the αB-, βB- and γB-duals for the sequence spaces l_{∞}, c and c_0.

Theorem 4. *Suppose that † denotes either of the symbols α, β or γ. We have*

$$c_0^{\dagger B} = c^{\dagger B} = l_{\infty}^{\dagger B} = l_1(B).$$

Proof. We only prove the statement for the case $\dagger = \beta$; the other cases are proved by Theorem 3. Obviously, $l_{\infty}^{\beta B} \subset c^{\beta B} \subset c_0^{\beta B}$ by Corollary 1(ii). So it is sufficient to show that $l_1(B) \subset l_{\infty}^{\beta B}$ and $c_0^{\beta B} \subset l_1(B)$.

Let $z \in l_1(B)$ and $x \in l_{\infty}$ be given. Hence

$$\sum_{n=1}^{\infty} \left| x_n \sum_{k=1}^{\infty} b_{n,k} z_k \right| \leq \sup |x_n| \sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} b_{n,k} z_k \right| < \infty, \tag{2}$$

which shows $(x_n \sum_{k=1}^{\infty} b_{n,k} z_k)_{n=1}^{\infty} \in cs$. Thus $z \in l_{\infty}^{\beta B}$ and $l_1(B) \subset l_{\infty}^{\beta B}$. Now let $z \notin l_1(B)$. We may choose an index subsequence (n_j) in $\mathbb{N}$ with $n_0 = 0$ and

$$\sum_{n=n_{j-1}}^{n_j - 1} \left| \sum_{k=1}^{\infty} b_{n,k} z_k \right| > j, \quad j = 1, 2, \ldots.$$

We define the sequence $x \in c_0$ such that

$$x_n = \begin{cases} \frac{1}{j} sgn \left(\sum_{k=1}^{\infty} b_{n,k} z_k \right), & \text{if } n_{j-1} \leq n < n_j \\ 0, & \text{otherwise} \end{cases}.$$

We get

$$\sum_{n=n_{j-1}}^{n_j - 1} \left(x_n \sum_{k=1}^{\infty} b_{n,k} z_k \right) = \frac{1}{j} \sum_{n=n_{j-1}}^{n_j - 1} \left| \sum_{k=1}^{\infty} b_{n,k} z_k \right| > 1,$$

for $j = 1, 2, \cdots$. Therefore $(x_n \sum_{k=1}^{\infty} b_{n,k} z_k)_{k=1}^{\infty} \notin cs$, and $z \notin c_0^{\beta B}$. This completes the proof of the theorem. $\square$

Remark 5. *If $B = I$ and $\dagger$ denotes either of the symbols α, β or γ. We have*

$$c_0^{\dagger} = c^{\dagger} = l_{\infty}^{\dagger} = l_1,$$

hence Theorem 1.29 from [12].

In the next theorem, we examine the αB-, βB- and γB-duals for the sequence space l_p.

Theorem 5. *If $1 < p < \infty$ and $q = p/(p-1)$, then*

$$l_p^{\alpha B} = l_p^{\beta B} = l_p^{\gamma B} = l_q(B).$$

Moreover for $p = 1$, we have $l_1^{\alpha B} = l_1^{\beta B} = l_1^{\gamma B} = l_{\infty}(B)$.

Proof. We only prove the statement for the case $1 < p < \infty$; the case $p = 1$ is proved similarly. Let $z \in l_q(B)$ be given. By Hölder's inequality, we have

$$\left|\sum_{k=1}^{\infty}\left(x_k \sum_{j=1}^{\infty} b_{k,j} z_j\right)\right| \leq \left(\sum_{k=1}^{\infty}\left|\sum_{j=1}^{\infty} b_{k,j} z_j\right|^q\right)^{1/q}\left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p} < \infty, \tag{3}$$

for all $x \in l_p$. This shows $z \in l_p^{\beta B}$ and hence $l_q(B) \subset l_p^{\beta B}$.

Now, let $z \in l_p^{\beta B}$ be given. We consider the linear functional $f_n : l_p \to \mathbb{R}$ defined by

$$f_n(x) = \sum_{k=1}^{n}\left(x_k \sum_{j=1}^{n} b_{k,j} z_j\right), \quad x \in l_p,$$

for $n = 1, 2, \cdots$. Similarly to (3), we obtain

$$|f_n(x)| \leq \left(\sum_{k=1}^{n}\left|\sum_{j=1}^{n} b_{k,j} z_j\right|^q\right)^{1/q}\left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p},$$

for every $x \in l_p$. So the linear functional f_n is bounded and

$$\|f_n\| \leq \left(\sum_{k=1}^{n}\left|\sum_{j=1}^{n} b_{k,j} z_j\right|^q\right)^{1/q},$$

for all n. We now prove the reverse of the above inequality. We define the sequence $x = (x_k)$ such that

$$x_k = \left(sgn \sum_{j=1}^{n} b_{k,j} z_j\right)\left|\sum_{j=1}^{n} b_{k,j} z_j\right|^{q-1},$$

for $1 \leq k \leq n$, and put the remaining elements zero. Obviously, $x \in l_p$, so

$$\|f_n\| \geq \frac{|f_n(x)|}{\|x\|_p} = \frac{\sum_{k=1}^{n} \left|\sum_{j=1}^{n} b_{k,j} z_j\right|^q}{\left(\sum_{k=1}^{n} \left|\sum_{j=1}^{n} b_{k,j} z_j\right|^q\right)^{1/p}} = \left(\sum_{k=1}^{n} \left|\sum_{j=1}^{n} b_{k,j} z_j\right|^q\right)^{1/q},$$

for $n = 1, 2, \cdots$. Since $z \in l_p^{\beta B}$, the map $f_z : l_p \to \mathbb{R}$ defined by

$$f_z(x) = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} b_{k,j} z_j\right) x_k, \quad x \in l_p\,,$$

is well-defined and linear, and also the sequence (f_n) is pointwise convergent to f_z. By using the Banach-Steinhaus theorem, it can be shown that $\|f_z\| \leq \sup_n \|f_n\| < \infty$, so

$$\left(\sum_{k=1}^{\infty} \left|\sum_{j=1}^{\infty} b_{k,j} z_j\right|^q\right)^{1/q} < \infty,$$

and $z \in l_q(B)$. This establishes the proof of the theorem. □

Remark 6. *If $B = I$ and $1 < p < \infty$ and $q = p/(p-1)$. Then we have*

$$l_p^{\alpha} = l_p^{\beta} = l_p^{\gamma} = l_q.$$

Moreover, for $p = 1$, we have $l_1^{\alpha} = l_1^{\beta} = l_1^{\gamma} = l_{\infty}$.

3. The αB-, βB- and NB-duals of sequence spaces $X(\Delta)$

The purpose of this section is to compute the $\dagger B$-dual of the difference sequence spaces $X(\Delta)$, where $X \in \{l_\infty, c, c_0\}$ and $\dagger \in \{\alpha, \beta, N\}$. In order to do this, we first give a preliminary lemma.

Lemma 2.

(i) *If $x \in l_\infty(\Delta)$, then $\sup_k \left|\frac{x_k}{k}\right| < \infty$.*

(ii) *If $x \in c(\Delta)$, then $\frac{x_k}{k} \to \xi\,(k \to \infty)$, where $\Delta x_k \to \xi\,(k \to \infty)$.*

(iii) *If $x \in c_0(\Delta)$, then $\frac{x_k}{k} \to 0\,(k \to \infty)$.*

Proof. The proof is trivial and so omitted. □

For convenience of the notations, we use $X^{\dagger B}(\Delta)$ instead of $X(\Delta)^{\dagger B}$, where $\dagger$ denotes either of the symbols α, β or N.

Theorem 6. *Define the set as follows:*

$$d_1 = \left\{ z = (z_k) \; : \; \left(n \sum_{k=1}^{\infty} b_{n,k} z_k\right)_{n=1}^{\infty} \in c_0 \right\}.$$

Then

$$c^{NB}(\Delta) = l_\infty^{NB}(\Delta) = d_1.$$

Proof. By using Corollary 1(ii), we have $l_\infty^{NB}(\Delta) \subset c^{NB}(\Delta)$. So it is sufficient to show that $d_1 \subset l_\infty^{NB}(\Delta)$ and $c^{NB}(\Delta) \subset d_1$.

Let $z \in d_1$ and $x \in l_\infty(\Delta)$. By Lemma 2 $\sup_n \left|\frac{x_n}{n}\right| < \infty$, so

$$\lim_{n\to\infty} x_n \sum_{k=1}^{\infty} b_{n,k} z_k = \lim_{n\to\infty} n \sum_{k=1}^{\infty} b_{n,k} z_k \frac{x_n}{n} = 0.$$

This implies that $z \in l_\infty^{NB}(\Delta)$. Now suppose that $z \in c^{NB}(\Delta)$, we have

$$\lim_{n\to\infty} x_n \sum_{k=1}^{\infty} b_{n,k} z_k = 0,$$

for all $x \in c(\Delta)$. If $x = (1, 2, 3, \cdots)$, we have $x \in c(\Delta)$ and

$$\lim_{n\to\infty} n \sum_{k=1}^{\infty} b_{n,k} z_k = 0.$$

So $z \in d_1$ and the proof of the theorem is finished. □

Remark 7. *If $B = I$, we have $c^N(\Delta) = l_\infty^N(\Delta) = \{z = (z_k) \;:\; (ka_k) \in c_0\}$, [9].*

Now, we recall the following theorem from [12] which is important to continue the discussion. Let $A = (a_{n,k})$ be an infinite matrix of real numbers $a_{n,k}$, where $n, k \in \mathbb{N} = \{1, 2, \cdots\}$. We consider the conditions

$$\sup_n \left(\sum_{k=1}^{\infty} |a_{n,k}| \right) < \infty, \tag{4}$$

$$\lim_{n\to\infty} a_{n,k} = 0, \quad k = 1, 2, \ldots, \tag{5}$$

$$\lim_{n\to\infty} a_{n,k} = l_k, \quad \text{for some } l_k \in \mathbb{R}, k = 1, 2, \ldots. \tag{6}$$

By (X, Y), we denote the class of all infinite matrices A such that $A : X \to Y$.

Theorem 7 (see [13]). *We have*

(*i*) *$A \in (c_0, c_0)$ if and only if conditions (4) and (5) hold;*

(*ii*) *$A \in (c_0, c)$ if and only if conditions (4) and (6) hold.*

Theorem 8. *Define the set as follows:*

$$d_2 = \left\{ z = (z_k) \;:\; \left(n \sum_{k=1}^{\infty} b_{n,k} z_k \right)_{n=1}^{\infty} \in l_\infty \right\}.$$

Then $c_0^{NB}(\Delta) = d_2$.

Proof. Suppose that $z \in d_2$. For $x \in c_0(\Delta)$, by Lemma 2 we have $\lim_{n\to\infty} \frac{x_n}{n} = 0$. So

$$\lim_{n\to\infty} x_n \sum_{k=1}^{\infty} b_{n,k} z_k = \lim_{n\to\infty} n \sum_{k=1}^{\infty} b_{n,k} z_k \frac{x_n}{n} = 0,$$

this implies that $z \in c_0^{NB}(\Delta)$.

Now let $z \in c_0^{NB}(\Delta)$. We define the matrix $D = (d_{n,j})$ by

$$d_{n,j} = \begin{cases} \sum_{k=1}^{\infty} b_{n,k} z_k, & \text{if } 1 \le j \le n \\ 0, & j > n, \end{cases}$$

and prove that $D \in (c_0, c_0)$. To do this, we show that $D_n = (d_{n,j})_{j=1}^{\infty} \in c_0^{\beta}$ for all n and moreover $Dy \in c_0$ for all $y \in c_0$.

Since $z \in c_0^{NB}(\Delta)$, we deduce that $\sum_{k=1}^{\infty} b_{n,k} z_k < \infty$ for all n; hence for $y \in c_0$

$$\sum_{j=1}^{\infty} d_{n,j} y_j = \sum_{j=1}^{n} \left(\sum_{k=1}^{\infty} b_{n,k} z_k \right) y_j = \left(\sum_{j=1}^{n} y_j \right) \left(\sum_{k=1}^{\infty} b_{n,k} z_k \right) < \infty,$$

for all n, so $D_n \in c_0^{\beta}$ for all n. Moreover, $z \in c_0^{NB}(\Delta)$ implies that

$$\lim_{n\to\infty} x_n \sum_{k=1}^{\infty} b_{n,k} z_k = 0,$$

for all $x \in c_0(\Delta)$. There exists one and only one $y = (y_k) \in c_0$ such that $x_n = \sum_{j=1}^{n} y_j$. So

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} d_{n,j} y_j = \lim_{n\to\infty} \sum_{j=1}^{n} \sum_{k=1}^{\infty} b_{n,k} z_k y_j = \lim_{n\to\infty} x_n \sum_{k=1}^{\infty} b_{n,k} z_k = 0,$$

for all $y \in c_0$. Hence $\lim_{n\to\infty} D_n y = 0$ and $Dy \in c_0$ for all $y \in c_0$.

By applying Theorem 7(i) for $D \in (c_0, c_0)$, we obtain

$$\sup_n \left| n \sum_{k=1}^{\infty} b_{n,k} z_k \right| = \sup_n \left| \sum_{j=1}^{n} \sum_{k=1}^{\infty} b_{n,k} z_k \right| = \sup_n \left| \sum_{j=1}^{\infty} d_{n,j} \right| < \infty.$$

This completes the proof of the theorem. $\square$

Remark 8. *If $B = I$, we have $c_0^N(\Delta) = \{z = (z_k) : (kz_k) \in l_\infty\}$, Lemma 2 from [9].*

In what follows, we consider the αB-dual for the sequence spaces $c(\Delta)$ and $l_\infty(\Delta)$.

Theorem 9. *Define the set d_3 as follows:*

$$d_3 = \left\{ z = (z_k) : \left(n \sum_{k=1}^{\infty} b_{n,k} z_k \right)_{k=1}^{\infty} \in l_1 \right\}.$$

Then $c^{\alpha B}(\Delta) = l_\infty^{\alpha B}(\Delta) = d_3$.

Proof. By applying Corollary 1(ii), we have $l_\infty^{\alpha B}(\Delta) \subset c^{\alpha B}(\Delta)$. So it is sufficient to show that $d_3 \subset l_\infty^{\alpha B}(\Delta)$ and $c^{\alpha B}(\Delta) \subset d_3$.

Let $z \in d_3$ and $x \in l_\infty(\Delta)$ be given. By Lemma 2 $\sup_n \left|\frac{x_n}{n}\right| < \infty$, so

$$\sum_{n=1}^{\infty}\left|x_n \sum_{k=1}^{\infty} b_{n,k} z_k\right| \leq \sup\left|\frac{x_n}{n}\right| \sum_{n=1}^{\infty}\left|n \sum_{k=1}^{\infty} b_{n,k} z_k\right| < \infty,$$

which shows $z \in c^{\alpha B}(\Delta)$ and $d_3 \subset l_\infty^{\alpha B}(\Delta)$. Now suppose that $z \in c^{\alpha B}(\Delta)$. Since $x = (1, 2, 3, \cdots) \in c(\Delta)$, we conclude that

$$\sum_{n=1}^{\infty}\left|n \sum_{k=1}^{\infty} b_{n,k} z_k\right| = \sum_{n=1}^{\infty}\left|x_n \sum_{k=1}^{\infty} b_{n,k} z_k\right| < \infty,$$

So $z \in d_3$, and this completes the proof of the theorem. □

Remark 9. *If $B = I$, we have $c^\alpha(\Delta) = l_\infty^\alpha(\Delta) = \{z = (z_k) : (kz_k) \in l_1\}$.*

In order to investigate the βB-dual of the difference sequence space $c_0(\Delta)$, we need the following lemma.

Lemma 3 (see [9, Lemma 1])**.** *If $z \in l_1$, $x \in c_0(\Delta)$ and $\lim_{k\to\infty} |z_k x_k| = L$, then $L = 0$.*

For the next result we introduce the sequence (R_k) given by

$$R_k = \sum_{t=k}^{\infty} \sum_{j=1}^{\infty} b_{t,j} z_j.$$

Theorem 10. *If*

$$d_4 = \{a = (a_k) \in l_1(B) : \ (R_k) \in l_1 \cap c_0^N(\Delta)\},$$

then we have $c_0^{\beta B}(\Delta) = d_4$.

Proof. Suppose that $z \in d_4$ and $x \in c_0(\Delta)$, by using Abel's summation formula, we have

$$\begin{aligned}\sum_{n=1}^{m}\left(x_n \sum_{k=1}^{\infty} b_{n,k} z_k\right) &= \sum_{n=1}^{m}\left(\sum_{t=1}^{n}\sum_{j=1}^{\infty} b_{t,j} z_j\right)(x_n - x_{n+1}) + \left(\sum_{n=1}^{m}\sum_{k=1}^{\infty} b_{n,k} z_k\right) x_{m+1} \\ &= \sum_{n=1}^{m}(R_1 - R_{n+1})(x_n - x_{n+1}) + (R_1 - R_{m+1}) x_{m+1} \\ &= \sum_{n=1}^{m+1} R_n (x_n - x_{n-1}) - R_{m+1} x_{m+1}. \qquad (7)\end{aligned}$$

This implies that

$$\sum_{n=1}^{\infty}\left(x_n \sum_{k=1}^{\infty} b_{n,k} z_k\right)$$

is convergent, so $z \in c_0^{\beta B}(\Delta)$.

Let $z \in c_0(\Delta)^{\beta B}$, by applying Corollary 1(ii) and Theorem 4 we have $c_0(\Delta)^{\beta B} \subset c_0^{\beta B} = l_1(B)$; hence $z \in l_1(B)$. If $x \in c_0(\Delta)$, then there exists $y = (y_k) \in c_0$ such that $x_n = \sum_{j=1}^{k} y_j$. By applying Abel's summation formula

$$\begin{aligned}\sum_{n=1}^{m} R_n y_n &= \sum_{n=1}^{m}(R_n - R_{n+1})\left(\sum_{j=1}^{n} y_j\right) + \sum_{n=1}^{m} R_{m+1} y_n \\ &= \sum_{n=1}^{m}\left(\sum_{j=1}^{n} y_j\right)\left(\sum_{j=1}^{\infty} b_{n,j} z_j\right) + \sum_{n=1}^{m} R_{m+1} y_n.\end{aligned}$$

Thus

$$\sum_{n=1}^{m}\left(\sum_{k=1}^{\infty} b_{n,k} z_k x_n\right) = \sum_{n=1}^{m}(R_n - R_{m+1}) y_n = \sum_{n=1}^{m}\left(\sum_{i=n}^{m}\sum_{j=1}^{\infty} b_{i,j} z_j\right) y_n. \tag{8}$$

Now we define the matrix $D = (d_{n,k})$ by

$$d_{n,k} = \begin{cases} \sum_{i=k}^{n}\sum_{j=1}^{\infty} b_{i,j} z_j, & \text{if } 1 \le k \le n \\ 0, & k > n, \end{cases}$$

and we prove that $D \in (c_0, c)$. To do this, we show that $D_n = (d_{n,j})_{j=1}^{\infty} \in c_0^{\beta}$ for all n, and moreover $Dy \in c$ for all $y \in c_0$.

Since $z \in c_0^{\beta B}(\Delta)$, we deduce that

$$\sum_{k=1}^{\infty} b_{n,k} z_k < \infty,$$

for all n; hence for $y \in c_0$

$$\sum_{k=1}^{\infty} d_{n,k} y_k = \sum_{k=1}^{n}\left(\sum_{i=k}^{n}\sum_{j=1}^{\infty} b_{i,j} z_j\right) y_k < \infty,$$

for all n. So $D_n \in c_0^{\beta}$ for all n. Moreover, $z \in c_0^{\beta B}(\Delta)$ implies that

$$\sum_{n=1}^{\infty}\left(x_n \sum_{k=1}^{\infty} b_{n,k} z_k\right)$$

is convergent for all $x \in c_0(\Delta)$. Hence by (8), we deduce that

$$\lim_{n\to\infty} D_n y = \lim_{n\to\infty}\sum_{k=1}^{n} d_{n,k} y_k = \lim_{n\to\infty}\sum_{i=1}^{n} x_i\left(\sum_{j=1}^{\infty} b_{i,j} z_j\right)$$

exists. So $Dy \in c_0$ for all $y \in c_0$ and $D \in (c_0, c)$. This implies that

$$\sup_n \sum_{k=1}^{\infty} |d_{n,k}| = \sup_n \sum_{k=1}^{n} \left| \sum_{i=k}^{n} \sum_{j=1}^{\infty} b_{i,j} z_j \right| < \infty,$$

by Theorem 7(ii). Thus we get

$$\sum_{k=1}^{\infty} |R_k| < \infty.$$

Furthermore, (7) implies that $\lim_{n\to\infty} R_{n+1} x_{n+1}$ exists for each $x \in c_0(\Delta)$; hence $(R_n) \in c_0^N(\Delta)$ by Lemma 3. This completes the proof of the theorem. □

Remark 10. *If $B = I$, then we have*

$$c_0^{\beta}(\Delta) = \{z = (z_k) \in l_1 \ : \ (R_k) \in l_1 \cap c_0^N(\Delta)\},$$

where the sequence (R_k) given by $R_k = \sum_{i=k}^{\infty} z_i$, hence Lemma 3 from [9] is resulted.

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