Two-stage optimisation of the multiphase production

JANEZ ARTEJAK† AND POLONA TOMINC‡

Abstract. Two approaches to the optimisation of the multiphase production process are presented in this paper. The first model - model I is a static one, using enterprise input/output modelling and linear programming simultaneously. The second approach - model II is a dynamic one and represents the combination between discrete dynamic deterministic programming in the first stage and the model I in the second stage. An application from building industry is also presented in the paper.

Key words: multiphase production models, enterprise modelling, production optimisation

AMS subject classifications: 90A11, 90B30, 90C05, 90C11, 90C39

1. Introduction

The paper deals with the determination of the optimal production quantities in the processing industries for a fixed number of periods assuming that the capacities are fixed and the demand for each period is known, but varies from one to another. Two linear optimisation models are presented. Optimisation Model I is a static approach to production problem using enterprise input/output modelling (EIOM) and linear programming (LP) simultaneously. Optimisation Model II is a two-stage model representing a partly dynamic approach by using dynamic programming in the first stage of optimisation and model I in the second stage.

The presented models have already been applied in practice in Slovenia and in this paper an application from the building industry is presented. This approach represents the continuation and extension of the work, which has started some years ago.

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2. Optimisation model I - static approach to the production problem. Enterprise input/output models (EIOM)

A general EIOM starts with a structural analysis of the multiphase production process, resulting in defining n production centres, each producing a single product which can be used as intermediate output in reproduction process and/or sold on the market (or markets) [1]. These relations are expressed in the matrix form as

\[ x = Ax + q \]  \hspace{1cm} (1)

where vector \( x \) stands for production quantities of production centres, vector \( q \) denotes selling quantities of products. Matrix \( A \) represents the matrix of input coefficients of production.

For the production quantities of production centres, denoted by vector \( x \), it is necessary to use appropriate input quantities of raw materials and labour hours, that is

\[ x^m = A^m x \]  \hspace{1cm} (2)

where vector \( x^m \) stands for input quantities and matrix \( A^m \) for input coefficients of raw materials and labour hours.

In each production centre of the enterprise it is therefore possible, using model (1), to decompose the quantity produced, to the part, which is consumed in production process and to the part, which is to be sold. If in (1), instead of vector \( x \), we put vector \( c \) for capacities of production centres, the relation

\[ q_{max} = c - Ac \]  \hspace{1cm} (3)

denotes the maximum possible selling quantities of products.

For analysing the impact of different selling quantities on produced quantities, it is possible to use model

\[ x = (E - A)^{-1} q \]  \hspace{1cm} (4)

which represents the solution of (1) for \( x \).

It is possible to extend this EIOM for analysing also the effects of changing production levels, of changing input prices or technological and institutional conditions, on the production prices.

Optimisation model (linear programming)

Considering relations (1), (2) and (4), the optimal selling quantities of products in period \( t \) can be obtained by solving the following optimisation model

\[
\max(z^T q_t) \\
\text{subject to } \\
x_t = (E - A)^{-1} q_t \leq c_t \\
x^m_t = A^m(E - A)^{-1} q_t \leq b_t \\
q^m_t A^m(E - A)^{-1} q_t \leq C_t \\
\text{Re} \leq q_t \leq \text{Im}
\]  \hspace{1cm} (5)

where
(5)- maximisation of the contribution to cover fixed costs, where the elements of vector \( z_t \) are defined as the difference between the selling price and cost price, calculated by using only variable costs, in period \( t \)

(6)- capacity limitations, where \( c_t \) is the vector of available capacities in period \( t \),

(7)- limited quantities of materials and working hours, where \( b_t \) is the vector of available quantities of materials and working hours in period \( t \),

(8)- financial limitations, where \( C_t \) is the vector of available amount of financial sources in period \( t \),

(9)- represents market limitations, where \( m_{lt} \) is the vector of minimum selling quantities of products and \( m_{ut} \) is the vector of maximum possible selling quantities of products.

After solving the model (5)-(9), the optimal production quantities of production centres can be obtained by using equation (4).

3. Optimisation model II - discrete dynamic deterministic approach to production problem

Let us assume to have an enterprise with \( n \) production centres, each producing a single product, which can be used as an intermediate output in the reproduction process and/or sold on the market (or markets).

This second approach starts with the selection of one or more products, which are of the greatest importance for business. Their optimal production quantities for the fixed number of periods \( T \) are obtained by using dynamic programming and assuming that the production and storage capacities are fixed and that the demand for each period \( t, t = 1, 2, \ldots, T \) is known, but may vary.

Let us assume that only one, the most important product, product A, is selected. By using dynamic programming, the following problem, presented in the form of recursion equations for \( t = 1, 2, \ldots, T \), is to be solved

\[
S_t(y_t) = \min_{x_t} [s(y_{t-1}) + s(x_t) + s_{t-1}(y_{t-1})] \tag{10}
\]

and \( s_0 = 0 \)

subject to

\[
y_0 = a \tag{11}
\]

\[
y_t = y_{t-1} + x_t - d_t \text{ for } t = 1, 2, \ldots, T \tag{12}
\]

\[
y_T = b \tag{13}
\]

\[
y_t \in E \text{ for } t = 1, 2, \ldots, T \tag{14}
\]

\[
x_t \in F \text{ for } t = 1, 2, \ldots, T \tag{15}
\]

where

\( y_{t-1} \) - stock at the beginning of period \( t \),

\( y_t \) - stock at the end of period \( t \),
$x_t$ - production quantity in period $t$, 
d_t - demand in period $t$, 
s(x_t) - production and set-up costs for period $t$, 
s(t, y_{t-1}) - storage costs at period $t$ for the stock at the beginning of period $t$, 
s_t(y_t) - minimum production, set-up and storage costs till the end of period $t$, meeting cumulative demands and stock at the end of period $t$, 
$E$ - set of possible states, which are all possible quantities of the stock for product A, 
$F$ - set of possible states, which are all possible production quantities of the product A.

By solving the dynamic model (10)-(15), the optimal production quantities of the selected product A for each period $t$ are obtained.

At the second stage of the model II the optimal selling and production quantities of other products, produced in other production centres, are to be obtained, by using the model (5)-(9), which must be modified and is expressed in the form

$$\max(\Sigma^T q_t)$$ (16)

subject to

$$x_t = (E - A)^{-1} q_t \leq \xi$$ (17)

$$x_t^m = A^m(E - A)^{-1} q_t \leq \bar{b}_t$$ (18)

$$pm^T A^m(E - A)^{-1} q_t \leq \bar{A}_t$$ (19)

$$m_{lt} \leq q_t \leq m_{ut}$$ (20)

where vectors $\xi$, $q_t$, $m_{lt}$ and $m_{ut}$ are reduced for one component. Vectors $\bar{b}_t$ and constant $\bar{A}_t$ are obtained by using relations

$$\xi = c_t - (E - A)^{-1} q_{t_{opt}}^i$$ (21)

$$\bar{b}_t = A_t - A^m(E - A)^{-1} q_{t_{opt}}^i$$ (22)

$$\bar{A}_t = A_t - pm^T A^m(E - A)^{-1} q_{t_{opt}}^i$$ (23)

where the value of the component $i$ of the vector $q_{t_{opt}}^i$ in period $t$ represents the optimal selling quantity of the discussed product (if the optimal selling quantity equals optimal producing quantity), but all other components are equal to zero.

4. An application from the building industry

The described procedure can be used in practice in many different ways. We get a simplified case if only one product is selected as the most important for the enterprise. But the same procedure may be used if two or more products are recognised as very important. We also get a special case, if the most important product (or products) is produced only from the raw materials, without any element, which is to be produced in the previous phase of the multiphase business process. This is also the case in the application presented in this section.
The multiphase business process of the enterprise TAME, which is the part of the joint-stock company Konstruktor d.d. Maribor, is presented in this paper. The ferro-concrete prefabricated elements are produced in this enterprise.

The multiphase business process is decomposed to the production, selling and purchasing activities. All inputs and outputs of the multiphase business process as well as the intermediate outputs, which are used in the further production process, are considered as elements. They are described in Table 1 as well as the corresponding purchasing or selling activities, purchasing or selling prices (in 1000 SIT) and also market limitation (maximal possible purchasing or selling quantities). Production activities, which can also be considered as the production centres, and costs (in 1000 SIT), that are not connected with the use of the elements in Table 1 (for example: water, electricity, ...) are given in Table 2.

<table>
<thead>
<tr>
<th>Elem.</th>
<th>Unit</th>
<th>Description</th>
<th>Purch. (y)/ Selling price (z)</th>
<th>Purchasing/ Selling price (in 1000 SIT)</th>
<th>Maxim. quant</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Concrete µB30</td>
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</tr>
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<td>Concrete-pipes machine</td>
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<td>Hollow-plate machine</td>
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</tr>
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<tr>
<td>E16</td>
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<td>Prec. Pillar 7m</td>
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<td>y23</td>
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<td>Prec. Roof T-Support 12m</td>
<td>y24</td>
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<td>y25</td>
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<td>y30</td>
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<td>550</td>
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<tr>
<td>E31</td>
<td>piece</td>
<td>Concrete Pipes Φ250</td>
<td>y31</td>
<td>9,990</td>
<td>550</td>
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<tr>
<td>E32</td>
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<td>Concrete Pipes Φ300</td>
<td>y32</td>
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</tr>
<tr>
<td>E33</td>
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<td>y33</td>
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<tr>
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<tr>
<td>E35</td>
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<td>y35</td>
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<td>550</td>
</tr>
<tr>
<td>E36</td>
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<td>y36</td>
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<td>550</td>
</tr>
<tr>
<td>E37</td>
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<td>y37</td>
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<td>550</td>
</tr>
<tr>
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<td>piece</td>
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<td>Tin-plates</td>
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<td>Roof manager</td>
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<td>Cutted Hollow-plate 200</td>
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<td>y43</td>
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<tr>
<td>E44</td>
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<td>Cutted Hollow-plate 400</td>
<td>y44</td>
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<td>y45</td>
<td>3750,2</td>
<td>1</td>
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<td>E46</td>
<td>piece</td>
<td>Prefabricated Hall 16m /8/9m</td>
<td>y46</td>
<td>4800,0</td>
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<tr>
<td>E47</td>
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<td>5683,2</td>
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<tr>
<td>E49</td>
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<td>y49</td>
<td>8337,6</td>
<td>1</td>
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</table>

Table 1. Elements of the multiphase business process

Production of the ferro-concrete prefabricated elements is therefore the multiphase business process with described production activities, which are considered as the production centres, each producing a single product, which can be used as an intermediate output in the production process and/or sold on the market. The element E23 - Ferro-concrete roof support, of 16 meters length, is an intermediate output, which causes high storage costs because of its special shape and other characteristics. But this ferro-concrete prefabricated element is at the same time at a...
very important place in the production program of the enterprise, since the demand for it on the market is almost permanent and very stable.

<table>
<thead>
<tr>
<th>Prod. Activity</th>
<th>Description</th>
<th>Costs</th>
</tr>
</thead>
<tbody>
<tr>
<td>X1</td>
<td>Production of the hollow-plates 200</td>
<td>218</td>
</tr>
<tr>
<td>X2</td>
<td>Production of the hollow-plates 265</td>
<td>220</td>
</tr>
<tr>
<td>X3</td>
<td>Production of the hollow-plates 400</td>
<td>340</td>
</tr>
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<td>X4</td>
<td>Production of the Fe-concrete pillar 9m</td>
<td>6.9</td>
</tr>
<tr>
<td>X5</td>
<td>Production of the Fe-concrete pillar 2m</td>
<td>2.5</td>
</tr>
<tr>
<td>X6</td>
<td>Production of the Fe-concrete pillar 7m</td>
<td>5.36</td>
</tr>
<tr>
<td>X7</td>
<td>Production of the Fe-concrete Roof support 18m</td>
<td>19,290</td>
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<tr>
<td>X8</td>
<td>Production of the Fe-concrete Roof support 16m</td>
<td>17,145</td>
</tr>
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<td>X9</td>
<td>Production of the Fe-concrete T-Roof support 12m</td>
<td>1,295</td>
</tr>
<tr>
<td>X10</td>
<td>Production of the Fe-concrete T-Roof support 8m</td>
<td>1,010</td>
</tr>
<tr>
<td>X12</td>
<td>Production of the paving plates</td>
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</tr>
<tr>
<td>X13</td>
<td>Production of the concrete borders 5 cm</td>
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</tr>
<tr>
<td>X14</td>
<td>Production of the concrete borders 15 cm</td>
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<td>X15</td>
<td>Production of the concrete pipes Φ20</td>
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<td>X16</td>
<td>Production of the concrete pipes Φ50</td>
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<td>X18</td>
<td>Production of the concrete pipes Φ100</td>
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<td>X19</td>
<td>Production of the concrete pipes Φ120</td>
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<td>Production of covers Φ50</td>
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<td>X21</td>
<td>Production of covers Φ80</td>
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<tr>
<td>X22</td>
<td>Production of covers Φ100</td>
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<td>X23</td>
<td>Production of covers Φ120</td>
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<td>X24</td>
<td>Cutting of the hollow-plates 200</td>
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</tr>
<tr>
<td>X25</td>
<td>Cutting of the hollow-plates 265</td>
<td>24,300</td>
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<td>X26</td>
<td>Cutting of the hollow-plates 400</td>
<td>37,300</td>
</tr>
<tr>
<td>X27</td>
<td>Production of the prefabricated-hall 16m/8/7m</td>
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<td>X28</td>
<td>Production of the prefabricated-hall 16m/8/9m</td>
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<tr>
<td>X29</td>
<td>Production of the prefabricated-hall 16m/12/7m</td>
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<td>Production of the prefabricated-hall 16m/12/9m</td>
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<td>X31</td>
<td>Production of the prefabricated-hall 18m/8/9m</td>
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</tr>
<tr>
<td>X32</td>
<td>Production of the prefabricated-hall 18m/12/9m</td>
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</table>

Table 2. Production activities

In the first stage of optimisation by Model II, the optimal production quantities of the element E23 for the fixed number of months $T$, $T = 3$, are obtained by using the dynamic programming and assuming that the production and storage capacities are fixed and that the demand for each month $t$, $t = 1, 2, 3$, is known and should be satisfied.

Let us assume that 10, 20 or 30 pieces of element E23 can be produced, which causes 400, 550 and 800 monetary units of the production costs. If the production quantity equals 0, the fixed costs of 200 monetary units arise. Each piece of the element E23 being in the storehouse one month causes storage costs of 6 monetary
units. Fixed storage costs are 50 monetary units per month. Maximal possible stock is 20 pieces. The market demand for the element E23 at the end of the first month equals 30, at the end of the second month 20 and at the end of the third month 35 pieces. Since the maximal produced quantity equals 30, the demand in the third month could not be satisfied without previously produced quantity of the element E23, therefore the need for dynamic programming to find the optimal producing quantity of the element for each period with minimal producing and storage costs. The stock at the beginning of the first month equals 20 pieces and the stock at the end of the third month should be minimal.

By using the methodology of discrete dynamic deterministic programming, briefly described in Section 3, the following results are obtained for production quantities of the element E23 for each month: 1. month 20 pieces, 2. month 20 pieces and 3. month 30 pieces.

The stock at the end of each month will be 10, 10 and 5 pieces of the element E23. The first stage of the Model II is therefore finished.

In the second stage, the optimal selling and production quantities of other products and optimal purchasing quantities of inputs are to be obtained for each month. First for month 1. Since the production centre/activity X8 where the element E23 is produced does not need any product from the other production centres/activities, this problem can be easily solved by the linear mixed integer optimisation model

\[
\begin{align*}
\text{max} & \quad (69z_{19} + 25z_{20} + 53z_{21} + 171.45z_{22} + 192.9z_{23} + 0.813z_{27} + 2.437z_{28}
+ 5.425z_{29} + 7.712z_{30} + 9.890z_{31} + 0.520z_{32} + 0.945z_{33} + 0.454z_{34} + 0.99z_{35}
+ 2.661z_{36} + 6.190z_{37} + 7.691z_{38} + 3763.2z_{44} + 4800z_{45} + 5145.6z_{46}
+ 5683.2z_{47} + 7300.8z_{48} + 8337.6z_{49} - 0.42y_8 - 1.6y_{10} - 3y_{11} - 1.8y_{12}
- 2.7y_{13} - 3.2y_{14} - 3.5y_{15} - 5.04y_{41} - 6.24y_{42} - 8.4y_{43} - 8.78y_1 - 9.3y_2
- 10.88y_3 - 14.21y_4 - 0.01y_5 - 0.011y_6 - 0.012y_7 - 0.35y_{10} - 0.15y_{40}
- 218y_1 - 220y_2 - 340y_3 - 6.9y_4 - 2.5y_5 - 5.36y_6 - 19.29y_7 - 17.145y_8
- 1.295y_9 - 1.01x_{10} - 0.023x_{12} - 0.034x_{13} - 0.045x_{14} - 0.155x_{15}
- 0.233x_{16} - 0.295x_{17} - 0.420x_{18} - 0.545x_{19} - 0.110x_{20} - 0.140x_{21}
- 0.180x_{22} - 0.205x_{23} - 19.1x_{24} - 24.3x_{25} - 37.3x_{26})
\end{align*}
\]

subject to

\[
\begin{align*}
z_{19} & \leq 100 \\
z_{27} & \leq 360 \\
z_{31} & \leq 550z_{32} \leq 190000 \\
z_{37} & \leq 5 \\
z_{32} & \leq 500 \\
z_{33} & \leq 500z_{35} \leq 5 \\
z_{34} & \leq 5 \\
z_{35} & \leq 5 \\
y_2 & \leq 0.0125x_{13} + 0.0375x_{11} - 0.125x_{15} - 0.156x_{16} - 0.2x_{17} - 0.31x_{18}
- 0.42x_{19} \geq 0 \\
y_3 & - 2x_2 - 0.45x_5 - 1.56x_6 - 0.03x_{20} - 0.08x_{21} - 0.13x_{22} - 0.21x_{23} \geq 0 \\
y_4 & - 4.6x_7 - 4.10x_8 - 0.6x_9 - 0.48x_{10} \geq 0 \\
y_5 & - 56000x_1 - 58000x_2 - 60000x_3 - 104x_4 - 24x_5 - 82x_6 - 460x_7 - 409x_8
- 114x_9 - 91x_{10} - 7x_{20} - 12x_{21} - 25x_{22} - 35x_{23} \geq 0 \\
y_6 & - 34000x_1 - 37000x_2 - 40000x_3 - 290x_4 - 64x_5 - 225x_6 - 160x_7 - 142x_8
- 82x_9 - 65x_{10} \geq 0 \\
\end{align*}
\]
\[ y_7 - 210x_7 - 187x_8 \geq 0 \]
\[ y_8 - 5x_4 - 2x_5 - 3.8x_6 - 31x_7 - 28x_8 - 4.75x_9 - 3.8x_{10} - 0.3x_{12} - 0.2x_{14} \]
\[ -0.3x_{14} - 0.1x_{20} - 0.4x_{21} - 0.95x_{22} - 1.35x_{23} \geq 0 \]
\[ y_{10} - 46.5x_{27} - 46.5x_{28} - 46.5x_{29} - 46.5x_{30} - 46.5x_{31} - 46.5x_{32} \geq 0 \]
\[ y_{18} - 0.2x_4 + 0.15x_5 - 0.2x_6 - x_7 - x_8 - 0.15x_9 - 0.15x_{10} \geq 0 \]
\[ y_{12} - 0.01x_{12} - 0.08x_{13} - 0.09x_{14} - 0.11x_{15} - 0.12x_{16} - 0.13x_{17} - 0.16x_{18} \]
\[ -0.2x_{19} - 0.01x_{20} - 0.05x_{21} - 0.25x_{22} - 0.35x_{23} \geq 0 \]
\[ y_{14} - 0.12x_{15} - 0.21x_{16} - 0.25x_{17} - 0.3x_{18} - 0.35x_{19} \geq 0 \]
\[ y_{15} - 36x_1 - 36x_2 - 36x_3 \geq 0 \]
\[ y_{14} - 20x_{24} - 30x_{24} - 40x_{26} \geq 0 \]
\[ x_1 - x_{24} \geq 0 \]
\[ x_2 - x_{25} \geq 0 \]
\[ x_3 - x_{26} \geq 0 \]
\[ x_4 - z_{19} - 48x_{28} - 32x_{30} - 32x_{32} - 32x_{32} \geq 0 \]
\[ x_5 - z_{20} \geq 0 \]
\[ x_6 - z_{21} - 48x_{27} - 32x_{29} \geq 0 \]
\[ x_7 - z_{22} - 4x_{31} - 3x_{32} \geq 0 \]
\[ x_8 - z_{23} - 4x_{27} - 4x_{28} - 3x_{29} - 3x_{30} \geq 0 \]
\[ x_9 - 16x_{29} - 16x_{30} - 16x_{32} \geq 0 \]
\[ x_{10} - 24x_{27} - 24x_{28} - 24x_{31} \geq 0 \]
\[ x_{15} - z_{27} \geq 0 \]
\[ x_{16} - z_{28} \geq 0 \]
\[ x_{17} - z_{29} \geq 0 \]
\[ x_{18} - z_{30} \geq 0 \]
\[ x_{19} - z_{31} \geq 0 \]
\[ x_{20} - z_{32} \geq 0 \]
\[ x_{21} - z_{33} \geq 0 \]
\[ x_{22} - z_{34} \geq 0 \]
\[ x_{23} - z_{35} \geq 0 \]
\[ x_{24} - z_{36} \geq 0 \]
\[ x_{25} - z_{37} \geq 0 \]
\[ x_{26} - z_{38} \geq 0 \]
\[ y_{30} - 432x_{27} - 432x_{28} - 432x_{29} - 432x_{30} - 480x_{31} - 480x_{32} \geq 0 \]
\[ y_{40} - 48x_{27} - 48x_{28} - 48x_{29} - 48x_{30} - 48x_{31} - 48x_{32} \geq 0 \]
\[ 1200x_{24} + y_{41} - 336x_{27} - 432x_{28} - 336x_{29} - 432x_{30} - 432x_{31} - 432x_{32} \geq 0 \]
\[ 1200x_{25} + y_{42} - 224x_{27} - 288x_{28} - 224x_{29} - 288x_{30} - 324x_{31} - 324x_{32} \geq 0 \]
\[ 1200x_{26} + y_{43} - 384x_{27} - 384x_{28} - 384x_{29} - 384x_{30} - 432x_{31} - 432x_{32} \geq 0 \]
\[ x_{27} - z_{46} \geq 0 \]
\[ x_{28} - z_{47} \geq 0 \]
\[ x_{29} - z_{44} \geq 0 \]
\[ x_{30} - z_{45} \geq 0 \]
\[ x_{31} - z_{49} \geq 0 \]
\[ x_{32} - z_{46} \geq 0 \]
\[ x_8 \geq 20 \]
\[ z_{23} = 20 \]
\[ x_8 \leq 30 \]

It is taken into account that the market demand for 30 pieces of element E23 in the first month will be satisfied by 20 produced pieces and 10 pieces will be taken from the warehouse. The result is

**OBJECTIVE FUNCTION VALUE 24139.4900**

\[ X_1 = 1.0 \quad Y_5 = 200306.70 \quad Z_{19} = 100.0 \]
\[ X_2 = 0.73 \quad Y_7 = 8058.00 \quad Z_{20} = 100.0 \]
\[ X_3 = 1.04 \quad Y_9 = 13838.80 \quad Z_{21} = 100.0 \]
\[ X_4 = 164.0 \quad Y_{10} = 139.5 \quad Z_{22} = 10.0 \]
Two-stage optimisation...

\[ X_5 = 100.0 \quad Y_11 = 128.0 \quad Z_{23} = 20.0 \]
\[ X_6 = 148.0 \quad Y_{12} = 360.0 \quad Z_{32} = 35995.0 \]
\[ X_7 = 17.0 \quad Y_1 = 620.96 \quad Z_{35} = 5.0 \]
\[ X_8 = 24.0 \quad Y_2 = 359.95 \quad Z_{46} = 1.0 \]
\[ X_9 = 16.0 \quad Y_5 = 21601.70 \quad Z_{48} = 1.0 \]
\[ X_{10} = 48.0 \quad Y_{14} = 99.6 \quad Z_{49} = 1.0 \]
\[ X_{12} = 35995.0 \quad Y_{15} = 91.6 \]
\[ X_{20} = 5.0 \quad Y_{40} = 144.0 \]
\[ X_{24} = 1.0 \quad Y_3 = 604.03 \]
\[ X_{25} = 0.73 \quad Y_4 = 209.24 \]
\[ X_{26} = 1.04 \quad Y_{39} = 1392.0 \]
\[ X_{27} = 1.0 \]
\[ X_{31} = 1.0 \]
\[ X_{32} = 1.0 \]

All other variables are 0.

This procedure must be repeated for the second and third month, where also all possible changes of market limitations (demand, prices, ...) can be taken into account.

5. Conclusion

The two-stage optimisation model discussed in this paper enables even higher degree of rationalisation in the business process, resulting in lower storage, production and purchasing costs.

This model has already been used also in the aluminium industry in Slovenia. It is suitable for every multiphase production, regardless of the number of production centres, especially in situations where storage costs represent an important part of business costs.

References
