Solving 2\textsuperscript{nd} order parabolic system by simulations of Markov jump processes\textsuperscript{*}

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\textbf{Abstract.} There are known methods of approximating the solution of parabolic 2\textsuperscript{nd} order systems by solving stochastic differential equations instead. The main idea is based on the fact that a stochastic differential equation defines a diffusion process, generated by an elliptic differential operator on $\mathbb{R}^d$. We propose a difference scheme for the elliptic operator, which possesses the structure of Markov (jump) process. The existence of such a scheme is proved, the proof relying on the choice of new coordinates in which the elliptic operator is “almost” Laplacian, and has the properties necessary for discretization.

Time discretization, which involves difference schemes for parabolic equations with known stability difficulties, can thus be replaced by space discretization and simulation of the associated Markov (jump) process.

\textbf{Key words:} parabolic system, simulation, Markov jump process

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\section{1. Introduction}

There is a well known statistical method for estimating solutions of the initial value problem of parabolic system of PDE of the 2\textsuperscript{nd} order \cite{5, 4}. The method is based on statistical estimation of sample paths of simulated solutions of the associated stochastic differential equation.

Here, another statistical method for estimating solutions of the parabolic system is proposed. It consists of two steps. In the first step the elliptic operator of the system is discretized on a space grid so that the resulting stiffness matrix is the generator of a Markov jump process on grid knots. In this way the parabolic system is approximated by the initial value problem for a system of ODE. In the second step solutions of ODE at the knots are statistically estimated by simulating the Markov jump process.

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The proposed method is advantageous in comparison to other methods only if the diffusion tensor and drift velocity are $t$-independent, while the extinction coefficient may be $t$-dependent. Due to this fact the time discretization is avoided. The proposed method simplifies the approach, which is extensively considered by Kuchner [8] in which Markov chains are used.

Results described here are more or less expectable. The necessary steps are gathered from various disciplines, and some lacking links are proved in order to get a consistent presentation. In Section 2 a proof is given that there exists a discretization of elliptic differential operator having the structure of a generator of a Markov jump process. In Section 3 the result about approximation is extended to the initial value problem for PDE. Statistical estimation of solutions by using Monte Carlo methods is described in Section 4. The initial value problem with homogeneous Dirichlet conditions is discussed in Section 5.

2. Numerical scheme for an elliptic operator having MJP–structure

Let the functions $a_{ij}, b_i$ and $c$ be uniformly bounded on $\mathbb{R}^d, i, j = 1, 2, ..., d,$ and have the following properties:

$$a_{ij}, b_i, c \in C^\alpha(\mathbb{R}^d), c \leq 0,$$

$$\sum_{i,j=1}^{d} a_{ij}(x)z_iz_j \geq \mu |z|^2, \mu > 0, x \in \mathbb{R}^d,$$

where $C^\alpha(\mathbb{R}^d)$ is the Hölder class of uniformly bounded functions with the parameter $\alpha \in (0, 1)$. Functions of $C^{k+\alpha}(\mathbb{R}^d)$ naturally define a real or a complex Banach space when endowed with the norm $\| \cdot \|_{k+\alpha}$ (see [2]). For the elliptic differential operator

$$A(x) = \sum_{i,j=1}^{d} a_{ij}(x)\partial_i \partial_j + \sum_{i=1}^{d} b_i(x)\partial_i + c(x),$$

the following result of classical or Shauder theory of PDE is used (see [2]). If (1) is valid and Re $\lambda > 0$, then: a) the differential operator $\lambda I - A$ is a one-to-one map between the Banach spaces $C^{2+\alpha}(\mathbb{R}^d)$ and $C^\alpha(\mathbb{R}^d)$, and b) for each $u \in C^{2+\alpha}(\mathbb{R}^d)$ the following inequality $\| (\lambda I - A)^{-1}u \|_0 \leq \| u \|_0 / \text{Re } \lambda$ holds.

For each natural $n$ the set of points $x = (k_1h, k_2h, \ldots, k_dh) \in \mathbb{R}^d, h = 2^{-n}$ defines the numerical grid $G_n$ on $\mathbb{R}^d$. Let $E_2 = C^{2+\alpha}(\mathbb{R}^d), E_2(n)$ be the closed subspace of all elements in $E_2$ vanishing at $G_n$. If the space $F_2 = E_2/E_2(n)$ is endowed with a quotient norm, it becomes isomorphic to a subspace of $l_\infty$. Analogously, $E_0 = C^\alpha(\mathbb{R}^d), E_0(n)$ and $F_0$ are defined. Because of the mentioned isomorphism, we identify $F_2$ with a subspace of $F_0$. The natural embedding from $E_k$ into $F_k, k = 2, 0,$ is denoted by $\phi_k(n)$.

We say that an element $u \in E_k$ is approximated with the order $\beta > 0$ by elements $u_n \in F_k$ if there exists an $n$-independent positive number $\kappa$ such that
\[ \| \phi(n)u - u_n \|_k < \kappa h^\beta. \]

In this case, we say that \( u_n \) factor converges to \( u \) with the order \( \beta \). Also we say that an operator \( A \in L(E_2, E_0) \) is approximated by the operators \( A_n \in L(F_2, F_0) \) with the order \( \beta \) if there exists an \( n \)-independent positive number \( \kappa \) such that

\[ \| \phi_0(n)A - A_n \phi_2(n) \|_{L(E_2, E_0)} \leq \kappa h^\beta. \]  

(3)

The number \( \beta \) is also called the order of approximation. The approximation is called stable if there exists an \( n \)-independent \( \rho > 0 \) such that

\[ \| (A_n)^{-1} \|_{L(F_0, F_2)} \leq \rho. \]

For \( \Re \lambda > 0 \), solutions \( u \) and \( u_n \) of respective equations,

\[ (\lambda I - A)u = f, \]
\[ (\lambda I - A_n)u_n = f_n, \quad n \in \mathbb{N}, \]

will be compared. For this purpose the following result [6] is used:

**Theorem 1.** Let \( A \in L(E_2, E_0), A_n \in L(F_2, F_0) \) be approximations of \( A \) with the order \( \beta \). Let \( f \in E_0 \), and let \( f_n \) be approximations of \( f \) with the order \( \beta \). If \( u \) and \( u_n \) are the unique solutions of (4), and \( \lambda I - A_n \) are stable, then \( u_n \) factor converges to \( u \) with the order equal to \( \beta \).

For an elliptic operator \( A \) satisfying (1), an approximation \( A_n \) is said to possess an MJP-structure if \( A_n \) are generators of (regular or irregular) Markov jump processes. Of course, the property is assumed to be valid for all natural \( n \) up to a finite number of them. These exceptional natural numbers are disregarded in the following.

The basic result of this section is the following theorem:

**Theorem 2.** Let \( A \) be defined by (1) and (2). Then there exist approximations \( A_n \in L(F_2, F_0) \) such that:

(a) each \( A_n \) possesses an MJP-structure and approximates \( A \) with the order \( \alpha \),

(b) there exists \( m_B \in \mathbb{N} \) such that \( A_n \) have the band structure with the band-width not larger than \( 1 + 2m_B \),

(c) equations \( (\lambda I - A_n)u_n = f_n, \Re \lambda > 0, \) have unique solutions in \( F_0 \) with the property \( \| u_n \|_\infty < \| f_n \|_\infty / \Re \lambda \),

(d) If \( f_n \) factor converges to \( f \) with the order \( \alpha \), then the sequence of solutions \( u_n \) of (4) also factor converges to \( u \) with the order \( \alpha \).

**Proof.** The statement (c) is a consequence of the MJP-structure of \( A_n \). This property ensures the stability of operators \( \lambda I - A_n \). The statement (d) follows from (a), (c) and **Theorem 1**.

To prove (a), let the approximation \( A_n \) of \( A \) have the property \( Ap(x) - A_n p(x) = 0 \) at \( x \in G_n \) whenever \( p \) is a polynomial of the second order. Then, by using standard methods of numerical analysis for PDE, the expression (3) can be easily derived with \( \beta = \alpha \). Hence, to finish the proof, we have to construct a sequence \( A_n \) having the mentioned property and the MJP-structure.
The partial differential operators of the first and second orders are approximated in the usual way:

\[
\begin{align*}
\partial f(x) \rightarrow \Box_i f(x) &= \frac{1}{2h} [f(x + e_i h) - f(x - e_i h)], \\
\partial^2_i f(x) \rightarrow \Box^2_i f(x) &= \frac{1}{h^2} [f(x + e_i h) - 2f(x) + f(x - e_i h)].
\end{align*}
\]

where \(e_i\) are unit vectors in the \(i\)th direction. The mixed partial differential operator can be approximated by one of the following four possibilities:

\[
\partial_i \partial_j f(x) \rightarrow \Box_i \Box_j f(x) = \frac{1}{h^2} \left\{ f(x + e_i h \pm e_j h) - f(x \pm e_i h) - f(x \pm e_j h) + f(x),
\right.
\]

\[
\left. - f(x \pm e_i h + e_j h) + f(x \pm e_i h) + f(x \pm e_j h) - f(x) \right\}.
\]

In this way, the quadratic operator \(a_{ij} (\partial_i)^2 + a_{jj} (\partial_j)^2\) is approximated by the matrix \(a_{ii} (\Box_i)^2 + a_{jj} (\Box_j)^2\) with negative diagonal elements and non-negative off-diagonal elements. If \(a_{ij} \leq 0\), then \(a_{ij} \partial_i \partial_j\) is approximated by the half sum of the first two possibilities, otherwise by the half sum of the second two possibilities. Let us assume that there exists \(0 < \rho < 1\), such that

\[
| a_{ij}(x) | \leq \rho \min\{a_{ii}(x), a_{jj}(x)\}, i \neq j, \tag{5}
\]

at the considered knot. Then the result, \(\sum_{ij} a_{ij} \Box_i \Box_j\), is the matrix having the MJP-structure. The approximation of \(\sum b_i \partial_i\) by \(\sum b_i \Box_i\) can violate the MJP-structure only for larger values of \(h\). Hence, if (5) is valid for all \(x \in \mathbb{R}^d\), the approximations \(A_n\) have the MJP-structure for large values of \(n\).

If there is no \(x \in \mathbb{R}^d\) with the property (5), a transformation of coordinates can be performed, and a redefinition of grids \(G_n\), so that the set \(D_0 \subset \mathbb{R}^d\) for which (5) holds is nonempty. To each \(x \in \mathbb{R}^d\) violating (5) there corresponds a new coordinate system centred at \(x\), and denoted by \(T(x)\), such that in the new coordinates the operator \(A(x)\) has the expression \(A(x) = \mu \Delta + \sum b_i \partial_i + c\). The coordinate transformation \(T(x)\) is defined by \(d(d - 1)\) angles in the set \(S^d\), where \(S \subset \mathbb{R}^d\) is the unit spherical surface. These angles are uniquely defined in case when \(T(0)\) is specified in advance, and if we demand the continuity of the map \(x \rightarrow T(x)\). By using the compactness of \(S^d\) and the uniform continuity of \(a_{ij}\), one proves that the set \(\mathbb{R}^d\) can be divided into a finite number of subsets \(D_m, m = 0, 1, 2, \ldots, M\), with the following properties: for each \(D_m\) there is a corresponding coordinate transformation \(T_m\) in which the transformed operator \(A\) has the property (5) for all \(x \in D_m\), and positive number \(\rho/2\) instead of \(\rho\). A coordinate axes of this system may miss knots of all \(G_n, n \in N\). Therefore, by a slight adjustment of axes, there can be constructed another set of \(M\) coordinate systems \(T_m\), such that the transformed operators \(A(\cdot)\) have the property (5) on \(D_m\). The approximations \(A_n\) in the transformed systems have the MJP structure. After turning to the original system, i.e. to the system \(T_0\), the obtained approximations preserve the MJP-structure. In this way \(A\) can be approximated by matrices \(A_n\) having the MJP-structure.

In order to prove (b) we point out that only a finite number of coordinate transformations are used for the discretization. Hence, for each grid-knot, irrespectively
of grid-step, only a finite number of various numerical neighbourhoods can be chosen. This proves the theorem.

The matrix $A_n$ may be reducible for each $n$. However, by omitting certain grid knots the irreducibility can be achieved:

**Lemma 1.** There exist subgrids $G_{n} \subset G_n$ such that $\bigcup G_n$ is dense in $\mathcal{R}^d$, and the approximations $A_n$ associated with $G_n$ are irreducible matrices.

### 3. Initial value problem for parabolic systems on $\mathcal{R}^d$

Let the functions $a_{ij}, b_i$ and $c$ have the following properties (the damping coefficient $c$ is generally $t$-dependent):

$$a_{ij}, b_i \in C^{1+\alpha}(\mathcal{R}^d), \quad c \in H^{\alpha/2,\alpha}([0, \infty) \times \mathcal{R}^d), \quad c \leq 0,$$

where the Hölder space $H^{\alpha/2,\alpha}$ is defined as in [3]. We consider the elliptic operator $B$ on $[0, \infty) \times \mathcal{R}^d$ defined by the expression:

$$B(t, x) = \sum_{i,j=1}^{d} \partial_i a_{ij}(x) \partial_j - \sum_{i=1}^{d} b_i(x) \partial_i + c(t, x).$$

(7)

Both differential operators, $B(t)$ and its formal adjoint $A(t) = B^+(t)$, have the form (2) with properties (1) for each $t \geq 0$. Therefore, the approximations $A_n(t) \in L(F_2, F_0)$ of the previous section can be constructed.

The objects of interest in this section is the following initial value problem:

$$\left( \frac{\partial}{\partial t} - B(t) \right) u(t) = f(t), \quad u(0) = u_0,$$

(8)

where $u_0 \in L_1(\mathcal{R}^d)$ and $f \in L_1([0, \infty), \mathcal{R}^d)$. Its approximations are:

$$\left( \frac{\partial}{\partial t} - B_n(t) \right) u_n(t) = f(t), \quad u_n(0) = u_{0n},$$

(9)

where $B_n(t) = A^+_n(t)$.

The existence of a solution of (8), with appropriate conditions on $u_0$ and $f$, can be obtained from the classical result [3]. Let $\hat{C}^{k+\alpha}(\mathcal{R}^d)$ be the completion in $\| \cdot \|_{k+\alpha}$-norm of compactly supported functions in $C^{k+\alpha}(\mathcal{R}^d)$. Let $u_0 \in \hat{C}^{2+\alpha}(\mathcal{R}^d)$ and $f \in H^{\alpha/2,\alpha}([0, \infty) \times \mathcal{R}^d)$. Then the problem (8) has a unique solution $u$ with the following properties. For each $t \geq 0$, $u(t) \in \hat{C}^{2+\alpha}(\mathcal{R}^d)$, $u'(t) \in \hat{C}^\alpha(\mathcal{R}^d)$, and for each $T > 0$, there exists $\kappa(T)$ such that $\| u(t) \|_{2+\alpha} \leq \kappa(T)(\| f \|_{\alpha/2,\alpha} + \| u_0 \|_{2+\alpha})$, $t \in [0, T]$.

The existence of the strong solutions of (8) in $C([0, \infty), L_1(\mathcal{R}^d))$ can be proved if we assume additional properties on the coefficients. Let $u_0 \in L_1(\mathcal{R}^d)$ and $f \in L_1([0, \infty) \times \mathcal{R}^d)$. It suffices to assume that the coefficients (6) differ from constant values on a compact set. Then, there exists an evolution family $U(t, s), 0 \leq s \leq t,
of positive and contraction operators, such that $u(t) = U(t, 0)u_0 + \int_0^t U(t, s)f(s)ds$ (basic steps of a proof can be found in [10]).

For the initial value problems for the column-valued functions $u_n(·)$ in (9), the following result is valid. The problems (9) are solvable in $l_1$. The bounded operator $B_n$ on $l_1$ generates an evolution family of operators $U_n(t, s), s, t \in [0, \infty)$. Solutions of the considered problems can be written as in the previous case.

In the remaining part of exposition it is assumed that (8) has a unique solution in $C([0, \infty) , L_1(\mathbb{R}^d))$, and for each $T > 0$, there exists an $n$-independent positive number $\kappa(T)$ such that the following error estimate is valid:

$$\| \phi_2(n)u(t) - u_n(t) \|_{l_1} \leq \kappa(T)h^n, t \leq T.$$ (10)

It is understood that the estimate (10) is valid for a subspace of initial states and nonhomogeneous terms $f(·)$.

4. Estimation of solutions by Monte Carlo simulation

The space discretization of an initial value problem for PDE results in a sequence of initial value problems for ODE associated with Markov jump processes. Therefore, only the basic facts of construction of relevant Markov jump processes on finite or denumerable state space will be described. All constructions must have the following structure. Let $S \subset G_n$ be a subset with the index set $\mathbb{R}$ and $\mathcal{P}(S)$ be the set of all probability densities on $S$. Let $G$ be a matrix with indices in $\mathbb{R}$, having the structure of a generator of a regular Markov process. The Markov process on $S$ with initial distribution $\nu \in \mathcal{P}(S)$ and generator $G$ is defined by $\{S, \nu, G\}$ and denoted by $t \rightarrow \xi(t)$. Its representation by means of an infinite sequence of independent standard exponential random variables and the corresponding Markov chain can directly be used for simulations.

A discretization of the initial value problem for parabolic system in terms of the resulting initial value problem for ODE has the form

$$\frac{du(t)}{dt} = B(t)u(t) + f(t),
\quad u(0) = u_0,$$ (11)

The matrix $A(t) = B(t)^+$ has an MJP-structure for each $t \geq 0$. We say that $A(·)$ is conservative if the sum of each of its rows is zero. In order to describe M.C. (Monte Carlo) method for estimating the $j$th component of the solution $u_j(t)$, it is useful to distinguish the following three cases:

• $t$-independent generator and $f = 0$. A problem with a non-conservative generator can be transformed to the problem with a conservative one by adding an abstract absorbing state $a$ to the state space $S$ and defining a new generator, $A$, for $S_a = S \cup \{a\}$. Therefore, it suffices to consider only the conservative generators. If the initial state $\nu = u_0$ of (11) is non-negative and $\| u_0 \|_1 = 1$, the object $\{S_a, \nu, A\}$ defines the Markov jump process to be used for simulations. The estimator of $u_j(t)$
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is defined by the statistics:

$$s_N(u_j(t)) = \frac{1}{N} \sum_{m=1}^{N} \chi_j(\xi_m(t)),$$

(12)

where $\chi_j$ is the function on $S$ defined by $\chi_j(x_i) = \delta_{ij}$, $x_j \in G_n$, and $\xi_m(\cdot)$ are independent Markov processes defined by $\{ S, \nu, A \}$. The first two statistical moments of (12) are $E[s_N(u_j(t))] = u_j(t)$ and $\text{Var}[s_N(u_j(t))] = N^{-1}u_j(t)(1 - u_j(t))$, so that the following upper bound is valid:

$$\text{Var}[s_N(u_j(t))] \leq \frac{1}{4N}.$$  

(13)

If the initial state $\nu = u_0$ is non-negative and $\| u_0 \|_1 \neq 1$, an appropriate scaling of the initial state must be performed first. If the components of $u_0$ have both signs, a decomposition $u_0 = u_+ - u_-$ can be made, where $u_{\pm}$ have non-negative components. The previous procedure can now be applied to the two processes defined by $\{ S, u_+, A \}$ and $\{ S, u_-, A \}$, and the results must be subtracted.

In the case of $u_0 \in \mathcal{P}(S)$, the error of the estimate consists of two parts. The first is caused by a space discretization and given by (10), and the second is the statistical error defined by (13). In other cases the scaling must be included in (13).

- $t$-independent generator and $u_0 = 0$. Let the column-valued function $f(\cdot)$ have non-trivial elements only for indices $i \in \mathbb{I} \subset \mathbb{R}$. The problem (11) has the solution:

$$u_j(t) = \sum_{i \in \mathbb{I}} \int_0^t w_{ij}(t - s)f_i(s)ds, j \in \mathbb{R},$$

where $W(t) = \exp(B^+ t)$.

Let the following objects be given: 1) Markov jump processes $\xi_i, i \in \mathbb{I}$, on $S$, starting at $x_i \in S$; 2) continuous random variables $\tau_i, i \in \mathbb{I}$, with values in $[0, T]$; and defined by means of densities $t \to p_i(t)$; 3) functions $g_i : [0, T] \to \mathbb{R}$, $g_i = f_i/p_i$, and the function $\chi_j$ defined as in (12). Then, the random variable

$$\eta_j(T) = \sum_{i \in \mathbb{I}} \chi_j(\xi_i(T - \tau_i))g(\tau_i),$$

has the moments $u_j(T) = E[\eta_j(T)]$ and:

$$\text{Var}[\eta_j(T)] = \sum_{i \in \mathbb{I}} \int_0^T u_{ij}(T - s)g_i(s)^2p_i(s)ds - E[\eta_j(T)]^2.$$

The following steps give rise to a simulation of $\eta_j$: for each $i \in \mathbb{I}$ a simulation of $\tau_i$ results in $t_i$; for each $i \in \mathbb{I}$ a simulation of $\xi_i$ on $[0, T - t_i)$ results in $x_i$; then, a simulation of $\eta_j(T)$ equals

$$\eta_j(T) = \sum_{i \in \mathbb{I}} \chi_j(x_i)f_i(\tau_i).$$
The function \( a \) is denoted by \( \tau \) to define the stopping time \( \xi \) path of \( \theta \) by it in the same way as in Section 3, the generator is non-conservative and can be represented in the following form:

\[
A(t) = \begin{bmatrix} 0_{11} & O_{13} \\ O_{31} & G \end{bmatrix} + \begin{bmatrix} O_{11} & O_{13} \\ R(t) & H(t) \end{bmatrix},
\]

where \( O_{11} = 0, O_{13} \) is the zero row of \( \text{card}(\mathcal{S}) \) - length, \( O_{31} \) the zero column of the same length, \( G \) a conservative generator, \( H(\cdot) \) a diagonal matrix-valued function with elements \( h_{ii}(t) = c(t, x_i) \), and \( R(\cdot) \) a column-valued function, \( r_{ii}(t) = -h_{ii}(t), i \in \mathcal{S} \). The abstract state associated with the first index of the matrix \( A(\cdot) \) is denoted by \( a \). We assume that \( c(t, x) < -c_0 \), where \( c_0 \) is a positive number.

Let \( \nu \in \mathcal{P}(\mathcal{S}) \) and \( \{S, \nu, A(\cdot)\} \) define a non-homogeneous Markov jump process \( \xi(\cdot) \), and \( \{S, \nu, G\} \) a homogeneous Markov jump process \( \theta(\cdot) \). In addition, we need to define the stopping time \( \tau \). Let \( \omega \rightarrow \theta(\omega, t) \) be a sample path of the process \( \theta(\cdot) \). The function

\[
F_{\omega}(t) = 1 - \exp \left( \int_0^t c(s, \theta(\omega, s))ds \right)
\]

is a distribution of a random variable, denoted by \( \tau_{\omega} \). It gives rise to the following stopping time \( \tau \), of the process \( \xi(\cdot) \):

\[
\xi(t) = \begin{cases} 
\theta(t) & \text{for } t < \tau, \\
a & \text{for } t \geq \tau.
\end{cases}
\]

A simulation of a sample path of \( \xi(\cdot) \) consists of two steps. Let a simulated sample path of \( \theta(\cdot) \) be \( \theta(\omega, \cdot) \), and \( T \) be a simulated value of \( \tau_{\omega} \). Then \( \xi(\omega, t) = \theta(\omega, t) \) for \( t < T \), and \( \xi(\omega, t) = a \) for \( t \geq T \).

5. Parabolic system with Dirichlet boundary condition

Let \( D \subset \mathbb{R}^d \) be an open, bounded and connected set with the boundary \( \partial D \) consisting of regular points with respect to the Laplacian. The so-called cone criterion is one amongst familiar criteria ensuring the regularity of a boundary. Then, the regularity with respect to elliptic operators is considered in [7]. We consider the parabolic system on \( D \) with the homogeneous Dirichlet boundary conditions on \( D \):

\[
\begin{align*}
\left( \frac{\partial}{\partial t} - B(t, x) \right) u(t, x) &= f(t, x), & x \in D, \\
\frac{\partial u(t, x)}{\partial n} &= 0, & x \in \partial D, \\
u(0, x) &= u_0(x),
\end{align*}
\]

where \( u_0 \in L_1(D) \) and \( f \in L_1((0, \infty), D) \). The problem can be solved if the conditions (6) are valid. In the case of \( t \)-independent coefficients, a contraction semigroup \( t \rightarrow U_D(t) \) on \( L_1(D) \) can be constructed, and solutions can be represented by it in the same way as in Section 3. An extension to a \( t \)-depending coefficient \( c \) is straightforward [10].

Solutions of (8) and (14) can be related by means of the diffusion process generated by the generator \( A_{\text{diff}} = \sum_{ij} a_{ij} \partial_i \partial_j + \sum_j b_j \partial_j \) [1] in terms of the first exit time from \( D \).
Let $\xi(\cdot)$ be a diffusion process generated by $A_{\text{dif}}$ and $\xi_n(\cdot)$ the family of Markov jump processes from the previous section. The following result is valid.

**Theorem 3.** Let (6) be valid and coefficients of $A_{\text{dif}}$ be different from constant values on a compact set. Then, the sequence of processes $\xi_n$ converges weakly to the process $\xi$.

This result enables us to estimate solutions of (14) by simulating the first exit time of Markov jump processes $\xi_n$ from the transient set $D \cap G_n$.

6. Conclusion

The initial value problem for the parabolic system on $\mathbb{R}^d$ and on a bounded domain with Dirichlet condition is considered. To estimate the solution by using an M.C. simulation of associated Markov jump processes seems to be very attractive. Its efficiency is demonstrated in [9]. The error of the estimate consists of two parts, one being the standard error of numerical analysis of PDE, and the other a statistical one.

References


