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ON THE INVERSE LIMITS OF T_0-ALEXANDROFF SPACES

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Abstract. We show that if X is a locally compact, paracompact and Hausdorff space, then X can be realised as the subspace of all maximal points of the inverse limit of an inverse system of partial orders with an appropriate topology (equivalently T_0-Alexandroff spaces). Then, the space X is homeomorphic to a deformation retract of that limit. Moreover, we extend results obtained by Clader and Thibault and show that if K is a simplicial complex, then its realisation $|K|$ can be obtained as the subspace of all maximals of the limit of an inverse system of T_0-Alexandroff spaces such that each of them is weakly homotopy equivalent to $|K|$. Moreover, if K is locally-finite-dimensional and $|K|$ is considered with the metric topology, then this inverse system can be replaced by an inverse sequence.

1. Introduction

Very often one describes a given class of spaces by means of other spaces. One way to do this is by inverse limits or more generally by limits of diagrams. For example, it is known ([13, p. 308]) that a topological space is a compact Hausdorff space if and only if it is homeomorphic to the inverse limit of an inverse system of compact polyhedra. Another known example ([18, Proposition 1.1.7]) says that every compact, Hausdorff and *totally disconnected space* (i.e., connected components are only one point sets) is homeomorphic to the inverse limit of an inverse system of discrete spaces. Discrete spaces are a particular case of Alexandroff spaces. So, it seems natural to consider inverse systems of these spaces as a generalization of the discrete case. Because of connections of these spaces with partial orders (more details in Section 2), we have a partial order on the product of T_0-Alexandroff spaces and therefore

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on the inverse limit as well. Moreover, T_0-Alexandroff spaces and simplicial complexes are closely related. This suggests that the inverse systems of T_0-Alexandroff spaces and their limits can be used to investigate the inverse limits of inverse systems of simplicial complexes, in particular, the compact Hausdorff spaces. We use the technique of the open covers and as it is well known, with every open cover one can associate a simplicial complex called its *nerve*. Such constructions like the nerve and others (like the Vietoris-Rips complex) are being used in the computational topology. More details can be found in [4]. All of this makes reasonable to consider the inverse systems of Alexandroff spaces and their inverse limits.

T_0-Alexandroff spaces seem to be not very interesting at the first glance. For this reason they were not investigated for a long time. This situation changed in the sixties when McCord and Stong published their papers [9] and [16] about finite topological spaces, their properties and relations between them and simplicial complexes. In particular, with every finite simplicial complex, we can associate a finite space that is weakly homotopy equivalent to its realisation and *vice versa*. This result is also true for arbitrary simplicial complexes, see [1, Chapter 1], [8, Theorem 5.1.2] and [9] for more details.

E. Clader ([2]) showed that the realisation $|K|$ of a finite simplicial complex K is homotopy equivalent to the inverse limit of an inverse sequence of finite T_0-spaces that are weakly homotopy equivalent to $|K|$. The same result was noticed by Wofsey in [20, p. 25]. Thibault ([17, Section 2.4]) discussed Clader's proof and extended her result to the class of locally finite simplicial complexes. However, in this case the inverse sequence of finite spaces is replaced by a similar sequence of T_0-Alexandroff spaces that are locally finite and also weakly homotopy equivalent to $|K|$. Also, Flachsmeyer ([6], [13, p. 415]) showed that a space X is a compact T_1-space if and only if it is homeomorphic to the subspace of all minimal points of the inverse limit of an inverse system of finite T_0-spaces[1]. Moreover, with every T_1-space X, he associated ([13, p. 416]) an inverse system of finite spaces where the subspace of all minimal points of its limit is homeomorphic to the *Wallman's compactification* of X. See [13, Chapter IV.2] for more details.

The aim of this paper is to give an extension of Clader and Thibault's results in two different ways. First, we show that if X is a locally compact, paracompact and Hausdorff space, then it can be realised as a strong deformation retract of the inverse limit of an inverse system of T_0-Alexandroff spaces which is the subspace of all maximal points. However, we do not get a sequence of T_0-Alexandroff spaces but an inverse system of such spaces indexed by the set of all locally finite open covers of the given space X. We use a technique similar to the Flachsmeyer's one.

[1] The order considered by Flachsmeyer is dual to the order considered in this paper.

Next, we consider the class of simplicial complexes. In this case, the system is indexed by all subdivisions of the given complex K. Here, every T_0-Alexandroff space is weakly homotopy equivalent to the realisation $|K|$. If K is locally-finite-dimensional and $|K|$ is considered with the metric topology, the system can be replaced by an inverse sequence. In this case, the result is a straightforward generalization of the one obtained by Clader and Thibault.

The paper consists of four sections. Except Introduction, Section 2 contains all necessary definitions and results about Alexandroff spaces and simplicial complexes. The case of locally compact, paracompact and Hausdorff spaces is considered in Section 3. The main result of this section is stated in Theorem 3.4 and Proposition 3.5 which state that every locally compact, paracompact and Hausdorff space can be realised as the subspace of all maximal points of the inverse limit of an inverse system of T_0-Alexandroff spaces (equivalently partial orders with an appropriate topology) that is a strong deformation retract of this limit. Moreover, Theorem 3.6 can be treated as an analog of the theorem ([13, p. 310]) by H. Freudenthal concerning the inverse limits of inverse sequences of compact metric spaces. The last Section 4 concerns the realisations of simplicial complexes. It is shown that the class of spaces that can be realised as in Section 3 is larger and contains realisations of all simplicial complexes. Moreover, the system can be replaced by the sequence constructed in [2] and [17] provided a simplicial complex K is locally-finite-dimensional and its realisation is considered with the metric topology.

2. Preliminaries

In this section we recall some properties of Alexandroff spaces and simplicial complexes that are needed in the sequel. It is primarily based on [1], [9], [16] and [17].

A topological space is called an *Alexandroff space* if the intersection of its arbitrary family of open sets is open. If (X, τ_X) is an Alexandroff space and $x \in X$, then $U_x = \bigcap_{U \in \tau_X, x \in U} U$ is the minimal open set containing x. The family $\{U_x\}_{x \in X}$ forms the minimal basis for the topology τ_X. A topology τ_X on a set X is called *Alexandroff* provided (X, τ_X) is an Alexandroff space.

Given an Alexandroff space (X, τ_X), we define the relation $\leq_{\tau_X}$, where $x \leq_{\tau_X} y$ provided $x \in U_y$ (equivalently $U_x \subseteq U_y$). This relation is reflexive and transitive. It is a partial order if and only if X is a T_0-space. Similarly, with every transitive and reflexive relation $\leq \subseteq X \times X$, we can associate a topology $\tau_\leq$ on X with a basis given by the sets $U_x = \{y \in X : y \leq x\}$. The pair $(X, \tau_\leq)$ is a T_0-space if and only if $\leq$ is a partial order. An Alexandroff space that is a T_0-space is called for short an *A-space.*

With an A-space X, we can associate an abstract simplicial complex $K(X)$, where the simplices are nonempty chains in X. By [9, Theorem 2], the space X is weakly homotopy equivalent to the realisation $|K(X)|$. Similarly, with every simplicial complex K, we can associate an A-space $\chi(K)$, where elements of $\chi(K)$ are simplices of K and the ordering is by the inclusion relation. If X is an A-space, then the space $\chi(K(X))$ is called the *barycentric subdivision* of the space X and is denoted by $X^{'}$. Inductively, we can form the n-th barycentric subdivision $X^{(n)} = (X^{(n-1)})^{'}$. Elements of the space $X^{'}$ are nonempty chains of X and the map $\sup : X^{'} \to X$ sending a chain $C \in X^{'}$ to its maximal element is a weak homotopy equivalence. More details can be found in [1], [8] and [17].

A map $f : X \to Y$ between A-spaces is continuous if and only if it is order preserving with respect to the orders associated with the topologies on X and Y. Moreover, the categories of partial orders and T_0-Alexandroff spaces are isomorphic. Since elements of X can be identified with vertices of $K(X)$, the map $f : X \to Y$ yields a simplicial map $K(f) : K(X) \to K(Y)$. Similarly, with a simplicial map $f : K \to L$, we can associate a continuous map $\chi(f) : \chi(K) \to \chi(L)$.

Reversing the order of an A-space X, we get the space X^{op}. These spaces have the same underlying set. Open sets in X correspond to closed sets of X^{op} and *vice versa*. In particular, the set $U_x \subseteq X$ is closed in X^{op}. The set $F_x = \{y \in X : y \geq x\}$ is open in X^{op} and therefore closed in X. If $f : X \to Y$ is a continuous function, then we have a function $f^{op} : X^{op} \to Y^{op}$ which is the same as f as the function of sets. Obviously, f is continuous if and only if f^{op} is so.

Let K be a simplicial complex. Denote by $V(K)$ the set of all its vertices. The realisation $|K|$ ([15, p. 110]) of K is the set of functions $x : V(K) \to [0, 1]$ such that for every $x \in |K|$ the set $\{v \in V(K) : x(v) \neq 0\}$ is a simplex of K. The *geometric interior* of a simplex $\sigma \in K$ is $\mathrm{Int}|\sigma| = \{x \in |K| : x(v) \neq 0$ if and only if $v \in \sigma\}$. The *open star* $\mathrm{St}\sigma$ of a simplex $\sigma \in K$ is the union of all geometric interiors $\mathrm{Int}|\tau|$, where $\sigma \subseteq \tau$. Clearly, $\mathrm{St}\sigma$ is the set of such $x \in |K|$ that $x(v) \neq 0$ for every $v \in \sigma$. Every point $x \in |K|$ lies in the geometric interior of a unique simplex $\mathrm{carr}(x)$ called the *carrier* of x. Notice that the geometric interiors of different simplices are disjoint. See [5, Chapter 2] for more details.

If K is a simplicial complex, the space $|K|$ is usually endowed with a topology in two different ways. The first one is called the *weak topology*, where a set $U \subseteq |K|$ is open if the intersection $U \cap |\sigma|$ is open in the realisation $|\sigma|$ for every simplex $\sigma \in K$. The second way is the *metric topology* given by the metric $d(x, y) = (\sum_{v \in V(K)} (x(v) - y(v))^2)^{\frac{1}{2}}$ for $x, y \in |K|$. The diameter of every positive-dimensional simplex is $\sqrt{2}$ with respect to this metric. See for example [15, Chapter 3] for more details. In view of [15, Theorem 8, p. 119], these two topologies coincide if and only if K is locally finite. In particular, they coincide when K is just a simplex. If K is not locally finite, then the metric topology is weaker than the weak topology. Write $|K|$ for the realisation of K with the weak topology and $|K|_m$ with the metric topology respectively. In view of [7, Proposition 3.3.7], the identity function $\mathrm{id}_{|K|} : |K| \to |K|_m$ is a homotopy equivalence.

3. Locally compact, paracompact and Hausdorff spaces

Let K be a simplicial complex and $K^{(n)}$ its n-th barycentric subdivision. In [2] and [17] elements of the space $X_n = \chi(K^{(n)})^{op}$ are the simplices of $K^{(n)}$ and the ordering is by the reversed inclusion relation. The functions $f_{n,m} : X_m \to X_n$ for $m \geq n$ sending an element from X_m to its carrier in $K^{(n)}$ form an inverse sequence $\{X,\mathbb{N}, f_{n,m}\}$, where $\mathbb{N}$ is the set of natural numbers.

With the complex $K^{(n)}$, we can associate an open cover $\{\mathrm{St}\sigma : \sigma \in K^{(n)}\}$ of the space $|K|$ given by open stars of simplices of $K^{(n)}$. If $\sigma \subseteq \tau$ for $\sigma, \tau \in K^{(n)}$, then $\mathrm{St}\tau \subseteq \mathrm{St}\sigma$. Hence, the space of open stars ordered by the inclusion is obviously homeomorphic to X_n. Second important point is that the space $\chi(K)^{op}$ is the quotient space of $|K|$ with respect to the relation given by: $x \sim y$ provided $\mathrm{carr}(x) = \mathrm{carr}(y)$ in $K^{(n)}$ (compare [20, p. 13]). Indeed, classes of $\sim$ are the geometric interiors of simplices of K. Since, a simplex determines its geometric interior and *vice versa*, we can identify elements of X_n with those of $|K|/\sim$. Let $p : |K| \to |K|/\sim$ be the quotient map. Notice that a set $U \subseteq |K|/\sim$ is open if and only if $p^{-1}(U) \cap |\sigma|$ is open in $|\sigma|$ for every $\sigma \in K$. Equivalently, it is open if and only if it contains the geometric interior of τ whenever $\sigma \subseteq \tau$ and $\mathrm{Int}|\sigma| \in U$. Hence, the spaces $|K|/\sim$ and $\chi(K)^{op}$ are homeomorphic.

Similar approach was used earlier by Flachsmeyer (compare [13, p. 416]). He associated a finite T_0-space with every finite open cover of a given space X.

Recall that a space X is called *locally compact* if every $x \in X$ has an open neighbourhood U such that its closure $\overline{U}$ is compact. An open cover $\mathcal{U}$ of a space X is *locally finite* if every point of X has an open neighbourhood that meets finitely many elements of $\mathcal{U}$. An open cover $\mathcal{U}$ of a space X is a *refinment* of an open cover $\mathcal{V}$ if every element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$. A space X is called *paracompact* if every open cover of X has an open refinment that is locally finite. Recall also, that every metric space is paracompact and every paracompact Hausdorff space is normal.

Let X be a locally compact, paracompact and Hausdorff space and let I_X be the set of all its locally finite open covers. Obviously, $I_X \neq \emptyset$ and for every finite family $U_1, \ldots, U_n$ of open subsets of X, there exists a locally finite, open cover that contains U_i for $i = 1, \ldots, n$. The set I_X is directed by the inclusion relation. It should not be confusing that an element $\mathcal{B}_\alpha$ of I_X will be identified with the supscript α. Let $\mathcal{B}_\alpha \in I_X$ and $x \in X$. The local finiteness implies that the set

$$U_x^\alpha = \bigcap_{U\in\mathcal{B}_\alpha,\ x\in U} U$$

is open for every $x \in X$. Define the relation $\sim_\alpha \subseteq X \times X$ such that $x \sim_\alpha y$ provided $U_x^\alpha = U_y^\alpha$. This is an equivalence relation. Let X_α be the quotient set of X with respect to the relation $\sim_\alpha$ and $p_\alpha : X \to X_\alpha$ the quotient function. Since the cover $\mathcal{B}_\alpha$ is locally finite, every class E of $\sim_\alpha$ is contained in finitely many elements of $\mathcal{B}_\alpha$. So, the intersection U_E^α of all open sets from $\mathcal{B}_\alpha$ containing E is open in X. Now, define the relation $\geq_\alpha \subseteq X_\alpha \times X_\alpha$ such that $E_1 \geq_\alpha E_2$ provided $E_2 \subseteq U_{E_1}^\alpha$ (equivalently $U_{E_2}^\alpha \subseteq U_{E_1}^\alpha$) for E_1, $E_2 \in X_\alpha$. It is obviously a partial order. As one can easily verify, $E_1 \geq_\alpha E_2$ if and only if every set from $\mathcal{B}_\alpha$ containing E_1 contains also E_2. Hence, we have an Alexandroff topology on the set X_α given by $\geq_\alpha$ (compare [13, p. 416]).

LEMMA 3.1. *Let $\mathcal{B}_\alpha$ and X be as above. Then, the quotient topology on X_α is finer than the topology given by $\geq_\alpha$.*

PROOF. Let U_E be the minimal open neighbourhood of E in the topology given by $\geq_\alpha$. By the definition of $\geq_\alpha$, the set U_E consists of all classes that are contained in U_E^α. So, $p_\alpha^{-1}(U_E) = U_E^\alpha$ and every open set from A-topology on X_α is open in the quotient topology. ∎

Lemma 3.1 shows that the quotient map $p_\alpha : X \to X_\alpha$ is continuous if X_α is considered with the A-topology. Considering X_α as a space, we mean that the topology is given by $\geq_\alpha$. In particular, it is an A-space. If $\mathcal{B}_\alpha$ were not locally finite, then Lemma 3.1 would not hold. For example, if $\mathcal{B}_\alpha$ consists of all nonempty open subsets of X, then every class is a one point set and the classes are not comparable with respect to $\geq_\alpha$. So, the A-space X_α is discrete. In particular, the function p_α does not have to be continuous.

Let $\mathcal{B}_\alpha \subseteq \mathcal{B}_\beta$. By the definitions of $\sim_\alpha$ and $\sim_\beta$, every class $E \in X_\beta$ is contained in a unique class $F \in X_\alpha$ that will be called the *carrier* of E in X_α and denoted by $\mathrm{carr}_\alpha(E)$. This yields a function $f_{\alpha,\beta} : X_\beta \to X_\alpha$, $E \mapsto \mathrm{carr}_\alpha(E)$. It is easy to see that $f_{\alpha,\beta} \circ p_\beta = p_\alpha$. Since every set from $\mathcal{B}_\alpha$ belongs to $\mathcal{B}_\beta$, we see that the function $f_{\alpha,\beta}$ is continuous by the definition of $\geq_\alpha$. So, we have an inverse system $\{X_\alpha, I_X, f_{\alpha,\beta}\}$. The space X and the maps p_α form a cone over this system. In particular, the inverse limit $X_\infty = \varprojlim X_\alpha$ is nonempty. The inverse sequence considered in [2] and [17] can be obtained in the similar fashion. Let K be a locally finite simplicial complex. Denote by $\mathcal{U}_n$ the cover of $|K|$ given by open stars of simplices from $K^{(n)}$ and let $\mathcal{V}_n = \bigcup_{i=0}^{n} \mathcal{U}_i$. Then, as it is easy to see the classes of $\sim_{\mathcal{V}_n}$ are geometric interiors of simplices from $K^{(n)}$.

The maps p_α give rise to a continuous map $p : X \to X_\infty$, $x \mapsto (p_\alpha(x))_{\alpha\in I_X}$. The partial orders on the sets X_α give a partial order $\geq$ on the set X_∞, where $A \geq B$ provided $A_\alpha \geq B_\alpha$ for every $\alpha \in I_X$. Notice that, if the inverse limit of A-spaces has two threads that are comparable, then it cannot be even a T_1-space. This is one of the reasons why we focus our attention on the subspace of all maximal elements.

Let $A = (A_\alpha)_{\alpha \in I_X} \in X_\infty$. For every finite subset $\{\alpha_1, \dots, \alpha_n\} \subset I_X$ there exists such $\beta \in I_X$ that $\beta \geq \alpha_i$ for every $i = 1, \dots, n$. By the definition of maps $f_{\alpha,\beta}$, the class A_β is contained in every A_{α_i} for $i = 1, \dots, n$. In other words, the coordinates A_α of the thread A have the so called *finite intersection property*. Obviously, the same property have their closures.

Since X is locally compact, for every $x \in X$ there exists an open set $V_x \ni x$ so that the closure $\overline{V}_x$ is compact. Hence, we have an open cover $\{V_x\}_{x \in X}$ of X such that every $\overline{V}_x$ is compact. By paracompactness, the space X has an open cover $\mathcal{B}_\beta$ which is locally finite and the closure of every member is compact. The intersection of all $\overline{A}_\alpha$ is a subset of a compact and closed set $V \subset X$. Sets $\overline{A}_\alpha \cap V$ are closed in V. In view of [3, Theorem 3.1.1], the intersection $F_A = \bigcap\limits_{\alpha \in I_X} \overline{A}_\alpha$ is nonempty for every $(A_\alpha)_{\alpha \in I_X} \in X_\infty$. Assume that there exist $x, y \in F_A$ with $x \neq y$. Since X is normal, there exist two open neighbourhoods U_1, U_2 of x and y respectively with $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. Consider $\mathcal{B}_\beta \in I_X$ such that $U_1, U_2 \in \mathcal{B}_\beta$. The class of $\sim_\beta$ containing x is a subset of U_1 and the one containing y is a subset of U_2. Hence, $\overline{A}_\beta$ may contain at most one element of $\{x, y\}$. So, the set F_A contains a unique point g_A. This allows one to define the function $G : X_\infty \to X$, $A \mapsto g_A$. This is an analog of the function G defined in [2] and [17].

LEMMA 3.2. *If $A = (A_\alpha)_{\alpha \in I_X} \in X_\infty$, then $p(g_A) \geq A$.*

PROOF. Assume to the contrary, that it is not true that $p(g_A) \geq A$. Equivalently, there exists $\mathcal{B}_\alpha \in I_X$ and a set $U \in \mathcal{B}_\alpha$ that contains $p_\alpha(g_A)$ but does not contain A_α. Hence, $\overline{A}_\alpha \subset X \setminus U$. Because X is normal, there exist open sets $V_1, V_2 \subset X$ such that $g_A \in V_1 \subset \overline{V}_1 \subset U$, $\overline{A}_\alpha \subset V_2$ and $\overline{V}_1 \cap \overline{V}_2 = \emptyset$. So, $g_A \notin \overline{A}_\alpha$. This contradiction completes the proof. ∎

So, $p(X)$ is precisely the subspace of all maximal elements of X_∞ with respect to the relation $\geq$. From Lemma 3.2, it follows that for a thread $A \in X_\infty$ the element $p(g_A)$ is greater or equal to A. This implies that X_∞ has an additional property: every thread is comparable to exactly one maximal element of X_∞. If $p(x) \geq A$, then $p_\alpha(x) \geq A_\alpha$ for every $\alpha \in I_X$. In particular, every open neighbourhood of $p(g_A)$ in X_∞ contains A. Moreover, if $A \geq B$ for $A, B \in X_\infty$, then $p(g_A) \geq A \geq B$ by Lemma 3.2. Hence, $p(g_A) = p(g_B)$ is the unique maximal point of X_∞ that B is comparable to.

LEMMA 3.3. *The function $G : X_\infty \to X$ defined above is continuous.*

PROOF. Let $U \subseteq X$ and $A = (A_\alpha)_{\alpha \in I_X} \in G^{-1}(U)$. This means that $g_A \in U$. Since X is normal, there exists an open set $V \subset X$ with $g_A \in$

$V \subset \overline{V} \subset U$. Let $C = p(g_A)$. From Lemma 3.2, we have $C \geq A$ and hence $g_C = g_A$. Now, consider the open cover $\mathcal{B}_\beta = \{V, X \setminus \{g_A\}\} \in I_X$. The only open set from $\mathcal{B}_\beta$ containing g_A is V. Let $\pi_\beta : X_\infty \to X_\beta$ be the canonical projection. Since $C_\beta \geq_\beta A_\beta$, it is obvious that $A \in \pi_\beta^{-1}(U_{C_\beta})$. We show that the set $\pi_\beta^{-1}(U_{C_\beta})$ is an open neighbourhood of A in $G^{-1}(U)$. Indeed, let $D = (D_\alpha)_{\alpha \in I_X} \in \pi_\beta^{-1}(U_{C_\beta})$. This means that $C_\beta \geq_\beta D_\beta$ in X_β and $U^\beta_{D_\beta} \subseteq U^\beta_{C_\beta}$. By the choice of C and $\mathcal{B}_\beta$, we have $C_\beta = \{g_A\}$ and $U^\beta_{C_\beta} = V$. Therefore, $g_D \in \overline{V} \subseteq U$ and the proof is complete. ∎

Let K be a locally finite simplicial complex. In the case of the inverse sequence considered in [17, Theorem 2.4.20] it was shown that the analogs of p and G form a homotopy equivalence of $|K|$ and X_∞. This result is also true if we replace the system from [17] by the one defined above.

THEOREM 3.4. (Generalization of [17, Theorem 2.4.20]) *Let X be a locally compact, paracompact and Hausdorff space. The maps $p : X \to X_\infty$ and $G : X_\infty \to X$ defined above are homotopy equivalences.*

PROOF. The proof follows *mutatis mutandis* the one of [17, Theorem 2.4.20]. Define the relation $\sim \subseteq X_\infty \times X_\infty$ where $A \sim B$ provided A and B are comparable with respect to $\geq$ with the same element of $p(X)$. By Lemma 3.2, $A \sim B$ if and only if $G(A) = G(B)$. Since for every $A \in X_\infty$ there exists a unique maximal element $\max(A) \in p(X)$ with $\max(A) \geq A$, the relation $\sim$ is an equivalence. Obviously, $\max(A) = pG(A)$. Let E be an equivalence class. Define, as in [17] the homotopy $h_E : E \times [0,1] \to E$, where

$$h_E(A,t) = \begin{cases} A & \text{if } t \in [0,1), \\ pG(A) & \text{if } t = 1 \end{cases}$$

for $(A,t) \in E \times [0,1]$. Since every open neighbourhood of $pG(A)$ contains A, this function is continuous. In particular, every equivalence class is contractible. All maps h_E give rise to a function $F : X_\infty \times [0,1] \to X_\infty$. We only need to show that F is continuous. Let $U \subseteq X_\infty$ be an open set. Note that $h_E^{-1}(U \cap E) = (U \cap E) \times [0,1)$ if $U \cap E$ does not contain $p(x)$ for some $x \in X$ and it is $E \times [0,1]$ otherwise. Hence, $F^{-1}(U) = (U \times [0,1]) \setminus (V \times \{1\})$, where $V = \{A \in U : pG(A) \notin U\}$. Since p and G are continuous, it means that $V = U \setminus (pG)^{-1}(U)$ which is closed in U and the proof is complete. ∎

In fact, the homotopy F constructed in the proof of Theorem 3.4 from the homotopies h_E is a deformation retraction of X_∞ onto $p(X)$.

In [17, Proposition 2.4.21] it is shown that for a locally finite simplicial complex K, the restriction $\tilde{G}|_{p(|K|)} : p(|K|) \to |K|$ for the analog $\tilde{G}$ of the map G, is a homeomorphism. We see that the same is true for the system defined above. Note that, the map $G|_{p(X)} : p(X) \to X$ is bijective. If $U \subseteq p(X)$ is open, then there exists an open set $V \subseteq X_\infty$ with $U = V \cap p(X)$. But every open neighbourhood of a point $p(x)$ for $x \in X$ contains every other point $A \leq p(x)$. So, $G|_{p(X)}(U) = p^{-1}(V)$. Therefore, we can state:

PROPOSITION 3.5. (Generalization of [17, Proposition 2.4.41]) *Let X be a locally compact, paracompact and Hausdorff space. Then, the space X is homeomorphic to the subspace of all maximal points of the inverse limit of the system $\{X_\alpha, I_X, f_{\alpha,\beta}\}$ defined above.*

In fact, like in [2] and [17] the proofs show that the space $p(X)$ is a strong deformation retract of X_∞.

The proof of Theorem 3.4 shows that if an arbitrary inverse system of A-spaces $\{X_\alpha, J, f_{\alpha,\beta}\}$ with the nonempty limit has the properties:

(P1) every element of $\varprojlim X_\alpha$ is comparable to exactly one element of the set $\max(\varprojlim X_\alpha) \subseteq \varprojlim X_\alpha$ of maximal elements;

(P2) the function $\max : \varprojlim X_\alpha \to \max(\varprojlim X_\alpha) \subset \varprojlim X_\alpha$ that maps an element A of $\varprojlim X_\alpha$ to the corresponding maximal element is continuous,

then, the set $\max(\varprojlim X_\alpha)$ is a strong deformation retract of the limit.

Unlike the system defined in [2] and [17], the system $\{X_\alpha, I_X, f_{\alpha,\beta}\}$ is not a sequence. Moreover, the spaces of this system do not have to be weakly homotopy equivalent to X.

In [13, p. 310] it is shown that X is a compact metric space if and only if it is homeomorphic to the inverse limit of an inverse sequence of compact polyhedra. This is a theorem by H. Freudenthal. One implication can be stated similarly in the language of A-spaces.

THEOREM 3.6. *If X is a locally compact and metric space then, it can be realised as the subspace of all maximal points of the inverse limit of an inverse sequence of A-spaces that is a strong deformation retract of the inverse limit.*

PROOF. Let $\mathcal{U}_n$ be an open cover of X given by the open balls $B(x, \frac{1}{n})$. Since every metric space is paracompact, there exists a locally finite open refinment $\mathcal{W}_n$ of $\mathcal{U}_n$. Every set from $\mathcal{W}_n$ has diameter not greater than $\frac{2}{n}$. Also, there exists an open cover $\mathcal{W}_0$ of X which is locally finite and $\overline{U}$ is compact for every $U \in \mathcal{W}_0$. Let $\mathcal{B}_n = \bigcup\limits_{i=0}^{n} \mathcal{W}_i$. Obviously, we have $\mathcal{B}_n \in I_X$ for every $n \in \mathbb{N}$ and $\mathcal{B}_n \subseteq \mathcal{B}_m$ for $m \geq n$. Hence, we have the spaces X_n and maps $p_n : X \to X_n$ and $f_{n,m} : X_m \to X_n$ for $m \geq n$. This gives rise to an inverse sequence $\{X_n, \mathbb{N}, f_{n,m}\}$. Since the space X and maps p_n form a cone over that system, we see that $\varprojlim X_n \neq \emptyset$. The space X is locally compact. So, we can define the function $G : \varprojlim X_n \to X$ that maps $A = (A_n)_{n\in\mathbb{N}}$ to the unique point $g_A = \bigcap\limits_{n=0}^{\infty} \overline{A}_n$ as before. For every point $x \in X$ and its open neighbourhood $U \subseteq X$, there exists $n \in \mathbb{N}$ such that there exists an open set from $\mathcal{B}_n$ containing x such that its closure is contained in U. By similar arguments as in the proof of Lemma 3.3, the function G is continuous.

Repeating the proofs of Theorem 3.4 and Proposition 3.5, we get the desired result. ∎

The converse is not true in general. Take an A-space X with more than one maximal element that is not discrete. Assume that $X_n = X$ for every $n \in \mathbb{N}$ and $f_{n,m} = \mathrm{id}_X$ for $m \geq n$. Then, the limit of such inverse sequence is X which is not even a T_1-space. Moreover, the subspace of all maximals may not be its strong deformation retract since the subspace of all maximal points is discrete.

4. Simplicial complexes

The aim of this section is to show that a space X can be realised as the subspace of all maximal points of the inverse limit of an inverse system of A-spaces also when X is not locally compact. This shows that the class of spaces that can be realised in this fashion is larger. We use techniques from [2] and [17].

Let K be a simplicial complex. In the inverse sequence defined in [2] and [17] every thread of the inverse limit can be identified with a nested sequence of realisations of simplices from the barycentric subdivisions of $|K|$. The function G is defined in such a way that a thread $A \in \varprojlim X_n$ is mapped to the intersection of realisations of all these simplices which is a one point set.[2] In order to prove continuity of G, one used the property that for every $x \in |K|$ and every open neighbourhood U of x, there exists a sufficiently fine barycentric subdivision of K that the star of $\mathrm{carr}(x)$ is contained in U.

If K is not locally finite, then the space $|K|$ cannot be realised as the subspace of all maximals of the inverse sequence of A-spaces constructed in [2] and [17]. It is easy to see that in that case the function G is not continuous. See [17, p. 35] for more details. Moreover, the space $|K|$ cannot be in that case even homeomorphic to a subspace of the inverse limit of an inverse sequence of A-spaces. Indeed, it is known [12, p. 14] that if K is not locally finite, then $|K|$ is neither locally compact nor first countable. But every A-space X has a basis at every $x \in X$ consisting of the minimal open set U_x. If $(x_n)_{n\in\mathbb{N}}$ is an element of the inverse limit of an inverse sequence of A-spaces then, the sets $\pi_n^{-1}(U_{x_n})$ form a basis at $(x_n)_{n\in\mathbb{N}}$.

However, $|K|$ can be realised as the subspace of all maximal elements of the inverse limit of an inverse system of A-spaces even though it is not locally compact.

If K_α is a subdivision of K, then we have an A-space $X_\alpha = \chi(K_\alpha)^{op}$. This is the space constructed in [2] and [17]. Elements of X_α are the simplices of K_α ordered by the reversed inclusion relation. The set I_K of all subdivisions of K is ordered by the subdivision relation, i.e., $K_\beta \geq K_\alpha$ provided K_β is

[2]Originally, it is considered a sequence of points. One from realisation of every simplex and its limit which is the common point of these realisations.

a subdivision of K_α. If K_β is a subdivision of K_α, then, we have a map $f_{\alpha,\beta} : X_\beta \to X_\alpha$ that maps $\sigma \in X_\beta$ to its carrier in K_α which we treat as an element of X_α. It is obviously continuous. So, we have an inverse system $\{X_\alpha, I_K, f_{\alpha,\beta}\}$. In [2] and [17], there were considered only barycentric subdivisions of K and the system was in fact an inverse sequence. It is easy to see that the function $p_\alpha : |K| \to X_\alpha$ which maps x to its carrier in K_α (treated as an element of X_α) is continuous for every $\alpha \in I_K$. These maps yield a unique continuous map $p : X \to \varprojlim X_\alpha$ and therefore $\varprojlim X_\alpha \neq \emptyset$. By the construction, every thread of $\varprojlim X_\alpha$ can be identified with a nested set of realisations of simplices indexed by the directed set I_K. Futhermore, the intersection of these realisations is a one point set and we can construct the function G.

Lemma 4.1. *Let K be a simplicial complex and $x \in |K|$. Then, there exists a subdivision $\tilde{K}$ of K which has x as a vertex.*

Proof. Let $\sigma \in K$ be the carrier of the point $x \in |K|$. In the construction of the barycentric subdivision $K^{'}$ ([15, p. 123]) of the simplicial complex K, we can replace the barycenter b_σ of σ by the point x obtaining a subdivision $\tilde{K}$ of K. This subdivision is isomorphic to the barycentric subdivision of K. However, the simplices of $K^{'}$ which have b_σ as a vertex correspond to the simplices of $\tilde{K}$ in which b_σ is replaced by x. □

The result [19, Theorem 35] states in particular that, if x is a vertex of a simplicial complex K and $U \subseteq |K|$ is an open set containing x, then there exists a subdivision K_α of K such that every simplex that has x as a vertex is contained in U. Hence, we can state the following:

Lemma 4.2. *Let K be a simplicial complex and $U \subseteq |K|$ an open neighbourhood of a point $x \in |K|$. Then, there exists a subdivision K_α of K having x as a vertex and with the property that every simplex of K_α that has x as a vertex is contained in U.*

Lemma 4.2 shows that, for every $A = (A_\alpha)_{\alpha \in I_K} \in \varprojlim X_\alpha$ and every open neighbourhood $U \subseteq |K|$ of g_A, there exists a subdivision of K such that the star of $\operatorname{carr}(g_A)$ is contained in U. Hence, using the same technique as before, we can prove the continuity of G. As before, mimicking the proofs of Theorem 3.4 and Proposition 3.5, we can state the following:

Theorem 4.3. *If K is a simplicial complex, then $|K|$ can be realised as the subspace of all maximal points of the inverse limit of an inverse system of A-spaces that is a strong deformation retract of the inverse limit.*

Moreover, as it is shown in [2], the functions $p_\alpha : |K| \to X_\alpha$ are weak homotopy equivalences. So, the system consists only of spaces that are weakly homotopy equivalent to $|K|$.

Now, we show that Clader and Thibault's result can be straightforward generalized to the class of locally-finite-dimensional simplicial complexes provided the realisation $|K|$ of a simplicial complex K is considered with the metric topology. Recall that a simplicial complex K is *locally-finite-dimensional* if for every vertex $v \in V(K)$ the star of v is finite dimensional. Consider, the sequence $K^{(n)}$ of barycentric subdivisions of K. In general, if $\tilde{K}$ is a subdivision of K, it is not true that $|K|_m = |\tilde{K}|_m$ as topological spaces. However, if $K^{'}$ is the barycentric subdivision, then $|K^{'}|_m = |K|_m$ as topological spaces, see [10]. We can construct an inverse sequence of A-spaces exactly like in [2] and [17]. If $A = (x_n)_{n\in\mathbb{N}}$ is a thread of $\varprojlim X_n = X_\infty$, then like above, we can construct the unique point

$$\{g_A\} = \bigcap_{n=1}^{\infty} |x_n|$$

and the function $G : X_\infty \to |K|_m$ that sends A to g_A. Let $p_n : |K|_m \to X_n$ be the maps as above for $n \geq 0$. In order to repeat the construction as before, we need the following result.

LEMMA 4.4. *If K is a locally-finite-dimensional simplicial complex, then the topology τ on the set $|K|$ generated by $\{\mathrm{St}\sigma : \sigma \in K^{(n)}$ for some $n \geq 0\}$ is the metric topology.*

PROOF. Let τ_m be the topology of $|K|_m$. By [7, p. 115], the open stars are open in the metric topology, so $\tau \subseteq \tau_m$. Now, let $B(x, \varepsilon) \subseteq |K|_m$ be an open ball and $y \in B(x, \varepsilon)$. Let $\delta_n = $carr$(y)$ in $K^{(n)}$. Since K is locally-finite-dimensional, the diameters of sets $\mathrm{St}(\delta_n) \subseteq |K|_m$ approach 0. So, there exists such $n \geq 0$ that $\mathrm{St}(\delta_n) \subseteq B(x, \varepsilon)$ and the proof follows. ◻

Once again, we have a bijection from the set $|K|$ onto the subspace of all maximal elements of X_∞. Repeating the process as before, we see that G restricted to the subspace of all maximals is a homeomorphism provided the topology of the set $|K|$ is generated by the open stars. In view of Lemma 4.4, this is the metric topology.

Since the space $|K|$ is not locally compact, the class of spaces that can be realised as the subspace of all maximal points of the inverse limit of an inverse system (in the sense that this subspace is a strong deformation retract od the limit) includes also other spaces besides locally compact, paracompact Hausdorff spaces. It is natural to ask which spaces can be realised in this fashion. Moreover, is it possible to obtain a homeomorphism of the whole limit with the spaces considered here? These seem to be interesting problems.

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