INTRINSIC STRONG SHAPE FOR PARACOMPACTA

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Dedicated to Sibe Mardešić

Abstract. In this paper an intrinsic definition of strong shape for paracompact topological spaces is presented. At first a coherent proximate net $f: X \rightarrow Y$ is defined, indexed by finite subsets of normal coverings of $Y$, and then there is a homotopy between two coherent proximate nets defined. A definition of composition of classes of homotopies between two coherent proximate nets $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ is given. Then it is proved that for any other choice of corresponding coverings, a function is obtained that is in the same class with the previously defined composition. The strong shape category is obtained, with paracompacta as objects, and the homotopy classes of coherent proximate nets as morphisms.

1. Introduction

The shape theory has shown to be more appropriate tool than homotopy theory when study of spaces with bad local behavior is involved ([2, 6–8, 10]). The strong shape theory is a strengthening of shape theory ([3, 6]). The first definition of strong shape for compact metric spaces is given in [9] by embedding metric compacta in Hilbert cube. In [6, 8] strong shape is described for topological spaces approximating the space by inverse system of polyhedra (ANRs). Both approaches follow the corresponding approaches in shape theory. For equivalence of different approaches for metric compacta we refer to [6].

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The intrinsic approach to shape does not use any approximation of spaces. The intrinsic definition of strong shape for compact metric spaces is presented in [11].

The definition of strong shape in [11] is based on the notion of strong proximate sequence.

The sequence of pairs \((f_n, f_{n,n+1})\) of functions \(f_n : X \to Y\) and \(f_{n,n+1} : X \times I \to Y\), is a strong proximate sequence from \(X\) to \(Y\), if there exists a cofinal sequence of finite coverings, \(V_1 \succ V_2 \succ \cdots\) of \(Y\), such that for each natural number \(n\), \(f_n : X \to Y\), is a \(V_n\)-continuous function and \(f_{n,n+1} : X \times I \to Y\) is a homotopy connecting \(V_n\)-continuous functions \(f_n : X \to Y\) and \(f_{n,n+1} : X \times I \to Y\).

We say that \((f_n, f_{n,n+1})\) is a strong proximate sequence over \((V_n)\).

If \((f_n, f_{n,n+1})\) and \((f'_n, f'_{n,n+1})\) are strong proximate sequences from \(X\) to \(Y\), then there exists a cofinal sequence of finite coverings \((V_n)\) such that \((f_n, f_{n,n+1})\) and \((f'_n, f'_{n,n+1})\) are strong proximate sequences over \((V_n)\).

In compact metric space, the existence of cofinal sequence of coverings \(V_1 \succ V_2 \succ \cdots\), allows to define strong shape theory using only homotopies of second order.

In more general case of paracompact spaces, homotopies of all orders must be considered. In [13] the construction for (strongly) paracompact spaces is described. We form all finite sets of coverings of \(Y\), \(a = \{V_0, V_1, \ldots, V_n\}\), having a maximal element (i.e. a covering that refines all other coverings of that finite set, and is not refined by any other covering of that finite set). The maximal element is denoted by \(\text{max } a\).

Finite sets of coverings with a maximal element are ordered by inclusion, and this ordering is cofinite, i.e. each \(a\) has only finite number of predecessors. This fact allows composition of coherent proximate nets to be defined, although such definition is technically rather complex. In this way category of strong shape is obtained for paracompact spaces. In [1] it is shown that strong shape category of metric compacta is a subcategory of the last category.

2. Coherent proximate nets

Let \(\Delta^n \subseteq \mathbb{R}^n\), \(\Delta^n = \{(t_1, t_2, \ldots, t_n) | 1 \geq t_1 \geq t_2 \geq \cdots \geq t_n \geq 0\}\) be the non standard \(n\)-simplex.

It is important to note that by a function \(f : X \to Y\) we do not necessarily mean continuous function.

Let \(Y\) be a paracompact space. We form all finite sets of coverings of \(Y\), \(a = \{V_0, V_1, \ldots, V_n\}\), having a maximal element (a covering that refines all other coverings of that finite set, and is not refined by any other covering of that finite set). The maximal element is denoted by \(\text{max } a\). If \(a \subseteq b\), then \(\text{max } a > \text{max } b\).

We define an ordering "<" by \(a < b\) if \(a \subset b\).
**Definition 2.1.** A coherent proximate net \( f : X \to Y \) consists of functions
\[
\mathfrak{f} = \{ f_{\omega} = (a_0, a_1, \ldots, a_n), a_0 < \cdots < a_n \}
\]
such that each \( f_{\omega} : X \times \Delta^n \to Y \) is \( \text{st}^n \max a_0 \)-continuous and is \( \text{st}^{n-1} \max a_0 \)-continuous on \( X \times \partial \Delta^n \), and the following coherence condition is satisfied:
\[
f_{\omega}(x, t_1, t_2, \ldots, t_n) = \left\{ \begin{array}{ll}
f_{a_1 \cdots a_n}(x, t_2, \ldots, t_n), & t_1 = 1 \\
f_{a_0 \cdots a_n}(x, t_1, \ldots, t_i, \ldots, t_n), & t_i = t_{i+1} \\
f_{a_0 \cdots a_{n-1}}(x, t_1, \ldots, t_{n-1}), & t_n = 0
\end{array} \right.
\]

The coherent proximate net will be shortly denoted by \( f = (f_{\omega}) \). Next we explain the definition in special cases \( n = 0 \) and \( n = 1 \). If \( n = 0 \), for each \( a_0 \), there exists \( f_{a_0} : X \to Y \), so that \( f_{a_0} \) is \( \text{max} a_0 \)-continuous. If \( n = 1 \), for each \( a_0 \), there exists \( f_{a_0} : X \to Y \), so that \( f_{a_0} \) is \( \text{max} a_0 \)-continuous and for every \( a_0, a_1 \), there exists \( f_{a_0 a_1} : \Delta^1 \times X \to Y \), such that \( f_{a_0 a_1} \) is \( \text{st}(\max a_0) \)-continuous and \( f_{\omega} \) is \( \text{max} a_0 \)-continuous on \( \partial \Delta^1 \times X \), and also
\[
f_{a_0 a_1}(0, x) = f_{a_0}(x), \quad f_{a_0 a_1}(1, x) = f_{a_1}(x).
\]

**Definition 2.2.** Coherent proximate nets \( f, g : X \to Y \) are homotopic (notation: \( f \approx g \)), if there exists a coherent proximate net \( H = (H_{\omega}) \), such that \( H_{\omega} : X \times \Delta^n \times I \to Y \) is \( \text{st}^{n+1} \max a_0 \)-continuous, \( H_{\omega} \) is \( \text{st}^n \max a_0 \)-continuous on \( X \times \partial (\Delta^n \times I) \), and the following conditions are satisfied:
\[
H_{\omega}(x, 0, 0) = f_{\omega}(x, 0),
H_{\omega}(x, 0, 1) = g_{\omega}(x, 0).
\]

The relation \( f \approx g \) is an equivalence relation. This is shown in [13].

3. Composition of coherent proximate nets - existence and uniqueness

The following two Theorems are needed for the determination of composition of coherent proximate nets. Theorem 3.2 is proved in [9] for the case \( k = 1 \).

**Theorem 3.1.** If \( f : X \to Y \) is \( W \)-continuous, then \( \text{id} \times f : K \times X \to K \times Y \) is \( K \times W \)-continuous, where \( K \) is compact and \( K \times W = \{ K \times W | W \in W \} \).

**Proof.** \( f : X \to Y \) is \( W \)-continuous, and therefore it follows \( \forall x \in X, \exists U_n \)-neighborhood of \( x \) and \( \exists W \in W \), so that \( f(U) \subseteq W \).

As usually \( \text{id} \times f \) is defined by \( \text{id} \times f(\{ k, x \}) = (k, f(x)) \). Hence, for \( (k, x) \in K \times X \), there exists \( U, U_n \) being a neighborhood of \( x \), and there exists \( K \times W \subseteq K \times W \), so that \( \text{id} \times f(K \times U) \subseteq K \times W \).

**Theorem 3.2.** Let \( W \) be a covering of \( Z \) and \( G : \Delta^k \times Y \to Z \) be a \( \text{st}^k(W) \)-continuous function and \( \text{st}^{k-1}(W) \)-continuous on \( \partial \Delta^k \times Y \). Then there exists \( V \), a covering of \( Y \), such that for each function \( f : X \to Y \) that is \( V \)-continuous, \( G(\text{id} \times f) : \Delta^k \times X \to Z \) is \( \text{st}^k(W) \)-continuous, and \( G(\text{id} \times f) \) is \( \text{st}^{k-1}(W) \)-continuous on \( \partial \Delta^k \times X \).
PROOF. Let \( y \in Y \) be a fixed point. For each \( s \in \Delta^k \setminus \partial \Delta^k \), there exists \( J_s \subseteq \Delta^k \setminus \partial \Delta^k \), a neighborhood of \( s \), and a neighborhood \( V^y_s \) of \( y \), so that \( G(J_s \times V^y_s) \subseteq W^y_s \), for some element \( W^y_s \in st^k(W) \). For each \( s \in \partial \Delta^k \), there exists \( J_s \subseteq N \), a neighborhood of \( s \), and a neighborhood \( V^y_s \) of \( y \), so that \( G(J_s \times V^y_s) \subseteq W^y_s \), for some element \( W^y_s \in st^{k-1}(W) \).

Then \( \{ J_s \mid s \in \Delta^k \} \) is an open covering of \( \Delta^k \). There exists a finite sub-covering \( J_{s_1}, J_{s_2}, \ldots, J_{s_n} \).

Let \( J_{s_1} = V^y_{s_1} \cap \ldots \cap V^y_{s_n} \). Then \( G(J_{s_1} \times V^y_{s_1}) \subseteq W^y_{s_1} \), for \( s_1 \in \Delta^k \setminus \partial \Delta^k \), and \( G(J_{s_1} \times V^y_{s_2}) \subseteq W^y_{s_2} \), for \( s_2 \in \partial \Delta^k \). \( \mathcal{V} = \{ V^y_s \mid y \in Y \} \) is a covering of \( Y \). Let \( V \in \mathcal{V} \). Then the following holds: \( G(J_{s_1} \times V^y_{s_1}) \subseteq W^y_{s_1} \), for \( s_1 \in \Delta^k \setminus \partial \Delta^k \), and \( G(J_{s_1} \times V^y_{s_2}) \subseteq W^y_{s_2} \), for \( s_2 \in \partial \Delta^k \). \( J_{s_1} \times V^y_{|s_1|,\ldots,p,V \in \mathcal{V}} \) is a covering of \( \Delta^k \times Y \).

If \( f : X \to Y \) is a \( \mathcal{V} \)-continuous function, then there exists a covering \( \mathfrak{V} \) of \( X \), so that \( f(\mathfrak{V}) \prec \mathcal{V} \). Now, if we define \( H : \Delta^k \times X \to Z \) by:

\[
H(\mathbf{y}, x) = G(\mathbf{y}, f(x)),
\]

then \( H \) is a \( st^k(W) \)-continuous function and \( H \) is \( st^{k-1}(W) \)-continuous on \( \partial \Delta^k \times X \).

We will now define a partitioning of the simplex

\[
\Delta^n = \{(t_1, t_2, \ldots, t_n) | 1 \geq t_1 \geq \ldots \geq t_n \geq 0\},
\]

by defining the sets \( K_m, 0 \leq m \leq n \) in the following way:

\[
K_m = \{(t_1, \ldots, t_n) | t_m \geq \frac{1}{2} \geq t_{m+1}\}.
\]

Let \( B \) be the denotation of the finite sets of coverings of \( Y \) with a maximal element and \( C \) be the denotation of the finite sets of coverings of \( Z \) with a maximal element. Let \( f = (f^y) : X \to Y \) and \( g = (g^y) : Y \to Z \) be coherent proximate nets. In order to proceed and define the composition \( h = (h^y) : X \to Z \) of \( f \) and \( g \), an induction by the height of the element \( c \in C \) is performed.

DEFINITION 3.3. Let \( c \in C \), \( h(c) = 0 \) be an ordered cofinite set. Then the height of \( a \) is defined as follows:

\[
h(a) = \max\{n | a_0 < a_1 < \cdots < a_{n-1} < a\}.
\]

A strictly increasing function \( g : C \to B \) is constructed as follows:

Case 0. Let \( c \in C \), \( h(c) = 1 \). We choose an element \( g(c) \), such that

\[
g(st \max b) \prec \max c.
\]

Let \( g(c) = b \). Now \( h_c : X \to Z \) may be defined by \( h_c = g_c f(g(c)) \). Then \( h_c \) is \( c \)-continuous.

Case 1. Let \( c \in C \), \( h(c) = 1 \). We define \( g(c) \), choosing \( b \in B \), so that the following conditions hold:

1. \( g(st \max b) \subseteq \max c \);
2. \( g(c_0) < b \), for all possible choices of \( c_0, c_0 < c \).
3. \(g_{0\alpha}(id \times f_b)\) is \(st\) \(max\ c\)-continuous and \(g_{0\alpha c}(id \times f_b)\) is \(max\ c\)-continuous on \(\partial \Delta^1 \times X\).

Let \(g(c) = b\). The functions \(h_c\) and \(h_{0\alpha c}\) are defined as follows:

\[ h_c = g_{c}f_{g(c)} \]

Then \(h_c\) is \(max\ c\)-continuous. The function \(h_{0\alpha c} : \Delta^1 \times X \to Z\) is defined by:

\[
\begin{align*}
  h_{0\alpha c}(t, x) &= \left\{ \begin{array}{ll}
  g_{0\alpha c}(2t - 1, f_b(x)), & t \in K_0 \\
  g_{0\alpha f_{b_{0\alpha}}}(2t, x), & t \in K_1
  \end{array} \right.
\end{align*}
\]

Theorem 3.1 provides that \(g_{0\alpha f_{b_{0\alpha}}}\) is \(st\) \(max\ c_0\)-continuous, and Theorem 2.2 and the condition 3 provide that \(g_{0\alpha}(id \times f_b)\) is \(st\) \(max\ c_0\)-continuous and \(g_{0\alpha c}(id \times f_b)\) is \(max\ c_0\)-continuous on \(\partial \Delta^1 \times X\). Then \(h_{0\alpha c}\) is \(st\) \(max\ c_0\)-continuous and \(h_{0\alpha c}\) is \(max\ c_0\)-continuous on \(\partial \Delta^1 \times X\). On Figure 1 below there is a given review of the mapping \(h_{0\alpha c}\).

\[
\begin{array}{ccc}
  g_{0\alpha f_{b_{0\alpha}}} & g_{0\alpha c}f_b & t \\
  0 & K_0 & 1/2 & K_1 & 1
\end{array}
\]

**Figure 1**

Case \(n - 1\) (inductive assumption). We assume that for each \(c\) having a height \(h(c) \leq n - 1, g(c) = b\) is defined, so that the following conditions hold:

1. \(g(st \max b) \prec \max c\);
2. \(g(c_0) < g(c_1) < \cdots < g(c_{n-2}) < g(c) = b\), for all possible choices of indices \(C_0 < c_1 < \cdots < c_{n-2} < c\);
3. The following mappings are defined: \(h_c, h_{0\alpha c}, h_{0\alpha c_{i_1} \alpha c_{i_2} \cdots \alpha c_{i_m}} : \Delta^{i+1} \times X \to Z\) is \(st\) \(i+1\) \(max\ c_0\)-continuous and \(h_{0\alpha c_{i_1} \alpha c_{i_2} \cdots \alpha c_{i_m}}\) is \(st\) \(i+1\) \(max\ c_0\)-continuous on \(\partial \Delta^{i+1} \times X\).

On Figure 2 below there is a given review of the mapping \(h_{0\alpha c_{i_1} \alpha c_{i_2} \cdots \alpha c_{i_m}}\), and the conditions 1-3 below hold.

1. \(g(st \max b) \prec \max c\);
2. \(g(c_0) < b\), for all possible choices of \(c_0, c_0 < c\);
3. \(g_{0\alpha c}(id \times f_b)\) is \(st\) \(max\ c\)-continuous and \(g_{0\alpha}(id \times f_b)\) is \(max\ c\)-continuous on \(\partial \Delta^1 \times X\).

Case \(n\). Let \(c \in C, h(c) = n\). Herein it is important to mention that there exists a linear homeomorphism between the sets \(K_1\) and \(\Delta^1 \times \Delta^{n-1}\) mapping vertices to vertices. We choose \(b\), so that:

1. \(g(st \max b) \prec \max c\);
2. \( g(c') < b \), for all possible choices of \( c', c < c \);
3. \( g_{c_0\ldots c_n}(id \times f_{b_{k+1} \ldots b_n}) : \Delta^k \times \Delta^{n-k} \times X \to Z \) is \( st^k \max c \)-continuous, for \( k = 1, 2, \ldots n \) and for max \( b, g_{c_0\ldots c_n}(id \times f_{b_{k+1} \ldots b_n}) \) is \( st^{k-1} \max c \)-continuous on \( \partial \Delta^k \times \Delta^{n-k} \times X \).

Let \( g(c) = b \). Because of the inductive assumption, the following holds: \( g(c_0) < g(c_1) < \cdots < g(c_{n-1}) < g(c) = b \) for each \( c_0 < c_1 < \cdots < c_{n-2} < c_{n-1} < c \), and the following mappings: \( h_{c}, h_{c_0c}, h_{c_0c_1}, \ldots, h_{c_0c_1\ldots c_{n-2}c} \) are defined. We define the function \( h_{c_0c_1\ldots c_{n-1}c} : \Delta^n \times X \to Z \) by:

\[
h_{c_0c_1\ldots c_{n-1}c}(l, x) =
\begin{cases}
g_{c_0}f_{g(c_0c_1\ldots c_{n-1}c)(2t_1, \ldots, 2t_n, x)}, & t_1 \leq \frac{1}{2} \\
g_{c_0c_1}(2t_1 - 1, \ldots, 2t_i - 1, f_{g(c_0c_1\ldots c_{n-1}c)(2t_{i+1}, \ldots, 2t_n, x)}), & t_i \geq \frac{1}{2} \geq t_{i+1}. \\
g_{c_0c_1\ldots c_{n-1}c}(2t_1 - 1, \ldots, 2t_i - 1, f_{g(c)}(x)), & t_n \geq \frac{1}{2}.
\end{cases}
\]

Theorem 3.1, Theorem 3.2 and the condition 3 provide that \( h_{c_0c_1\ldots c_{n-1}c} \) is \( st^n \max c \)-continuous and \( h_{c_0c_1\ldots c_{n-1}c} \) is \( st^{n-1} \max c_0 \)-continuous on \( \partial \Delta^n \times X \). We check that \( h_{c_0c_1\ldots c_{n-1}c} \) is well defined. As in previous cases, \( g(c_0) = b_0, g(c_1) = b_1, \ldots, g(c_{n-1}) = b_{n-1}, g(c) = b \). Let \( t_1 = \frac{1}{2} \).

\[
g_{c_0}f_{b_0b_1\ldots b_{n-1}b}(2 \cdot \frac{1}{2}, 2t_2, \ldots, 2t_n, x) = g_{c_0}f_{b_0b_1\ldots b_{n-1}b}(1, 2t_2, \ldots, 2t_n, x)
= g_{c_0}f_{b_0b_1\ldots b_{n-1}b}(2t_2, \ldots, 2t_n, x).
\]

\[
g_{c_0c_1}(2 \cdot \frac{1}{2}, f_{b_0b_1\ldots b_{n-1}b}(2t_2, \ldots, 2t_n, x)) = g_{c_0c_1}(0, f_{b_1\ldots b_{n-1}b}(2t_2, \ldots, 2t_n, x))
= g_{c_0}(f_{b_1\ldots b_{n-1}b}(2t_2, \ldots, 2t_n, x))
= g_{c_0}(f_{b_1\ldots b_{n-1}b}(2t_2, \ldots, 2t_n, x)).
\]
Let \( t_i = t_{i+1} = \frac{1}{2} \): \( t_i = t_{i+1} \) implicates the following:

\[
g_{c_0 \cdots c_i}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_i - 1, f_{b_i b_{i+1} \cdots b_{n-1} b}(2t_{i+1}, \ldots, 2t_n, x))
\]

\[
= g_{c_0 \cdots c_{i-1}}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_i - 1, f_{b_i b_{i+1} \cdots b_{n-1} b}(2t_{i+1}, \ldots, 2t_n, x)),
\]

whereas \( t_i = \frac{1}{2} \) implicates:

\[
g_{c_0 \cdots c_i}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_i - 1, f_{b_i b_{i+1} \cdots b_{n-1} b}(2t_{i+1}, \ldots, 2t_n, x))
\]

\[
= g_{c_0 \cdots c_i}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_i - 1, 2\frac{1}{2} - 1, f_{b_i b_{i+1} \cdots b_{n-1} b}(2t_{i+1}, \ldots, 2t_n, x))
\]

\[
= g_{c_0 \cdots c_i}(2t_1 - 1, 2t_2 - 1, \ldots, 0, f_{b_i b_{i+1} \cdots b_{n-1} b}(2t_{i+1}, \ldots, 2t_n, x))
\]

\[
= g_{c_0 \cdots c_{i-1}}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_i - 1, f_{b_i b_{i+1} \cdots b_{n-1} b}(2t_{i+1}, \ldots, 2t_n, x)).
\]

Let \( t_n = \frac{1}{2} \):

\[
g_{c_0 \cdots c_{n-1}}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_n - 1)
\]

\[
= g_{c_0 \cdots c_{n-1} c}(2t_1 - 1, 2t_2 - 1, \ldots, 2\frac{1}{2} - 1, f_0(x)),
\]

\[
g_{c_0 \cdots c_{n-1} c}(2t_1 - 1, 2t_2 - 1, \ldots, 0, f_0(x))
\]

\[
= g_{c_0 \cdots c_{n-1}}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_{n-1} - 1, f_0(x)).
\]

\[
g_{c_0 \cdots c_0}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_i - 1, f_{b_i b_{i+1} \cdots b_{n-1} b}(2t_{i+1}, \ldots, 2t_n, x))
\]

\[
= g_{c_0 \cdots c_{n-1}}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_n - 1, f_{b_{n-1} b}(2\cdot \frac{1}{2}, x))
\]

\[
= g_{c_0 \cdots c_{n-1}}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_n - 1, f_{b_{n-1} b}(1, x))
\]

\[
= g_{c_0 \cdots c_{n-1}}(2t_1 - 1, 2t_2 - 1, \ldots, 2t_n - 1, f_0(x)).
\]

Therefore it is obtained that the function is well defined. Consequently it is obtained that the composition of the coherent proximate nets \( f = (f_0): X \rightarrow Y \) and \( g = (g_0): Y \rightarrow Z \) is a coherent proximate net \( h = (h_0): Y \rightarrow Z \). Let this composition be denoted by \( f \circ g : X \rightarrow Y \).

4. COMPOSITION OF HOMOTOPY CLASSES OF COHERENT PROXIMATE NETS

We define a composition of homotopy classes of coherent proximate nets by \([(g_0)]([f_0]) = [(g_0 f_0(c))])\). In order for this definition to be valid, it is necessary to be proved that it does not depend on the choice of strictly increasing function \( g: C \rightarrow B \). It is enough to show that for another choice of a strictly increasing function \( g': C \rightarrow B \), such that it satisfies the same required conditions 1 - 3 from the definition of composition in 2, the corresponding coherent proximate net \( h' = (h'_0): X \rightarrow Z \) is in the same homotopy class with \( h = (h_0): X \rightarrow Z \). In fact, as the way \( g: C \rightarrow B \), and \( (h_0): X \rightarrow Z \), and \( g': C \rightarrow B \), \( (h'_0): X \rightarrow Z \) are obtained, similarly we may obtain another strictly increasing function \( g'' : C \rightarrow B \), with additional condition \( g''(c) > g(c) \), \( g'(c) \) for all \( c \in C \), and a coherent proximate net \( (h''_0): X \rightarrow Z \).
Now, by induction, a homotopy $H = (H_{c^*}) : X \times I \to Z$, connecting $(h_{c^*})$ and $(h''_{c^*})$, is constructed.

Case 0. If $c \in C, h(c) = 1, H_c : I \times X \to Z$, is defined by $H_c(t, x) = g_c f_{h(c)}(t, x)$. This homotopy connects $h_c = g_c f_{h(c)}$ and $h''_c = g_c f_{h(c^*)}$.

Case 1. Let $c \in C, h(c) = 1$. The homotopy $H_c : I \times X \to Z$ is defined by $H_c(t, x) = g_c f_{h(c^*)}(t, x)$. The following step is to show that $h_{c_{abc}}$ is homotopic to $h''_{c_{abc}}$. Therefore, $H_{c_{abc}} : \Delta^1 \times I \times X \to Z$ is defined in the following way (Figure 3):

\[
H_{c_{abc}}(t, s, x) = \begin{cases} 
  g_{c_{abc}} f_{h_{c_{abc}}}(s, 2t, x), & 0 \leq t \leq \frac{s}{2} \\
  g_{c_{abc}} f_{h_{c_{abc}}}(s, 2t, x), & \frac{s}{2} \leq t \leq \frac{3s}{4} \\
  g_{c_{abc}} (2t - 1, f_{h_{c_{abc}}}(s, x)), & \frac{3s}{4} \leq t \leq 1
\end{cases}
\]

$H_{c_{abc}}$ is well defined on the edges, and it is shown as follows: - If $t = \frac{s}{2}$, then

\[
g_{c_{abc}} f_{h_{b^*}}(s, 2 \cdot \frac{s}{2}, x) = g_{c_{abc}} f_{h_{b^*}}(s, s, x) = g_{c_{abc}} f_{h_{b^*}}(s, x).
\]

On the other hand,

\[
g_{c_{abc}} f_{h_{b^*}}(2 \cdot \frac{s}{2}, s, x) = g_{c_{abc}} f_{h_{b^*}}(s, s, x) = g_{c_{abc}} f_{h_{b^*}}(s, x).
\]
If $t = \frac{1}{2}$, then
\[ g_{ca}(f_{bb}^{-1}(2 \cdot \frac{1}{2}, s, x)) = g_{ca}(f_{bb}^{-1}(1, s, x)) = g_{ca}(f_{bb}^{-1}(s, x)). \]

On the other hand,
\[ g_{ca}(2 - f_{bb}^{-1}(s, x)) = g_{ca}(0, f_{bb}^{-1}(s, x)) = g_{ca}(f_{bb}^{-1}(s, x)). \]

The following also holds: If $s = 0$, then
\[
H_{eac}(t, 0, x) = \begin{cases} 
  g_{ca}(f_{bb}^{-1}(0, 2t, x)), & t = 0 \\
  g_{ca}(f_{bb}^{-1}(2t, 0, x)), & 0 \leq t \leq \frac{1}{2} \\
  g_{ca}(2t - 1, f_{bb}^{-1}(0, x)), & \frac{1}{2} \leq t \leq 1 
\end{cases}
\]
\[
= \begin{cases} 
  g_{ca}(f_{bb}^{-1}(0, x)), & t = 0 \\
  g_{ca}(f_{bb}^{-1}(2t, x)), & 0 \leq t \leq \frac{1}{2} \\
  g_{ca}(2t - 1, f_{bb}^{-1}(x)), & \frac{1}{2} \leq t \leq 1 
\end{cases}
\]
\[
= \begin{cases} 
  g_{ca}(f_{bb}^{-1}(x)), & t = 0 \\
  g_{ca}(f_{bb}^{-1}(2t, x)), & 0 \leq t \leq \frac{1}{2} \\
  g_{ca}(2t - 1, f_{bb}^{-1}(x)), & \frac{1}{2} \leq t \leq 1 
\end{cases}
\]
\[
= \begin{cases} 
  g_{ca}(f_{bb}^{-1}(2t, x)), & 0 \leq t \leq \frac{1}{2} \\
  g_{ca}(2t - 1, f_{bb}^{-1}(x)), & \frac{1}{2} \leq t \leq 1 
\end{cases} = h_{eac}(t, x).
\]

If $s = 1$, then
\[
H_{eac}(t, 1, x) = \begin{cases} 
  g_{ca}(f_{bb}^{-1}(1, 2t, x)), & 0 \leq t \leq \frac{1}{2} \\
  g_{ca}(f_{bb}^{-1}(2t, 1, x)), & t = \frac{1}{2} \\
  g_{ca}(2t - 1, f_{bb}^{-1}(1, x)), & \frac{1}{2} \leq t \leq 1 
\end{cases}
\]
\[
= \begin{cases} 
  g_{ca}(f_{bb}^{-1}(1, 2t, x)), & 0 \leq t \leq \frac{1}{2} \\
  g_{ca}(f_{bb}^{-1}(1, 1, x)), & t = \frac{1}{2} \\
  g_{ca}(2t - 1, f_{bb}^{-1}(1, x)), & \frac{1}{2} \leq t \leq 1 
\end{cases}
\]
\[
= \begin{cases} 
  g_{ca}(f_{bb}^{-1}(1, x)), & t = \frac{1}{2} \\
  g_{ca}(2t - 1, f_{bb}^{-1}(x)), & \frac{1}{2} \leq t \leq 1 
\end{cases}
\]
\[
= \begin{cases} 
  g_{ca}(f_{bb}^{-1}(2t, x)), & 0 \leq t \leq \frac{1}{2} \\
  g_{ca}(2t - 1, f_{bb}^{-1}(x)), & \frac{1}{2} \leq t \leq 1 
\end{cases} = h_{eac}^{-1}(t, x).
\]

Hence it is proved that $H_{eac}$ is a homotopy between $h_{eac}$ and $h_{eac}^{-1}$. 
Case $n$. The homotopy $H_{e_{0}c_{1}...c_{n-1}c} : \Delta^n \times I \times X \rightarrow Z$ is defined in the following way: We have defined the partitioning of the non-standard simplex $\Delta^n = \{(t_1, t_2, \ldots, t_n) \mid 1 \geq t_1 \geq t_2 \geq \ldots \geq t_n \geq 0\}$ by the sets $K_i, 0 \leq i \leq n$ where $K_i = \{(t_1, t_2, \ldots, t_n) \mid t_i \geq \frac{1}{2} \geq t_{i+1}\}$. Now we need a partitioning of the sets, and therefore we define the sets for each, in the following way:

$$K_i^j = \{(t_1, \ldots, t_i, \ldots, t_n, s) \mid (t_1, \ldots, t_n, s) \in K_i \times I$$

so that $\forall m, i \leq m \leq n - j, \frac{s}{2} \leq t_m \leq \frac{1}{2} \forall l, n - j < l \leq n, 0 \leq t_l \leq \frac{1}{2}$.

Now $H_{e_{0}c_{1}...c_{n-1}c} : \Delta^n \times I \times X \rightarrow Z$ is defined on $K_i \times I, \forall i = 0, 1, \ldots, n$, by:

$$H_{e_{0}c_{1}...c_{n-1}c}(t_1, \ldots, t_n, s, x) = g_{e_{0}c_{1}...c}(2t_1 - 1, \ldots, 2t_i - 1, f_{g(e_{i-1}, \ldots, c_{n-j})} g'(e_{n-j}c_{n-j+1}c)) (2t_{i+1}, \ldots, s, 2t_{n-j+1}, \ldots, 2t_n, x)$$

for $(t_1, \ldots, t_n, s) \in K^j_i$. Next we show that $H_{e_{0}c_{1}...c_{n-1}c}$ is well defined. In fact, we need to observe several cases, and for each we will show that $H_{e_{0}c_{1}...c_{n-1}c}$ is well defined:

1. For $(t_1, \ldots, t_n, s, x) \in K^j_{i-1} \cap K^j_i, j = 1, \ldots, n - i$, we will show that the definition of $H_{e_{0}c_{1}...c_{n-1}c}$ is unique.
2. For $(t_1, \ldots, t_n, s, x) \in K^j_{i-1} \cap K^j_i, j = 1, \ldots, n - i$, we will show that the definition of $H_{e_{0}c_{1}...c_{n-1}c}$ is unique.
3. We will show that the definition of $H_{e_{0}c_{1}...c_{n-1}c}$ on the edges of $\Delta^n \times I$ coincides with the corresponding homotopies on $\Delta^{n-1} \times I$, i.e., with $H_{e_{0}c_{1}...c_{n-1}c}, i = 1, \ldots, n$.

1. Let $(t_1, \ldots, t_n, s, x) \in K^j_{i-1} \cap K^j_i$, i.e. $t_i = \frac{1}{2}$. Because of $(t_1, \ldots, t_n, s, x) \in K^j_{i-1}$ and $t_i = \frac{1}{2}$, we obtain the following:

$$H_{e_{0}c_{1}...c}(t_1, \ldots, s, x) = g_{e_{0}c_{1}...c}(2t_1 - 1, \ldots, 2t_{i-1} - 1, f_{b_{0}b_{1}...b_{n-j}} b_{b_{n-j+1}} \ldots b_{2t_{n-j+1}, \ldots, 2t_n, x})$$

$$= g_{e_{0}c_{1}...c}(2t_1 - 1, \ldots, 2t_{i-1} - 1, f_{b_{0}b_{1}...b_{n-j}} b_{b_{n-j+1}} \ldots b_{2t_{n-j+1}, \ldots, 2t_n, x})$$

$$= g_{e_{1}c_{1}...c}(2t_1 - 1, \ldots, 2t_{i-1} - 1, f_{b_{0}b_{1}...b_{n-j}} b_{b_{n-j+1}} \ldots b_{2t_{n-j+1}, \ldots, 2t_n, x})$$

$$= g_{e_{0}c_{1}...c}(2t_1 - 1, \ldots, 2t_{i-1} - 1, f_{b_{0}b_{1}...b_{n-j}} b_{b_{n-j+1}} \ldots b_{2t_{n-j+1}, \ldots, 2t_n, x}).$$
On the other hand, because of \((t_1, \ldots, t_n, s, x) \in K_i^j\) and \(t_i = \frac{1}{2}\), we obtain:

\[
H_{c_0 \cdots c_i}(t_1, \ldots, t_n, s, x) \\
= g_{c_0 \cdots c_i}(2t_1 - 1, \ldots, 2t_i - 1, \\
\quad \quad \quad f_{b_1 \cdots b_{n-j} b_{n-j+1}^{\cdots} b^v}(2t_{i+1}, \ldots, s, 2t_{n-j+2}, \ldots, 2t_n, x)) \\
= g_{c_0 \cdots c_i}(2t_1 - 1, \ldots, 0, \\
\quad \quad \quad f_{b_1 \cdots b_{n-j} b_{n-j+1}^{\cdots} b^v}(2t_{i+1}, \ldots, s, 2t_{n-j+2}, \ldots, 2t_n, x)) \\
= g_{c_0 \cdots c_i}(2t_1 - 1, \ldots, t_{i-1} - 1, \\
\quad \quad \quad f_{b_1 \cdots b_{n-j} b_{n-j+1}^{\cdots} b^v}(2t_{i+1}, \ldots, s, 2t_{n-j+2}, \ldots, 2t_n, x))
\]

by which we show case 1.

2. Let \((t_1, \ldots, t_n, s, x) \in K_i^{j-1} \cup K_i^j\), i.e. \(t_{n-j+1} = \frac{s}{2}\). Because of \((t_1, \ldots, t_n, s, x) \in K_i^{j-1}\) and \(t_{n-j+1} = \frac{s}{2}\), we obtain:

\[
H_{c_0 \cdots c_i}(t_1, \ldots, t_n, s, x) \\
= g_{c_0 \cdots c_i}(2t_1 - 1, \ldots, 2t_i - 1, \\
\quad \quad \quad f_{b_1 \cdots b_{n-j} b_{n-j+1}^{\cdots} b^v}(2t_{i+1}, \ldots, 2 \cdot \frac{s}{2}, s, 2t_{n-j+2}, \ldots, 2t_n, x)) \\
= c_0 \cdots c_i(2t_1 - 1, \ldots, 2t_i - 1, \\
\quad \quad \quad f_{b_1 \cdots b_{n-j} b_{n-j+1}^{\cdots} b^v}(2t_{i+1}, \ldots, s, 2t_{n-j+2}, \ldots, 2t_n, x)) \\
= g_{c_0 \cdots c_i}(2t_1 - 1, \ldots, 2t_i - 1, \\
\quad \quad \quad f_{b_1 \cdots b_{n-j} b_{n-j+1}^{\cdots} b^v}(2t_{i+1}, \ldots, 2t_{n-j}, s, 2t_{n-j+2}, \ldots, 2t_n, x))
\]

On the other hand, because of \((t_1, \ldots, t_n, s, x) \in K_i^j\) and \(t_{n-j+1} = \frac{s}{2}\), we obtain:

\[
H_{c_0 \cdots c_i}(t_1, \ldots, t_n, s, x) \\
= g_{c_0 \cdots c_i}(2t_1 - 1, \ldots, 2t_i - 1, \\
\quad \quad \quad f_{b_1 \cdots b_{n-j} b_{n-j+1}^{\cdots} b^v}(2t_{i+1}, \ldots, \frac{2s}{2}, s, 2t_{n-j+2}, \ldots, 2t_n, x)) \\
= g_{c_0 \cdots c_i}(2t_1 - 1, \ldots, 2t_i - 1, \\
\quad \quad \quad f_{b_1 \cdots b_{n-j} b_{n-j+1}^{\cdots} b^v}(2t_{i+1}, \ldots, s, \frac{2s}{2}, 2t_{n-j+2}, \ldots, 2t_n, x)) \\
= g_{c_0 \cdots c_i}(2t_1 - 1, \ldots, 2t_i - 1, \\
\quad \quad \quad f_{b_1 \cdots b_{n-j} b_{n-j+1}^{\cdots} b^v}(2t_{i+1}, \ldots, 2t_{n-j}, s, 2t_{n-j+2}, \ldots, 2t_n, x))
\]

by which case 2 is completed.
3. The edges of $\Delta^n \times I$, because of the definition of $\Delta^n$, are of a type:

$$
\partial^n_0 = \{(1, t_2, \ldots, t_n, s) (1, t_2, \ldots, t_n) \in \Delta^n\},
$$

$$
\partial^n_1 = \{(1, t_2, \ldots, t_n, s) [(t_1, t_2, \ldots, t_n) \in \Delta^n, t_l = t_{l+1}], l = 1, \ldots, n - 1,
$$

$$
\partial^n_n = \{(t_1, \ldots, t_{n-1}, 0, t, s) [(t_1, t_2, \ldots, t_{n-1}, 0) \in \Delta^n]\}.
$$

On $\partial^n_0$ there are the sets $K_1, K_2, \ldots, K_n$. On $\partial^n_1$ there are the sets $K_0, K_{i-1}, K_{i+1}, \ldots, K_n$. On $\partial^n_n$ there are the sets $K_0, K_1, \ldots, K_{n-1}$. For $\partial^n_0$, we obtain:

$$
H_{c_{0 \ldots c}}(t_1, \ldots, t_n, s, x)
= g_{c_{0 \ldots c}}(2 \cdot 1 - 1, \ldots, 2t_i - 1,
\quad f_{b_{0 \ldots b}}(2t_{i+1}, \ldots, s, 2t_{n-j+1}, \ldots, 2t_n, x))
= g_{c_{0 \ldots c}}(1, \ldots, 2t_i - 1,
\quad f_{b_{0 \ldots b}}(2t_{i+1}, \ldots, s, 2t_{n-j+1}, \ldots, 2t_n, x))
= g_{c_{0 \ldots c}}(2t_{i-1}, \ldots, 2t_i - 1,
\quad f_{b_{0 \ldots b}}(2t_{i+1}, \ldots, s, 2t_{n-j+1}, \ldots, 2t_n, x))
= H_{c_{1 \ldots c}}(t_2, \ldots, t_n, s, x),
$$

for $(t_2, \ldots, t_n, s) \in K^1_i, \forall i = 1, \ldots, n$.

For $\partial^n_1$, there are two cases, either $l < i$ or $l > i$ in the general formula of $H_{c_{0 \ldots c_{l-1} \ldots c_{i-1} \ldots c}}$. It is impossible that $l = i$, because on the edge $\partial^n_1$ there is no set $K_i$. For $l < i$, we obtain:

$$
H_{c_{0 \ldots c}}(t_1, \ldots, t_n, s, x)
= g_{c_{0 \ldots c}}(2t_1 - 1, \ldots, 2t_i - 1, 2t_{i+1} - 1, \ldots, 2t_i - 1,
\quad f_{b_{0 \ldots b}}(2t_{i+1}, \ldots, s, 2t_{n-j+1}, \ldots, 2t_n, x))
= g_{c_{0 \ldots c}}(2t_1 - 1, \ldots, 2t_i - 1, 2t_i - 1,
\quad f_{b_{0 \ldots b}}(2t_{i+1}, \ldots, s, 2t_{n-j+1}, \ldots, 2t_n, x))
= g_{c_{0 \ldots c_{l-1} \ldots c}}(2t_{l-1}, \ldots, 2t_i - 1,
\quad f_{b_{0 \ldots b}}(2t_{i+1}, \ldots, s, 2t_{n-j+1}, \ldots, 2t_n, x))
= H_{c_{l+1 \ldots c_{l} \ldots c_{i-1} \ldots c}}(t_1, \ldots, t_{i+1}, t_{i+2}, \ldots, t_i, \ldots, t_n, s, x),
$$

for $(t_1, \ldots, t_{i+1}, t_{i+2}, \ldots, t_i, \ldots, t_n, s) \in K^1_i, i = 1, \ldots, n$. For $l > i$, there are also two cases possible: either both $t_l$ and $t_{l+1}$ are before $s$, or both $t_l$ and $t_{l+1}$ are after $s$. We will show the case when both $t_l$ and $t_{l+1}$ are before
s, and the other can be obtained similarly:

\[
H_{c_0c_1...c_{n-1}}(t_1, ..., t_n, s, x)
= g_{c_0c_1...c_{n-1}}(2t_1 - 1, ..., 2t_i - 1,
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\hat{H}_{b_i...b_n-j...b_n-j+1}^n(2t_{i+1}, ..., 2t_l, 2t_l, 2t_{n-j+1}, ..., 2t_n, x))
\end{array}
\]

\[
= g_{c_0c_1...c_{n-1}}(2t_1 - 1, ..., 2t_i - 1, 1,
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\hat{H}_{b_i...b_n-j...b_n-j+1}^n(2t_{i+1}, ..., 2t_l, 2t_l, 2t_l, 2t_{n-j+1}, ..., 2t_n, x))
\end{array}
\]

\[
= g_{c_0c_1...c_{n-1}}(2t_1 - 1, ..., 2t_i - 1,
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\hat{H}_{b_i...b_n-j...b_n-j+1}^n(2t_{i+1}, ..., 2t_l, 2t_l, 2t_l, 2t_{n-j+1}, ..., 2t_n, x))
\end{array}
\]

\[
= g_{c_0c_1...c_{n-1}}(2t_1 - 1, ..., 2t_i - 1,
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\hat{H}_{b_i...b_n-j...b_n-j+1}^n(2t_{i+1}, ..., 2t_l, 2t_l, 2t_l, 2t_{n-j+1}, ..., 2t_n, x))
\end{array}
\]

for \((t_1, ..., t_i, ..., t_{n-1}, s) \in K_i, i = 1, ..., l, ..., n-1, n\). For \(\partial_n\), we obtain:

\[
H_{c_0c_1...c_{n-1}}(t_1, ..., t_n, s, x)
= g_{c_0c_1...c_{n-1}}(2t_1 - 1, ..., 2t_i - 1,
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\hat{H}_{b_i...b_n-j...b_n-j+1}^n(2t_{i+1}, ..., 2t_l, 2t_l, 2t_{n-j+1}, ..., 0, x))
\end{array}
\]

\[
= g_{c_0c_1...c_{n-1}}(2t_1 - 1, ..., 2t_i - 1,
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\hat{H}_{b_i...b_n-j...b_n-j+1}^n(2t_{i+1}, ..., 2t_l, 2t_l, 2t_{n-j+1}, ..., 0, x))
\end{array}
\]

\[
= g_{c_0c_1...c_{n-1}}(2t_1 - 1, ..., 2t_i - 1,
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\hat{H}_{b_i...b_n-j...b_n-j+1}^n(2t_{i+1}, ..., 2t_l, 2t_l, 2t_{n-j+1}, ..., 0, x))
\end{array}
\]

\[
= g_{c_0c_1...c_{n-1}}(2t_1 - 1, ..., 2t_i - 1,
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\hat{H}_{b_i...b_n-j...b_n-j+1}^n(2t_{i+1}, ..., 2t_l, 2t_l, 2t_{n-j+1}, ..., 0, x))
\end{array}
\]

\[
= g_{c_0c_1...c_{n-1}}(2t_1 - 1, ..., 2t_i - 1,
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\hat{H}_{b_i...b_n-j...b_n-j+1}^n(2t_{i+1}, ..., 2t_l, 2t_l, 2t_{n-j+1}, ..., 0, x))
\end{array}
\]

\[
= H_{c_0c_1...c_{n-1}}(t_1, ..., t_i, ..., t_{n-1}, 0, x),
\]

\((t_1, ..., t_i, ..., t_{n-1}, s) \in K_i, i = 0, ..., n - 1\). Showing case 3 is in fact showing the coherence condition for \(H_{c_0c_1...c_{n-1}}\). We have also shown that the homotopy \(H_{c_0c_1...c_{n-1}}\) is well defined. Next we show that \(H_{c_0c_1...c_{n-1}}\) connects \(h_{c_0c_1...c_{n-1}}\) and \(h^n_{c_0c_1...c_{n-1}}\). For \(s = 0\), by the definition of the sets \(K_i\), it follows that for each \(n - j < l \leq n, t_l = 0\), and therefore we obtain:

\[
H_{c_0c_1...c_{n-1}}(t_1, ..., t_n, 0, x)
= g_{c_0c_1...c_{n-1}}(2t_1 - 1, ..., 2t_i - 1,
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\hat{H}_{b_i...b_n-j...b_n-j+1}^n(2t_{i+1}, ..., 2t_l, 2t_l, 2t_{n-j+1}, 0, ..., 0, x))
\end{array}
\]

\[
= g_{c_0c_1...c_{n-1}}(2t_1 - 1, ..., 2t_i - 1,
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\hat{H}_{b_i...b_n-j...b_n-j+1}^n(2t_{i+1}, ..., 2t_l, 2t_l, 2t_{n-j+1}, 0, ..., 0, x))
\end{array}
\]

\[
= g_{c_0c_1...c_{n-1}}(2t_1 - 1, ..., 2t_i - 1,
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\hat{H}_{b_i...b_n-j...b_n-j+1}^n(2t_{i+1}, ..., 2t_l, 2t_l, 2t_{n-j+1}, 0, ..., 0, x))
\end{array}
\]

\[
= g_{c_0c_1...c_{n-1}}(2t_1 - 1, ..., 2t_i - 1,
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\hat{H}_{b_i...b_n-j...b_n-j+1}^n(2t_{i+1}, ..., 2t_l, 2t_l, 2t_{n-j+1}, 0, ..., 0, x))
\end{array}
\]

for \((t_1, ..., t_n, 0) \in K_i\), which corresponds to the definition of \(h_{c_0c_1...c_{n-1}}\).
For $s = 1$, by the definition of the sets $K_i^j$, it follows that $\forall m, i < m \leq n - j, t_m = \frac{1}{2}$, and therefore we obtain:

$$H_{c_0c_1\ldots c_{n-1}c}(t_1, \ldots, t_n, 1, x) = g_{c_0\ldots c_i}(2t_1 - 1, \ldots, 2t_i - 1, f_{b_1\ldots b_{n-j}b_{n-j+1}\ldots b'}(2\cdot \frac{1}{2}, \ldots, 2\cdot \frac{1}{2}, 2t_{n-j+1}, \ldots, x))$$

for $(t_1, \ldots, t_n, 1) \in K_i^j$, which corresponds to the definition of $h''_{c_0c_1\ldots c_{n-1}c}$.

Next we observe the example when $n = 2$, in order to illustrate the above. $h_{c_0c_1c}$ and $h_{c_0c_1c}'$ were previously defined in the following way (Figure 2):

$$h_{c_0c_1c}(t_1, t_2, x) = \begin{cases} 
g_{c_0f_{b_1b_2b}(2t_1, 2t_2, x)}, & t_1 \leq \frac{1}{2} \ ((t_1, t_2) \in K_0) 
g_{c_0c_1}(2t_1 - 1, f_{b_1b}(2t_2, x)), & t_1 \geq \frac{1}{2} \geq t_2 ((t_1, t_2) \in K_1) , 
g_{c_0c_1c}(2t_1, 2t_2, f_b(x)), & t_2 \leq \frac{1}{2} \ ((t_1, t_2) \in K_2) 
\end{cases}$$

FIGURE 4
The homotopy $H_{c_0c_1c} : \Delta^2 \times I \times X \rightarrow Z$, which connects $h_{c_0c_1c}$ and $h^*_{c_0c_1c}$, is defined on $K_i \times I$, for each $i = 0, 1, 2$, by:

$$H_{c_0c_1c}(t_1, t_2, s, x) = \begin{cases} 
  g_{c_0, f_{b_0}^i b_{-i}^i} (2t_1, 2t_2, x), & t_1 \leq \frac{t_2}{2} \ (t_1, t_2) \in K_0 \ 
  g_{c_0c_1c}(2t_1 - 1, f_{b_0}^i b_{-i}^i (2t_2, x)), & t_1 \geq \frac{t_2}{2} \ (t_1, t_2) \in K_1 
\end{cases}$$

$$g_{c_0c_1c}(2t_1, 2t_2, f_{b_0}^i (x)), \ t_2 \leq \frac{t_1}{2} \ (t_1, t_2) \in K_2$$

for $(t_1, t_2, s) \in K_i^j, j = 0, \ldots, 2 - i$. More specifically, $H_{c_0c_1c}$ is defined as follows, for each $K_i \times I, i = 0, 1, 2$:

- For $K_0 \times I$:

$$H_{c_0c_1c}(t_1, t_2, s, x) = \begin{cases} 
  g_{c_0, f_{b_0}^i b_{-i}^i} (2t_1, 2t_2, s, x), & (t_1, t_2, s) \in K_0^0 
  g_{c_0, f_{b_0}^i b_{-i}^i} (2t_1, 2t_2, s, 2t_2 - 2t_1, 2t_2, x)), & (t_1, t_2, s) \in K_0^1 
\end{cases}$$

- For $K_1 \times I$:

$$H_{c_0c_1c}(t_1, t_2, s, x) = \begin{cases} 
  g_{c_0c_1c}(2t_1 - 1, f_{b_0}^i (2t_2, s, x)), & (t_1, t_2, s) \in K_1^0 
  g_{c_0c_1c}(2t_1, 2t_2, f_{b_0}^i (x)), & (t_1, t_2, s) \in K_1^1 
\end{cases}$$

- For $K_2 \times I$:

$$H_{c_0c_1c}(t_1, t_2, s, x) = g_{c_0c_1c}(2t_1, 2t_2 - 1, f_{b_0}^i (s, x)), (t_1, t_2, s) \in K_2^0$$

On Figure 4 we see the way $\Delta^2 \times I$ divides itself into three prisms $K_0 \times I$, $K_1 \times I$, and $K_2 \times I$. On Figures 5, 6 and 7 below we can see the corresponding partitioning of $K_i \times I$ onto $K_i^j, i = 0, 1, 2$.

- $K_0 \times I$ divides into three parts, $K_0^0$, $K_0^1$ and $K_0^2$.
- $K_1 \times I$ divides into two parts, $K_1^0$ and $K_1^1$.
- $K_2 \times I$ does not divide and the one part is $K_2^0$.

**Figure 5**
Hence it is shown that \((h_c)\) and \((h''_c)\) are homotopic. In an analogue way we prove that \((h_c)\) and \((h''_c)\) are homotopic. The relation of homotopy of coherent nets is an equivalence relation, therefore \((h_c) = (g_c f_{g(c)})\) and \((h''_c) = (g_c f_{g'(c)})\) and are in the same homotopy class. Now we may define a composition of homotopy classes of coherent proximate nets by

\[
[(g_c)][(f_c)] = [(g_c f_{g(c)})],
\]
and this definition does not depend on the choice of strictly increasing function \( g : C \to B \).

5. The category of strong shape

**Theorem 5.1.** If \( f : X \to Y \), \( g : Y \to Z \), \( h : Z \to W = (W_d, w_d, D) \) are proximate coherent nets, then the proximate coherent nets \( h(gf) \) and \((hg)f\) are homotopic.

**Proof.** Suppose \( f = (f_d), g = (g_d) \), and \( h = (h_d) \). In order to obtain an explicit formula for the proximate coherent net \( h(gf) \), we define a decomposition of \( \Delta^n \) into subpolyhedra \( K_{l,m} \) for any pair of integers \( l, m \) such that \( 0 \leq l \leq m \leq n \), \( K_{l,m} = \{(t_1, t_2, \ldots, t_n) | t_l \geq \frac{1}{4} \geq t_{l+1}, t_m \geq \frac{1}{2} \geq t_{m+1} \} \). By applying the composition formula twice, for \((t_1, \ldots, t_n) \in K_{l,m} \) we have

\[
(h(gf))(t, x) = h_{d_0 \ldots d_l}(2t_1 - 1, \ldots, 2t_l - 1, gh_{d_1 \ldots d_m}(4t_{l+1} - 1, \ldots, 4t_m - 1, f_{gh_{d_1 \ldots d_m}}(d_{l+1}, \ldots, d_k, x))).
\]

Similarly, to obtain an explicit formula for the proximate coherent net \((hg)f\), we define a decomposition of \( \Delta^n \) into subpolyhedra \( Q_{l,m} \) for any pair of integers \( l, m \) such that \( 0 \leq l \leq m \leq n \), \( Q_{l,m} = \{(t_1, \ldots, t_n) | t_l \geq \frac{1}{4} \geq t_{l+1}, t_m \geq \frac{1}{2} \geq t_{m+1} \} \). Then, for \((t_1, \ldots, t_k) \in Q_{l,m} \) we have

\[
(h(gf))(t, x) = h_{d_0 \ldots d_l}(4t_1 - 3, \ldots, 4t_l - 3, gh_{d_1 \ldots d_m}(4t_{l+1} - m, \ldots, 4t_m - 2, f_{gh_{d_1 \ldots d_m}}(d_{l+1}, \ldots, 2t_k, x))).
\]

We define a partition of \( \Delta^n \times I \) into subpolyhedra \( M_{l,m} \) for any pair of integers \( l, m \) such that \( 0 \leq l \leq m \leq n \),

\[
M_{l,m} = \{(t_1, \ldots, t_n, s) | t_l \geq \frac{2 + s}{4} \geq t_{l+1}, t_m \geq \frac{1 + s}{2} \geq t_{m+1} \}.
\]

We define a homotopy \( H : I \times X \to W \) which connects \( h(gf) \) and \((hg)f\). This map will be given by the function \( fgh : D \to A \) and by the maps \( H_d : \Delta^n \times I \to W_{d_0} \) defined in the following way:

\[
H_d(t, s, x) = h_{d_0 \ldots d_l}(\frac{4t_1 - 2 - s}{2 - s}, \ldots, \frac{4t_l - 2 - s}{2 - s}, gh_{d_1 \ldots d_m}(4t_{l+1} - m - 2, \ldots, 4t_m - 2, f_{gh_{d_1 \ldots d_m}}(d_{l+1}, \ldots, 2t_k, x))).
\]

We mention that \( K_{1,m} \times 0 = \{(t_1, \ldots, t_n, 0) | (t_1, \ldots, t_n, 0) \in M_{1,m} \} \) and then, for \((t_1, \ldots, t_n) \in K_{1,m} \), it is easily checked that \( H_d(t, 0, x) = (h(gf))(t, x) \). Also, \( Q_{1,m} \times 1 = \{(t_1, \ldots, t_n, 1) | (t_1, \ldots, t_n, 1) \in M_{1,m} \} \) and for \((t_1, \ldots, t_n) \in Q_{1,m} \), \( H_d(t, 1, x) = (h(gf))(t, x) \). To complete the proof we will check the
well defining and the coherence conditions of the map $H^2_2$. To check that the
definition is well, suppose that $(t_1, \ldots, t_n, s) \in M_{t_i, m} \cap M_{t_i, m-1}$, i.e., $t_i = \frac{2 + s}{2}$. For these points $H^2_2$ is defined in two ways. If we compute the formula for
$t = (t_1, \ldots, t_n, s) \in M_{t_i, m}$ and $t_i = \frac{2 + s}{2}$, then we have:

$$H_d(t, s, x) = h_{d_0 \cdots d_{i-1}} \left( \frac{4t_1 - 2 - s}{2 - s}, \frac{4t_{i-1} - 2 - s}{2 - s}, \ldots, \frac{4t_{i+1} - 2 - s}{2 - s}, \frac{4t_m - 2 - s}{2 - s} \right).$$

The same expression is obtained if we compute the formula for $(t_1, \ldots, t_n, s) \in M_{t_i, 1}$ and $t_i = \frac{2 + s}{2}$. Similarly, we can check that the definition is
well for $(t_1, \ldots, t_n, s) \in M_{t_i, 1} \cap M_{t_i, m-1}$, and the other cases can be deduced
to one of these two cases. To check the coherence conditions of
the homotopy $H_d$, suppose that $(t_1, \ldots, t_n, s) \in M_{t_i, 1}$ and $t_n = 0$. Then
$f_{gh(d_{m-1} \cdots d_n)}(\frac{4t_{m-1} + 1}{1 + s}, \ldots, \frac{4t_n}{1 + s}, x)$, and it follows that for $t_n = 0, H_d = H_{d_0 \cdots d_{i-1}}(t_1, \ldots, t_n, x)$. The case when $t_i = 0$ is treated similarly. If $t_i = t_{i+1}$, and $i < l$, then

$$H_d(t, s, x) = h_{d_0 \cdots d_i}(\frac{4t_1 - 2 - s}{2 - s}, \ldots, \frac{4t_{i-1} - 2 - s}{2 - s}, \frac{4t_{i+1} - 2 - s}{2 - s}, \ldots, \frac{4t_m - 2 - s}{2 - s}).$$

The cases $l < i < m$ and $m < i < n$ are treated similarly.

**Theorem 5.2.** If proximate coherent nets $f, f' : X \to Y$ are homotopic,
and coherent maps $g, g' : Y \to Z$ are level homotopic, then the coherent maps
$gf, g'f' : X \to Y$ are level homotopic.

**Proof.** Let $f, f' : X \to Y$ be homotopic by a homotopy $F : I \times X \to Y$
given by a strictly increasing function $g : B \to A$. Then the proximate coherent nets $gf, g'f' : X \to Z$ are homotopic by the homotopy $gF : I \times X \to Z$
given by a strictly increasing function $fg : C \to A$. Let $g, g' : Y \to Z$ be
homotopic by a homotopy $G : I \times Y \to Z$ given by the strictly increasing function $g : C \to B$. Then the proximate coherent nets $gf, g'f' : Z \to Z$ are homotopic by the homotopy $G(1 \times f') : I \times X \to Z$
given by strictly increasing function $fg : C \to A$. It follows that the proximate coherent nets $gf, g'f' : X \to Z$ are homotopic. \qed
Theorem 5.3. The proximate coherent nets $f$ and $f1_X$ are homotopic; $f$ and $1_Y f$ are homotopic.

Proof. We will prove that $f$ and $1_Y$ are homotopic, and the other statement is treated in the similar way. First we define a partition of $\Delta^n \times I$ into subpolyhedra $L_l, l = 0, 1, \ldots, n$, by $L_l = \{(t_1, \ldots, t_n, s)| t_l \geq \frac{s}{2} + \frac{1}{2} \geq t_{l+1}\}$. We define a homotopy $F : I \times X \to Y$. This map will be given by the function $f : B \to A$ and by the maps $F_b : \Delta^n \times I \times X \to Y$ defined for $(t, s) \in L_l$ by $F_b(t, s, x) = f_{b_1 \ldots b_n} (\frac{2s}{t_l + 1}, \ldots, \frac{2s}{t_n + 1}, x)$. We mention that $K_l \times 0 = \{(t_1, \ldots, t_n, 0)| (t_1, \ldots, t_n, 0) \in L_l\}$ and then, for $t = (t_1, \ldots, t_n) \in K_l$, we have $F_b(t, 0, x) = (1_Y f)_b(t, x)$. Also, $\{(t_1, \ldots, t_n, 1)| (t_1, \ldots, t_n) \in \Delta^n\} = L_0$ and $F_0(t, 1, x) = f_2(t, x)$. Category of strong shape is obtained. The objects are paracompact topological spaces, and the morphisms are the classes of the coherent proximate nets. For the isomorphic objects in this category we say they have the same strong shape.

References


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