ORTHOGONALITY, SATURATION AND SHAPE

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To Sibe Mardesić on his 80th birthday:
amicus et vitae magister

Abstract. The class of shape equivalences for a pair \((C, K)\) of categories is the orthogonal of \(K\), that is \(\Sigma = K^\perp\). Then \(\Sigma\) is internally saturated (\(\Sigma = \Sigma^\perp\)). On the other hand, every internally saturated class of morphisms \(\Sigma \subset \text{Mor}(C)\), is the class of shape equivalences for some pair \((C, K)\). Moreover, every class of shape equivalences \(\Sigma\) enjoys a calculus of left fractions and such a fact allows one to use techniques from categories of fractions to obtain conditions for \(\Sigma^\perp\) to be reflective or proreflective in \(C\).

1. Introduction

Shape equivalences can be defined for any pair \((C, K)\) of categories. They form the class \(\Sigma\) of morphisms of \(C\) that are orthogonal to the class of objects of \(K\), in symbols \(\Sigma = K^\perp\). \(\Sigma\) is always internally saturated, that is it coincides with its double orthogonal, moreover it is true that every internally saturated class of morphisms \(\Sigma \subset \text{Mor}(C)\) is the class of shape equivalences for some pair \((C, K)\). It is worth noting that every internally saturated class of morphisms, hence every class of shape equivalences enjoys a calculus of left fractions. This suggests connections between the shape category of \((C, K)\) and the category of left fraction \(C[\Sigma^{-1}]\). On the other hand one is faced with the problem of the possible existence of other objects in \(C\), out of \(K\), which every shape equivalence for \((C, K)\) is orthogonal to. This amounts to determine the internal saturation of the class of objects of \(K\). We do this when \(K\) is reflective or proreflective in \(C\), then we consider the general case. We conclude giving

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conditions for an internally saturated class \( K \subset \text{Ob}(\mathcal{C}) \) to be reflective or proreflective in \( \mathcal{C} \).

2. Orthogonality and Saturation

All categories considered are finitely complete and cocomplete. \( T \) will denote the terminal object.

Let \( F : \mathcal{C} \to \mathcal{D} \) be a functor. There are a category \( \mathcal{C}_F \) and functors \( F_0 : \mathcal{C} \to \mathcal{C}_F \), \( F_1 : \mathcal{C}_F \to \mathcal{D} \) with the following properties:

(i) \( F_0 \) is the identity on objects and \( F_1 \) is fully faithful,
(ii) \( F = F_1 \circ F_0 \) and such a factorization is uniquely determined, up to isomorphisms, among all factorizations of \( F \) by a functor bijective on objects followed by a fully faithful functor.

\( \mathcal{C}_F \) is the full image of \( F \). It has the same objects as \( \mathcal{C} \) while \( \mathcal{C}_F(X, Y) \) is identified with \( \mathcal{D}(F(X), F(Y)) \), for all \( X, Y \in \text{Ob}(\mathcal{C}) \) ([10], 21.2).

The following easy lemma gives some important and useful properties of the full image.

**Lemma 2.1.** Let \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{D} \to \mathcal{E} \) be functors.

(i) There is a unique functor \( T : \mathcal{C}_F \to \mathcal{C}_G \circ F \) such that \( T \circ F_0 = (G \circ F)_0 \) and \( (G \circ F)_1 \circ T = G \circ F_1 \). If \( G \) is fully faithful, then \( T \) is an isomorphism.

(ii) There is a unique functor \( V : \mathcal{C}_G \circ F \to \mathcal{D}_G \) such that \( V \circ (G \circ F)_0 = G_0 \circ F \) and \( G_1 \circ V = (G \circ F)_1 \). If \( F \) is bijective on objects, then \( V \) is an isomorphism.

If \( \Sigma \subset \text{Mor}(\mathcal{C}) \), then \( \mathcal{C}[\Sigma^{-1}] \) denotes the category of left fractions of \( \mathcal{C} \) and \( P_\Sigma : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}] \) the canonical functor. \( \Sigma \) is externally saturated when \( \Sigma = S(P_\Sigma) \), that is, \( \Sigma \) is the class of all morphisms that are turned into isomorphisms by \( P_\Sigma \). Equivalently, \( \Sigma \) is externally saturated if \( \Sigma = S(F) \), for some functor \( F : \mathcal{C} \to \mathcal{D} \). We refer to [6] as a classical source on such matters.

**Proposition 2.2** ([10, page 267]). A class \( \Sigma \subset \text{Mor}(\mathcal{C}) \) has a calculus of left fractions (CLF) whenever

(i) \( \Sigma \) contains all isomorphisms,
(ii) if two of the morphisms \( s, t, t \circ s \) are in \( \Sigma \), the third is also in \( \Sigma \). If \( v \circ s \) and \( t \circ v \) are in \( \Sigma \), then \( v \) also belongs to \( \Sigma \),
(iii) \( \Sigma \) is pushout closed,
(iv) if \( f \circ s = g \circ s, s \in \Sigma \) then \( \text{coeq}(f, g) \in \Sigma \).

\( \Sigma \) has a terminal calculus of left fractions (TCLF) if, besides (i)-(iv), the comma category \( X \downarrow \Sigma \) has a terminal object, for every \( X \in \text{Ob}(\mathcal{C}) \).
When $\Sigma$ has a CLF each morphism $\varphi \in \mathcal{C}[\Sigma^{-1}](X, Y)$ can be represented as $\varphi = P_{\Sigma}(s)^{-1} \circ P_{\Sigma}(f)$, where $s \in \Sigma$. In presence of a TCLF, $\mathcal{C}[\Sigma^{-1}]$ is also a legitimate category.

From now on $(\mathcal{C}, \mathcal{K})$ will be a pair of categories, where $\mathcal{K}$ is a full subcategory of $\mathcal{C}$ and $E : \mathcal{K} \to \mathcal{C}$ is the inclusion functor.

Let $Y_{\mathcal{K}} : \mathcal{K} \to [\mathcal{K}^\circ, \text{SET}]$ denote the Yoneda embedding and let $\gamma_E : \mathcal{C} \to [\mathcal{K}^\circ, \text{SET}]$ be its natural extension to $\mathcal{C}$, that is:

- $\gamma_E(X) = \mathcal{C}(X, E(-))$,
- $\gamma_E(f : X \to Y) = f^* : \mathcal{C}(Y, E(-)) \to \mathcal{C}(X, E(-))$ is the natural transformation defined by composition with $f$.

The shape category $\mathcal{Sh}(\mathcal{C}, \mathcal{K})$ of the pair $(\mathcal{C}, \mathcal{K})$ [11] is the full image of the functor $\gamma_E$, as described by the commutative diagram:

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\gamma_E} & [\mathcal{K}^\circ, \text{SET}] \\
\downarrow \mathcal{Sh} & & \downarrow \gamma_E \mathcal{Sh} \\
\mathcal{Sh}(\mathcal{C}, \mathcal{K}) & \xrightarrow{(\gamma_E)_1} & [\mathcal{K}^\circ, \text{SET}] \\
\end{array}
\]

Then $\mathcal{Sh}(\mathcal{C}, \mathcal{K})$ has the same objects as $\mathcal{C}$ and morphisms given by

$\mathcal{Sh}(\mathcal{C}, \mathcal{K})(X, Y) = \text{Nat}(\mathcal{C}(Y, E(-)), \mathcal{C}(X, E(-)))$.

The shape functor $\mathcal{Sh} : \mathcal{C} \to \mathcal{Sh}(\mathcal{C}, \mathcal{K})$ is the identity on objects and $\mathcal{Sh}(f) = f^*$.

The class $\Sigma = \mathcal{S}(\mathcal{Sh})$ of all morphisms in $\mathcal{C}$ that are turned into isomorphisms by $\mathcal{Sh}$, is called the class of shape equivalences for $(\mathcal{C}, \mathcal{K})$.

Note that, by the very definition of shape equivalences, one has

$\Sigma = \mathcal{K}^\perp = \{ s \in \text{Mor}(\mathcal{C}) \mid f_P^* : \mathcal{C}(Y, P) \to \mathcal{C}(X, P) \text{ is bijective for all } P \in \mathcal{K} \}$,

where $f_P^*$ is the map defined by composition with $f$. By a standard abuse of notation we often denote by $\mathcal{K}$ the class of objects and the full subcategory of $\mathcal{C}$ with these objects.

$\mathcal{K}^\perp$ is called the orthogonal of $\mathcal{K}$. Symmetrically, the orthogonal of $\Sigma \subset \text{Mor}(\mathcal{C})$ is given by

$\Sigma^\perp = \{ P \in \text{Ob}(\mathcal{C}) \mid s_P^* : \mathcal{C}(Y, P) \to \mathcal{C}(X, P) \text{ is bijective for all } s \in \Sigma \}$.

$\mathcal{K}$ (resp. $\Sigma$) will be called internally saturated whenever $\mathcal{K} = \mathcal{K}^{\perp \perp}$ (resp. $\Sigma = \Sigma^{\perp \perp}$) [1, 2]. It is worth noting that $(\cdot)^{\perp \perp}$ and $(\cdot)^{\perp}$ are both closure operators for the classes of morphisms and for the classes of objects of $\mathcal{C}$, respectively.
Given morphisms $f : X \to Y$ and $g : V \to Z$ in $C$, we write $f \uparrow g$ to mean that every commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{r} & & \downarrow{s} \\
V & \xleftarrow{g} & Z
\end{array}
\]

has a unique diagonal $d : Y \to V$ such that $g \circ d = s$ and $d \circ f = r$.

If $\Sigma$ is a class of morphisms in $C$, then

\[
\Sigma^\perp = \{ f \in \text{Mor}(C) | f \uparrow g, \text{ for all } g \in \Sigma \},
\]

\[
\Sigma^\perp = \{ g \in \text{Mor}(C) | g \uparrow f, \text{ for all } f \in \Sigma \}.
\]

The following properties are easy verified, for $K, H \subset \text{Ob}(C)$ and $\Sigma, \Gamma \subset \text{Mor}(C)$:

1. $K \subset K^\perp \cap K^\perp$ and $K^\perp = K^\perp \cap K^\perp$,
2. $\Sigma \subset \Sigma^\perp \cap \Sigma^\perp$ and $\Sigma^\perp = \Sigma^\perp \cap \Sigma^\perp$,
3. $K \subset H \Rightarrow H^\perp \subset K^\perp$,
4. $\Sigma \subset \Gamma \Rightarrow \Gamma^\perp \subset \Sigma^\perp$,
5. $(\Sigma \cup \Gamma)^\perp = \Sigma^\perp \cap \Gamma^\perp$ and $\Sigma^\perp \cup \Gamma^\perp \subset (\Sigma \cap \Gamma)^\perp$,
6. $(K \cup H)^\perp = K^\perp \cap H^\perp$ and $K^\perp \cup H^\perp \subset (K \cap H)^\perp$,
7. $(\text{Ob}(C))^\perp = \text{Iso}(C)$ and $(\text{Mor}(C))^\perp = T$.

**Proposition 2.3.** Let $\Sigma = K^\perp$, then

(i) $\Sigma$ is internally saturated,
(ii) $\Sigma$ has a CLF,
(iii) $\Sigma^\perp$ is closed under finite colimits.

**Proof.** (i) is immediate. (ii) is an easy verification of the properties of Proposition 2.2, considering the fact that contravariant representable functors take colimits into limits. For (iii) see [2].

**Theorem 2.4.** Let $\Sigma \subset \text{Mor}(C)$. $\Sigma$ is the class of shape equivalences for a pair $(C, K)$ if and only if it is an internally saturated class of morphisms.

**Proof.** The class of shape equivalences for $(C, K)$ is given by $\Sigma = K^\perp$, so that the assertion follows from (i) of the proposition above. Conversely, if $\Sigma$ is internally saturated, then it is the class of shape equivalences for the pair $(C, \Sigma^\perp)$.

Hence every shape theory determines and is determined by its class of shape equivalences.

The following result was obtained in [1].

**Corollary 2.5.** An internally saturated class of morphisms is also externally saturated.
Let now $\mathcal{C} \subset \mathcal{E}$ be two categories and let $\mathcal{K} \subset \text{Ob}(\mathcal{E})$. It makes sense to consider the orthogonal of $\mathcal{K}$ with respect to $\mathcal{E}$, written $\mathcal{K}^{\perp \mathcal{E}}$, and with respect to $\mathcal{C}$, written $\mathcal{K}^{\perp \mathcal{C}}$. Symmetrically $\Sigma^{\top \mathcal{E}}$ and $\Sigma^{\top \mathcal{C}}$, for a class $\Sigma \subset \text{Mor}(\mathcal{E})$. It is clear that

- $\mathcal{K}^{\perp \mathcal{C}} \subset \mathcal{K}^{\perp \mathcal{E}}$ and $\Sigma^{\top \mathcal{C}} \subset \Sigma^{\top \mathcal{E}}$,
- $\mathcal{K}^{\perp \mathcal{C}} = \mathcal{K}^{\perp \mathcal{E}} \cap \text{Mor}(\mathcal{C})$ and $\Sigma^{\top \mathcal{C}} = \Sigma^{\top \mathcal{E}} \cap \text{Ob}(\mathcal{C})$,

moreover the following proposition holds.

**Proposition 2.6.**

(i) $\mathcal{K}^{\perp \mathcal{C}} = \mathcal{K}^{\perp \mathcal{E}} \cap \text{Ob}(\mathcal{C})$, when $T \in \mathcal{K}$

(ii) $\Sigma^{\top \mathcal{C}} = \Sigma^{\top \mathcal{E}} \cap \text{Mor}(\mathcal{C})$.

**Proof.** Consider that

\[
\mathcal{K}^{\perp \mathcal{C}} = \mathcal{K}^{\perp \mathcal{E}} \cap \text{Ob}(\mathcal{C}) = (\mathcal{K}^{\perp \mathcal{E}} \cap \text{Mor}(\mathcal{C})) \cap \text{Ob}(\mathcal{C}) \supset \mathcal{K}^{\perp \mathcal{E}} \cap \text{Ob}(\mathcal{C}),
\]

and

\[
\mathcal{K}^{\perp \mathcal{C}} \subset \mathcal{K}^{\perp \mathcal{E}} \Rightarrow \mathcal{K}^{\perp \mathcal{C}} \cap \text{Ob}(\mathcal{C}) \subset \mathcal{K}^{\perp \mathcal{E}} \cap \text{Ob}(\mathcal{C}).
\]

This proves (i). The proof of (ii) goes along the same lines.

Let us denote by $\text{Pro}\mathcal{C}$ the category of inverse systems in $\mathcal{C}$. Objects of $\text{Pro}\mathcal{C}$ are covariant functors of type $X : I \to \mathcal{C}$, where $I$ is an essentially small cofiltering category, denoted $X = (X_i)_{i \in I}$. The category of direct systems in $\mathcal{C}$ is defined by $\text{Ind}\mathcal{C} = (\text{Pro}\mathcal{C})^{\text{op}}$. Its objects are covariant functors $X : I \to \mathcal{C}$, where $I$ is an essentially small filtering category, also denoted $X = (X_i)_{i \in I}$.

A direct system $X : I \to \mathcal{C}$ gives an inverse system $X : I^{\text{op}} \to \mathcal{C}$, and vice-versa.

We refer to [8, 7, 5], for constructions and terminology concerning inverse and direct systems.

Let $\Sigma \subset \text{Mor}(\mathcal{C})$. A $\Sigma$-level morphism in $\text{Pro}\mathcal{C}$ (resp. $\text{Ind}\mathcal{C}$) is a morphism $f : X \to Y$ (resp. $f : X \to Y$) which belongs levelwise to $\Sigma$. Here we assume, as it is possible [7], that $X$, $Y$ are indexed over the same set and that $f$ (resp. $f$) is a natural transformation. Let us denote by $\text{Pro}\Sigma$ (resp. $\text{Ind}\Sigma$) the class of $\Sigma$-level morphisms in $\text{Pro}\mathcal{C}$ (resp. $\text{Ind}\mathcal{C}$).

**Lemma 2.7.**

(i) $(\text{Pro}\mathcal{K})^{\perp \text{Pro}\mathcal{C}} = \mathcal{K}^{\perp \text{Pro}\mathcal{C}}$.

(ii) $(\text{Pro}\mathcal{K})^{\perp \mathcal{C}} = \mathcal{K}^{\perp \mathcal{C}}$.

(iii) $(\text{Ind}\mathcal{K})^{\perp \text{Ind}\mathcal{C}} = \mathcal{K}^{\perp \text{Ind}\mathcal{C}}$.

(iv) $(\text{Ind}\mathcal{K})^{\perp \mathcal{C}} = \mathcal{K}^{\perp \mathcal{C}}$.

**Proof.** (i) From $\mathcal{K} \subset \text{Pro}\mathcal{K}$ it follows $(\text{Pro}\mathcal{K})^{\perp \text{Pro}\mathcal{C}} \subset \mathcal{K}^{\perp \text{Pro}\mathcal{C}}$. Conversely, let $f : X \to Y$, $f \in \mathcal{K}^{\perp \text{Pro}\mathcal{C}}$, and let $P = (P_\lambda)_{\lambda \in \Lambda}$ be an object of $\text{Pro}\mathcal{K}$. $f_{\lambda}^\alpha$ is bijective for each $\lambda \in \Lambda$, hence $f_{\lambda}^\alpha = \lim \lambda f_{\lambda}^\alpha$ is bijective, so
that $f \in (\text{Pro} \mathcal{C})^{\perp \text{pro} \mathcal{C}}$. (ii) and (iv) are immediate, while the proof of (iii) is similar to that of (i).

**Proposition 2.8.**

(i) $\mathcal{K}$ is internally saturated in $\mathcal{C}$ iff $\text{Pro} \mathcal{K}$ (resp. $\text{Ind} \mathcal{K}$) is internally saturated in $\text{Pro} \mathcal{C}$ (resp. $\text{Ind} \mathcal{C}$),

(ii) $\Sigma$ is internally saturated in $\mathcal{C}$ iff $\text{Pro} \Sigma$ (resp. $\text{Ind} \Sigma$) is internally saturated in $\text{Pro} \mathcal{C}$ (resp. $\text{Ind} \mathcal{C}$).

**Proof.** We only prove that $\mathcal{K} = \mathcal{K}^{\perp \text{pro} \mathcal{C}} \iff \text{Pro} \mathcal{K} = \mathcal{K}^{\perp \text{pro} \mathcal{C}}$, the other statements are proved similarly.

First of all, notice that $\mathcal{K}^{\perp \text{pro} \mathcal{C}} \supset \mathcal{K}^{\perp \text{pro} \mathcal{C}}$ and, since $\mathcal{K} \subset \mathcal{K}^{\perp \text{pro} \mathcal{C}}$, then $\mathcal{K} \subset \mathcal{K}^{\perp \text{pro} \mathcal{C}}$. Moreover, from $\mathcal{K}^{\perp \text{pro} \mathcal{C}} \subset \mathcal{K}^{\perp \mathcal{C}}$, it follows $\mathcal{K} = \mathcal{K}^{\perp \text{pro} \mathcal{C}}$. Finally

$$\text{Pro} \mathcal{K} = \text{Pro} (\mathcal{K}^{\perp \text{pro} \mathcal{C}}) =$$

$$= \text{Pro} (\mathcal{K}^{\perp \text{pro} \mathcal{C}} \cap \text{Ob}(\mathcal{C})) = \mathcal{K}^{\perp \text{pro} \mathcal{C}} \cap \text{Ob}(\text{Pro} \mathcal{C}) = \mathcal{K}^{\perp \text{pro} \mathcal{C}}.$$

The converse follows from Proposition 2.6(i).

3. Categorical shape

Let $(\mathcal{C}, \mathcal{K})$ be a pair of categories as in the previous section. For each $X \in \text{Ob}(\mathcal{C})$ let $X \downarrow \mathcal{K}$ be the comma category of $X$ over $\mathcal{K}$ and let $\sigma_X : X \downarrow \mathcal{K} \to \mathcal{K}$ be the codomain functor. Then one can form the category $\mathcal{C} \downarrow \mathcal{K}$ whose objects are the comma categories $X \downarrow \mathcal{K}$, $X \in \text{Ob}(\mathcal{C})$, and morphisms the functors $t : X \downarrow \mathcal{K} \to Y \downarrow \mathcal{K}$ such that $\sigma_Y \circ t = \sigma_X$. There is an evident functor

In the sequel we shall need the following [6, 10, 1, 3]:

**Theorem 2.9.** Let $R : \mathcal{C} \to \mathcal{K}$ be left adjoint to the inclusion $E : \mathcal{K} \to \mathcal{C}$ and let $\Sigma = S(\mathcal{R})$. The following properties hold

(i) $\Sigma$ has a TCLF,

(ii) the unique functor $R' : \mathcal{C}[\Sigma^{-1}] \to \mathcal{K}$, such that $R' \circ P_\Sigma = R$, is an equivalence of categories,

(iii) $\mathcal{K} = \Sigma^\perp$ and $\Sigma = \mathcal{K}^\perp = (\text{Mor}(\mathcal{K}))^\uparrow$.

(iv) $\mathcal{K} = \mathcal{K}^{\perp \perp}$, $\Sigma = \Sigma^{\perp \perp}$

(v) $\Sigma = \Sigma \downarrow \uparrow$.

Notice that (ii) implies that $(\Sigma, \mathcal{K})$ is an orthogonal pair [2], while (v) says that $(\Sigma, \Sigma \downarrow \uparrow)$ is a prefactorization system in $\mathcal{C}$ [3].
σ : C → C ↓ K and an isomorphism Φ : Sh(C, K) → C ↓ K which makes the following diagram commutative

\[
\begin{array}{cc}
C & \xrightarrow{\gamma_E} & \mathcal{K} \\
\downarrow{\sigma} & & \downarrow{\gamma_K} \\
C ↓ K & & \mathcal{K}
\end{array}
\]

See [4] for a complete explanation of this fact.

If E : K → C has left adjoint R : C → K, then there is, up to isomorphisms, a decomposition

\[
\begin{array}{cc}
C & \xrightarrow{\gamma_E} & \mathcal{K}^\circ, \text{SET} \\
\downarrow{R} & & \downarrow{\gamma_K} \\
\mathcal{K} & & \mathcal{K}
\end{array}
\]

in fact, for each \( X \in \text{Ob}(\mathcal{C}) \), there is a natural isomorphisms \( \mathcal{C}(X, E(-)) \cong K(RX, -) \). Since \( \gamma_K \) is fully faithful, it follows that \( Sh(C, K) \) can be obtained as the full image of \( R \). By Lemma 2.1 there is an equivalence

\[ Sh(C, K) \cong \mathcal{K} \]

and \( \Sigma = S(R) = S(\gamma_E) \) has a TCLF.

Note that the unique functor \( R' : \mathcal{C}[\Sigma^{-1}] \to K \) such that \( R = R' \circ P_\Sigma \), being an equivalence, gives the decomposition \( \gamma_E = \gamma_K \circ R' \circ P_\Sigma \), with \( \gamma_K \circ R' \) fully faithful, from which one gets an isomorphism

\[ Sh(C, K) \cong \mathcal{C}[\Sigma^{-1}] \]

A more interesting case, related to classical topological shape, is when \( E \) has a proadjoint \( P : \mathcal{C} \to \text{Pro}\mathcal{K} \) [5]. This means that, if \( \mathcal{E} : \text{Pro}\mathcal{K} \to \text{Pro}\mathcal{C} \) denotes the inclusion of the categories of inverse systems, then \( \mathcal{E} \) has a left adjoint \( \mathcal{P} : \text{Pro}\mathcal{C} \to \text{Pro}\mathcal{K} \) and \( P \) is its restriction to \( \mathcal{C} \). \( \mathcal{P} \) can be recovered from \( P \) as \( \mathcal{P} = \lim_{\leftarrow} \text{Pro} \mathcal{P} \) [11]. \( P \) is proadjoint to \( E \) iff there is a natural isomorphism

\[ \text{Pro}\mathcal{K}(P(X), -) \cong \mathcal{C}(X, E(-)) \]

for every \( X \in \text{Ob}(\mathcal{C}) \).

Let \( L : \text{Pro}\mathcal{K} \to [\mathcal{K}^\circ, \text{SET}] \) be the Grothendieck functor which sends an inverse system \( (X_i)_{i \in I} \) to the colimit of representables \( \lim_{\rightarrow} \mathcal{K}(X_i, -) \). In this case \( \gamma_E \) can be decomposed as shown

\[
\begin{array}{cc}
\mathcal{C} & \xrightarrow{\gamma_E} & [\mathcal{K}^\circ, \text{SET}] \\
\downarrow{P} & & \downarrow{L} \\
\text{Pro}\mathcal{K} & & \mathcal{K}^\circ, \text{SET}
\end{array}
\]
and, since $L$ is fully faithful, it turns out that $Sh(C, K)$ can be obtained as the full image of $P$.

If $\Sigma = S(P)$ then $\Sigma = S(P) = S(\gamma_E)$. From the previous observations, one has that

- $\Sigma$ has a TCLF in $ProC$,
- there is an equivalence

$$Sh(ProC, ProK) \cong ProK.$$ 

and an isomorphism

$$Sh(ProC, ProK) \cong ProC[\Sigma^{-1}].$$

**Lemma 3.1.** Let $K \subset C \subset E$ be categories. Then

(i) $Sh(ProC, ProK) \cong Sh(ProC, K),$

(ii) $Sh(C, K) \subset Sh(C, K).$

As a consequence of the above the shape category of $(C, K)$ can be viewed as the full subcategory of $ProC[\Sigma^{-1}]$ with objects those of $C$, which we write

$$Sh(C, K) \cong C[\Sigma^{-1}].$$

It follows that a shape morphism $\varphi : X \to Y$ is a left fraction

$$\varphi = P_{\Sigma}(s)^{-1} \circ P_{\Sigma}(f),$$

for $f : X \to Z, s : Y \to Z$, with $s \in \Sigma$ and $Z \in Ob(ProC).$

Passing from the reflective to the proreflective case both $\Sigma$ and $K$ inherit good properties:

**Theorem 3.2.** Let $K$ be proreflective in $C$ and let $\Sigma$ be the class of shape equivalences for $(C, K)$. Then

(i) $\Sigma$ has a CLF and is internally saturated w.r.t. $C$,

(ii) $K$ is internally saturated w.r.t. $C$.

**Proof.** $\Sigma$ is internally saturated and has a CLF in $ProC$. By Proposition 2.2, $\Sigma = \Sigma \cap Mor(C)$ has a CLF in $C$. Moreover, from Proposition 2.6, one has

$$\Sigma = \Sigma \cap Mor(C) = \Sigma^\top_{ProC} \cap Mor(C) = \Sigma^\top_{ProC} \cap Mor(C).$$

The proof of (ii) follows again from Proposition 2.6, since $ProK$ is is internally saturated w.r.t. $ProC$.

Assume now that $K$ is an internally saturated class of objects of $C$. Recall that under such assumptions

- $\Sigma = K^\bot$ is also internally saturated,
- $\Sigma$ has a CLF and, for every $X \in Ob(C)$, the comma category $X \downarrow \Sigma$ is filtered ([10], 19.2.3)
Theorem 3.3. Let $\mathcal{K} \subset \mathcal{C}$ be internally saturated and let $\Sigma = \mathcal{K}^\perp_{\mathcal{C}}$. $\mathcal{K}$ is proreflective in $\mathcal{C}$ whenever the comma category $X \downarrow \Sigma$ is essentially small, for all $X \in \text{Ob}(\mathcal{C})$.

Proof. First of all note that, by Proposition 2.8, Pro$\mathcal{K}$ (resp. Ind$\mathcal{K}$) is internally saturated in Pro$\mathcal{C}$ (resp. Ind$\mathcal{C}$). For the same reason Ind$\Sigma$ is also internally saturated in Ind$\mathcal{C}$, so that it has a CLF in Ind$\mathcal{C}$. For each $X \in \text{Ob}(\mathcal{C})$, let $I_X$ denote the small final subcategory of the filtered category $X \downarrow \Sigma$. The range functor $I_X \rightarrow X \downarrow \Sigma \rightarrow \mathcal{C}$ determines a direct system $X = (X_i)_{i \in I_X}$ and a structural morphism $\tau : X \rightarrow \overline{X}$, $\tau \in \text{Ind} \Sigma$, which actually is a final object for $X \downarrow \text{Ind} \Sigma$. From [9, 15.3, p.110] it follows that $X \in \Sigma^{\perp_{\text{Ind} \mathcal{C}}}$, hence $\overline{X} \in \text{Ind} \mathcal{K}$ and $X_i \in \mathcal{K}$, for all $i \in I_X$. Dualizing, the morphism $x : X \rightarrow \overline{X} = (X_i)_{i \in I_X}$ is an initial object in $X \downarrow \text{Pro} \Sigma$: for every $s : X \rightarrow S$, $s \in \Sigma$, there is an $i \in I_X$ and an $s_i : X_i \rightarrow S$ such that $s_i \circ x_i = s$, in particular $s_i \in \Sigma$. Let now $f : X \rightarrow K$, $K \in \mathcal{K}$. Since $\mathcal{K} = \Sigma^{\perp_{\mathcal{C}}}$, then there is a bijection $x_{i,K}^* : \mathcal{C}(X_i, K) \rightarrow \mathcal{C}(X, K)$, for all $i \in I_X$. Passing to the colimit one obtains a bijection $\varphi_K^* : \text{Pro} \mathcal{C}(X, K) \rightarrow \mathcal{C}(X, K)$ which gives the proreflectiveness of $\mathcal{K}$ in $\mathcal{C}$.

Corollary 3.4. Let $\mathcal{K} \subset \mathcal{C}$ be internally saturated class and let $\Sigma = \mathcal{K}^\perp$. $\mathcal{K}$ is reflective in $\mathcal{C}$, whenever the comma category $X \downarrow \Sigma$ has an initial object, for all $X \in \text{Ob}(\mathcal{C})$.

References
