SEQUENCES OF ITERATES OF RANDOM-VALUED VECTOR FUNCTIONS AND CONTINUOUS SOLUTIONS OF RELATED EQUATIONS

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Abstract. Given a probability space \((\Omega, \mathcal{A}, P)\), a separable metric space \(X\), and a random-valued vector function \(f : X \times \Omega \to X\), we obtain some theorems on the existence and on the uniqueness of continuous solutions \(\varphi : X \to \mathbb{R}\) of the equation \(\varphi(x) = \int_{\Omega} \varphi(f(x, \omega)) P(d\omega)\).

1. Introduction

The basic technique for getting a solution of functional equations in a single variable is iteration. However, it may happen that instead of the exact value of a function at a point we know only some parameters of this value. The iterates of such functions were defined independently by K. Baron and M. Kuczma [4] and Ph. Diamond [5]. In [3] and [6, 8] these iterates were applied (for the first time in [3]) to equations of the form

\[ \varphi(x) = \int_{\Omega} \varphi(f(x, \omega)) P(d\omega). \]

Equation (1.1) appears in many branches of mathematics and its solutions \(\varphi\) are extensively studied (see [2, Part 4] and [1, Part 3]). A very particular case of (1.1) was studied by W. Sierpiński in [15] (cf. [9, Theorem 11.11]) to characterize Cantor’s function. A more general equation, but still much less general than (1.1), was considered by S. Paganoni Marzegalli [14]. J. Morawiec elaborated on her method in [12] and [13] to the case of (1.1) but on the real line only. The aim of this paper is to enlarge the procedure of J. Morawiec to

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get the continuity of the solution given via probability distribution of a limit of the sequence of iterates \( (f^n(x, \cdot)) \) of the given function \( f \) in the vector case.

2. Random-valued functions and their iterates

Fix a probability space \((\Omega, \mathcal{A}, P)\) and a separable metric space \(X\). Let \(\mathcal{B}(X)\) denote the \(\sigma\)-algebra of all Borel subsets of \(X\). We say that \(f : X \times \Omega \to X\) is a random-valued function if it is measurable with respect to the product \(\sigma\)-algebra \(\mathcal{B}(X) \otimes \mathcal{A}\). The iterates of such a function \(f\) are defined by

\[
f^1(x, \omega_1, \omega_2, \ldots) = f(x, \omega_1), \quad f^{n+1}(x, \omega_1, \omega_2, \ldots) = f(f^n(x, \omega_1, \omega_2, \ldots), \omega_{n+1})
\]

for \(x\) from \(X\) and \((\omega_1, \omega_2, \ldots)\) from \(\Omega^\infty\) defined as \(\Omega^N\). Note that \(f^n : X \times \Omega^\infty \to X\) is a random-valued function on the product probability space \((\Omega^\infty, \mathcal{A}^\infty, P^\infty)\). More exactly, the \(n\)-th iterate \(f^n\) is \(\mathcal{B}(X) \otimes \mathcal{A}_n\)-measurable, where \(\mathcal{A}_n\) denotes the \(\sigma\)-algebra of all the sets of the form

\[
\{ (\omega_1, \omega_2, \ldots) \in \Omega^\infty : (\omega_1, \omega_2, \ldots, \omega_n) \in A \}
\]

with \(A\) from the product \(\sigma\)-algebra \(\mathcal{A}^n\). (See [4, 7]; also [10, Sec. 1.4]). Since, in fact, \(f^n(\cdot, \omega)\) depends only on the first \(n\) coordinates of \(\omega\), instead of \(f^n(x, \omega_1, \omega_2, \ldots)\) we will write also \(f^n(x, \omega_1, \ldots, \omega_n)\).

3. Main results

Being motivated by the paper [3] (especially by [3, Proposition 2.2]) we will get continuity of the solution of (1.1) given via the probability distribution of the limit of \((f^n(x, \cdot))\) (cf. also [8]). For this purpose we will obtain the vector counterparts of [12, Proposition 1, Theorem 1] adopting methods of S. Paganoni Marzegalli and J. Morawiec.

Fix a nonempty set \(S\), and for every \(s \in S\) fix a nonempty subset \(X_s\) of \(X\) and a function \(u_s : X_s \to \mathbb{R}\). We are interested in solutions \(\varphi : X \to \mathbb{R}\) of (1.1) in the class \(\mathcal{F}\) defined by

\[
\mathcal{F} = \{ \varphi : X \to \mathbb{R} \mid \varphi \text{ is a bounded function,} \quad \varphi(x) = u_s(x) \text{ for } x \in X_s \text{ and } s \in S \}.
\]

First we prove a theorem on the existence and uniqueness of such solutions accepting the following assumptions:

(A) For every \(s \in S\) there exist: an open set \(U_s \subset X\), an event \(A_s \in \mathcal{A}\) of positive probability and a positive integer \(m\) such that

\[
f^m(U_s \times A_s^N) \subset X_s;
\]

moreover, for some \(s_0 \in S\) the function \(f(\cdot, \omega)\) is continuous for \(\omega \in A_{s_0}\) and there exists an \(m_0 \in \mathbb{N}\) such that

\[
f^{m_0}(X \setminus \bigcup_{s \in S} U_s) \times A_{m_0}^N \subset \bigcup_{s \in S} U_s.
\]
The following theorem is an extension of [12, Proposition 1].

**Theorem 3.1.** Assume (A). If the closure of $X \setminus \bigcup_{s \in S} X_s$ is compact, then equation (1.1) has in the class $\mathcal{F}$ at most one solution.

**Proof.** Assume that $\varphi_1, \varphi_2 \in \mathcal{F}$ are solutions of (1.1) and put $\varphi = \varphi_1 - \varphi_2$. Clearly $\varphi$ is a solution of (1.1) and

\[(3.3) \quad \varphi(x) = 0 \quad \text{for } x \in \bigcup_{s \in S} X_s.\]

Suppose that $M := \sup \{|\varphi(x)| : x \in X\} > 0$ and consider the set

\[Y = \{x \in X : \text{there exists a sequence } (x_n) \text{ such that } \lim_{n \to \infty} x_n = x \text{ and } \lim_{n \to \infty} |\varphi(x_n)| = M\}.\]

Since $M > 0$, (3.3) and compactness of $\overline{X \setminus \bigcup_{s \in S} X_s}$ show that the set $Y$ is nonempty. We will prove that $U_s \cap Y = \emptyset$ for every $s \in S$. To get this suppose that $x \in U_s \cap Y$ for some $s \in S$. Then

\[(3.4) \quad \lim_{n \to \infty} x_n = x \quad \text{and } \lim_{n \to \infty} |\varphi(x_n)| = M\]

for some sequence $(x_n)$ of points of $U_s$. Applying (1.1), (3.1) and (3.3) we see that

\[
|\varphi(x_n)| = \left| \int_{\Omega} \left( \cdots \left( \int_{\Omega} \varphi(f^m(x_n, \omega_1, \ldots, \omega_m)) P(d\omega_m) \right) \cdots \right) P(d\omega_1) \right| \\
\leq \int_{A_s} \left( \cdots \left( \int_{A_s} |\varphi(f^m(x_n, \omega_1, \ldots, \omega_m))| P(d\omega_m) \right) \cdots \right) P(d\omega_1) \\
+ M P^{\infty}\{(\omega_1, \omega_2, \ldots) \in \Omega^\infty : (\omega_1, \ldots, \omega_m) \notin A_s^m \} \\
= M(1 - P(A_s)^m)
\]

for every $n \in \mathbb{N}$, which is a contradiction. Consequently,

\[(3.5) \quad Y \subset X \setminus \bigcup_{s \in S} U_s.\]

Now fix an $x \in Y$ and an $(x_n)$ satisfying (3.4). Applying Fatou’s Lemma and (1.1) we obtain

\[
0 \leq \int_{\Omega} \liminf_{n \to \infty} \left( M - |\varphi(f(x_n, \omega))| \right) P(d\omega) \\
\leq \liminf_{n \to \infty} \int_{\Omega} \left( M - |\varphi(f(x_n, \omega))| \right) P(d\omega) \\
\leq \liminf_{n \to \infty} \left( M - |\varphi(x_n)| \right) = 0.
\]
This gives \( \liminf_{n \to \infty} \left( M - |\varphi(f(x_n, \omega))| \right) = 0 \) a.e. In particular,
\[
\limsup_{n \to \infty} |\varphi(f(x_n, \omega_1))| = M
\]
for some \( \omega_1 \in A_{s_0} \). By the continuity of \( f(\cdot, \omega_1) \) we have \( f(x, \omega_1) \in Y \). Replacing \( x \) by \( f(x, \omega) \) we can find \( \omega_2 \in A_{s_0} \) such that \( f(f(x, \omega_1), \omega_2) \in Y \), i.e. \( f^2(x, \omega_1, \omega_2) \in Y \). After \( m_0 \) steps we obtain a sequence \( \omega_1, \ldots, \omega_{m_0} \) of elements of \( A_{s_0} \) such that
\[
f^{m_0}(x, \omega_1, \ldots, \omega_{m_0}) \in Y.
\]
On the other hand, on account of (3.5) and (3.2), \( f^{m_0}(x, \omega_1, \ldots, \omega_{m_0}) \) belongs to \( \bigcup_{s \in S} U_s \) which is a contradiction.

Now fix a family \( F_0 \subset F \). We will prove a theorem on the existence and on the uniqueness of solutions of (1.1) in the class \( F_0 \) under the following assumptions:

(B) There exist an \( m \in \mathbb{N} \) and \( U_s \subset X, A_s \in A \) for \( s \in S \) such that
\[
\inf \{ P(A_s) : s \in S \} > 0,
\]
condition (3.1) holds for every \( s \in S \), and for some \( s_0 \in S \) we have

\[
f^m((X \setminus \bigcup_{s \in S} U_s) \times A_{s_0}) \subset \bigcup_{s \in S} X_s.
\]

(C) For every \( \varphi \in F_0 \) the function \( \varphi \circ f(x, \cdot) \) is measurable for \( x \in X \), and the function \( \psi \) given by
\[
\psi(x) = \int_{\Omega} \varphi(f(x, \omega)) P(d\omega)
\]
belongs to \( F_0 \).

In the proof of the next theorem we will integrate nonnegative functions possibly nonmeasurable. If \( A \in A \) and \( h : A \to [0, \infty) \), then
\[
\int_A h(\omega) P(d\omega) = \sup_{\Pi} \sum_{E \in \Pi} P(E) \inf E h(E)
\]
where the supremum is taken over all partitions \( \Pi \) of \( A \) into a countable number of pairwise disjoint members of \( A \) (cf. [11, p. 117]).

**Theorem 3.2.** Assume (B) and (C). If \( F_0 \) is nonempty and closed in uniform convergence, then equation (1.1) has in \( F_0 \) exactly one solution.

**Proof.** Consider the operator \( L : F_0 \to F_0 \) given by
\[
L\varphi(x) = \int_{\Omega} \varphi(f(x, \omega)) P(d\omega).
\]
It is enough to prove that \(L^m : \mathcal{F}_0 \to \mathcal{F}_0\) is a contraction in the supremum metric \(\tau\). To this end we will show (by induction) that for every \(n \in \mathbb{N}\), \(\varphi_1, \varphi_2 \in \mathcal{F}_0\), \(x \in X\) and \(A \in \mathcal{A}\) the following inequality holds:
\[
|L^n \varphi_1(x) - L^n \varphi_2(x)| \leq \tau(\varphi_1, \varphi_2)(1 - P(A)^n)
\]
(3.8) + \(\int_A \ldots (\int_A |(\varphi_1 - \varphi_2)(f^n(x, \omega_1, \ldots, \omega_n))|P(d\omega_n)) \ldots P(d\omega_1)\).

In fact, if \(\varphi_1, \varphi_2 \in \mathcal{F}_0\), then putting \(\varphi = \varphi_1 - \varphi_2\), for every \(x \in X\) and \(A \in \mathcal{A}\) we have
\[
|L\varphi_1(x) - L\varphi_2(x)| \leq \int_{\Omega, A} \|\varphi(f(x, \omega))\|P(d\omega) + \int_A \|\varphi(f(x, \omega))\|P(d\omega)
\]
\[
\leq \tau(\varphi_1, \varphi_2)(1 - P(A)) + \int_A \|\varphi(f(x, \omega))\|P(d\omega)
\]
and
\[
|L^{n+1} \varphi_1(x) - L^{n+1} \varphi_2(x)| = |L^n L\varphi_1(x) - L^n L\varphi_2(x)|
\]
\[
\leq \tau(L\varphi_1, L\varphi_2)(1 - P(A)^n)
\]
\[
+ \int_A \ldots (\int_A |(L\varphi_1 - L\varphi_2)(f^n(x, \omega_1, \ldots, \omega_n))|P(d\omega_n)) \ldots P(d\omega_1)
\]
\[
\leq \tau(\varphi_1, \varphi_2)(1 - P(A)^n)
\]
\[
+ \int_A \ldots (\int_A \tau(\varphi_1, \varphi_2)(1 - P(A))
\]
\[
+ \int_A |\varphi(f^n(x, \omega_1, \ldots, \omega_n, \omega_{n+1}))|P(d\omega_{n+1}) \ldots P(d\omega_1)
\]
\[
= \tau(\varphi_1, \varphi_2)(1 - P(A)^n) + \tau(\varphi_1, \varphi_2)(1 - P(A))P(A)^n
\]
\[
+ \int_A \ldots (\int_A |\varphi(f^{n+1}(x, \omega_1, \ldots, \omega_{n+1}))|P(d\omega_{n+1})) \ldots P(d\omega_1).
\]

Fix \(\varphi_1, \varphi_2 \in \mathcal{F}_0\) and, using (B), fix also an \(m \in \mathbb{N}\) satisfying (3.1) and (3.6). If \(s \in S\) and \(x \in U_s\), then by (3.8) and (3.1) we have
\[
|L^m \varphi_1(x) - L^m \varphi_2(x)| \leq \tau(\varphi_1, \varphi_2)(1 - P(A)^m),
\]
whilst if \(x \in X \setminus \bigcup_{s \in S} U_s\), then (3.8) and (3.6) give
\[
|L^m \varphi_1(x) - L^m \varphi_2(x)| \leq \tau(\varphi_1, \varphi_2)(1 - P(A_0)^m).
\]
By this we obtain
\[
|L^m \varphi_1(x) - L^m \varphi_2(x)| \leq \tau(\varphi_1, \varphi_2)\sup\{1 - P(A_s)^m : s \in S\}
\]
for every \(x \in X\) and, consequently,
\[
\tau(L^m \varphi_1, L^m \varphi_2) \leq \tau(\varphi_1, \varphi_2)\sup\{1 - P(A_s)^m : s \in S\}.
\]
Remark 3.3. Under the assumptions of Theorems 3.1 and 3.2 equation (1.1) has in \( F \) exactly one solution and this solution belongs to \( F_0 \).

Now we proceed to the case where

\[ F_0 = \{ \varphi : X \to \mathbb{R} \mid \varphi \text{ is a bounded continuous function,} \]

\[ \varphi(x) = 0 \text{ for } x \in X_1, \varphi(x) = 1 \text{ for } x \in X_2 \}

for some Borel subsets \( X_1, X_2 \subset X \), assuming the following:

(D) There exist open sets \( U_1, U_2 \subset X \), events \( A_1, A_2 \) of positive probability, and an \( m \in \mathbb{N} \) such that (3.1) holds for \( s \in \{1, 2\} \),

\[ f^m((X \setminus (U_1 \cup U_2)) \times \mathbb{R}^n) \subset (X_1 \cup X_2) \cap (U_1 \cup U_2), \]

(3.9) \[ f(X_1 \times \Omega) \subset X_1, f(X_2 \times \Omega) \subset X_2, \]

\( f(\cdot, \omega) \) is continuous for every \( \omega \in A_1 \) and \( f \) is \( P \)-continuous (i.e., if \( x_n \to x \), then \( f(x_n, \cdot) \to f(x, \cdot) \) in probability).

The main result of this paper, which is a generalization of [3, Proposition 2.2], reads as follows.

Theorem 3.4. Assume (D), \( \text{dist}(X_1, X_2) > 0 \) and that \( \text{cl}(X \setminus (X_1 \cup X_2)) \) is compact. Then:

(i) Equation (1.1) has exactly one bounded solution \( \varphi : X \to \mathbb{R} \) such that

\[ \varphi(x) = 0 \text{ for } x \in X_1, \varphi(x) = 1 \text{ for } x \in X_2; \]

this solution is a continuous function.

(ii) If \( X \) is complete and the function \( \pi : X \times \mathcal{B}(X) \to [0, 1] \) given by

\[ \pi(x, B) = P^{\infty}\left( \{ \omega \in \Omega^\infty : \text{the sequence } (f^n(x, \omega)) \text{ converges and its limit belongs to } B \} \right) \]

(3.11) satisfies

\[ \pi(x, X_2) = 0 \text{ for } x \in X_1, \pi(x, X_2) = 1 \text{ for } x \in X_2, \]

then \( \pi(\cdot, X_2) \) is a continuous solution of (1.1).

(iii) If for every \( x \in X \) the sequence \( (f^n(x, \cdot)) \) converges in probability to a random variable \( \xi(x, \cdot) \), and the function \( \pi : X \times \mathcal{B}(X) \to [0, 1] \) given by

\[ \pi(x, B) = P^{\infty}(\xi(x, \cdot) \in B) \]

(3.12) satisfies

\[ \pi(x, X_1) = 1 \text{ for } x \in X_1, \pi(x, X_2) = 1 \text{ for } x \in X_2, \]
then for every bounded and continuous function $u : X \to \mathbb{R}$ such that
\begin{equation}
(3.14)
\quad u(x) = 0 \text{ for } x \in X_1, \quad u(x) = 1 \text{ for } x \in X_2,
\end{equation}
the function $\varphi : X \to \mathbb{R}$ defined by
\begin{equation}
(3.15)
\quad \varphi(x) = \int_X u(y)\pi(x,dy) = \int_{\Omega} u(\xi(x,\omega))P^\infty(d\omega)
\end{equation}
is a continuous solution of equation (1.1) and has property (3.10).

**Proof.** Since $\text{cl}X_1$ and $\text{cl}X_2$ are disjoint, the family $\mathcal{F}_0$ is nonempty. It
is also closed in the uniform convergence. Fix a $\varphi \in \mathcal{F}_0$. By the continuity
of $\varphi$ the function $\varphi \circ f(x,\cdot)$ is measurable for every $x \in X$. Consider the
function $\psi : X \to \mathbb{R}$ defined by (3.7). Obviously $\psi$ is a bounded function,
$\psi(x) = 0$ for $x \in X_1$ and $\psi(x) = 1$ for $x \in X_2$. We will prove that $\psi$ is
continuous. If the sequence $(x_n)$ of points of $X$ converges to an $x$, then the
sequence $(\varphi \circ f(x_n,\cdot))$ of uniformly bounded functions converges in probability
to $\varphi \circ f(x,\cdot)$ and on account of the Lebesgue-Vitali Dominated Convergence
Theorem the sequence $(\psi(x_n))$ converges to $\psi(x)$. This shows (C) with
\begin{equation*}
S = \{1, 2\}, \quad u_1 = 0, \quad u_2 = 1.
\end{equation*}
Clearly, conditions (A) and (B) are fulfilled. Applying Remark 3.3 we get the
first assertion.

To prove the second one it is enough to observe that by [8, Theorem 1]
(for $u = 1_{X_2}$) the function $\pi(\cdot,X_2)$ is a (bounded) solution of (1.1) and to
apply (i).

Passing to a proof of the third assertion fix a $u \in \mathcal{F}_0$. According to [8,
Theorem 2.(i)] the function $\varphi : X \to \mathbb{R}$ given by (3.15) is a bounded solution
of (1.1). In view of the first part of Theorem 3.4 it is enough to verify that $\varphi$
satisfies (3.10). This however follows immediately from (3.13) and (3.14): if
$x \in X_1$, then
\begin{equation*}
\varphi(x) = \int_{X_1} u(y)\pi(x,dy) = 0,
\end{equation*}
and for $x \in X_2$ we have
\begin{equation*}
\varphi(x) = \int_{X_2} u(y)\pi(x,dy) = 1.
\end{equation*}

4. Examples

The following shows a possible application of Theorem 3.4.

Fix an $N \in \mathbb{N}$ and let $X = [0,1]^N$.

Denoting the set $\{1,\ldots,N\}$ by $I$, define the subsets $X_1, X_2$ and $U_1, U_2$
of $X$ as follows:
\begin{align*}
X_1 &= \{0\}, \quad X_2 = \{x \in X : x_n = 1 \text{ for some } n \in I\},
\end{align*}
\begin{align}
U_1 = \{ x \in X : x_n < b \text{ for } n \in I \}, \quad U_2 = \{ x \in X : x_n > a \text{ for some } n \in I \},
\end{align}
where \( 0 < b < a < 1 \) are fixed. Assume that \( \alpha_1, \ldots, \alpha_N : [0,1] \to [0,1] \) are nondecreasing continuous functions such that
\begin{align}
(4.1) \quad \alpha_n(t) = 0 \quad \text{for } t \in [0,b], \quad \alpha_n(1) = 1 \quad \text{and} \quad \alpha_n(t) < t \quad \text{for } t \in (0,1),
\end{align}
and let \( v_1, \ldots, v_N, w_1, \ldots, w_N : X \to [0,1] \) be continuous functions. Given \( p_1 > 0 \) and \( p_2 > 0 \) summing up to 1, consider also \( \Omega = \{ \omega_1, \omega_2 \} \) and define the function \( f : X \times \Omega \to X \) by
\begin{align}
f(x, \omega_i) = f_i(x),
\end{align}
where
\begin{align}
f_1(x) = (\alpha_1(v_1(x)), \ldots, \alpha_N(v_N(x))), \quad f_2(x) = (w_1(x), \ldots, w_N(x)).
\end{align}
Since \( f_1, f_2 \) are continuous, it follows that \( f \) is random-valued. Equation (1.1) takes the form
\begin{align}
(4.2) \quad \varphi(x) = p_1 \varphi(\alpha_1(v_1(x)), \ldots, \alpha_N(v_N(x))) + p_2 \varphi(w_1(x), \ldots, w_N(x)).
\end{align}
(1) Assume that
\begin{align}
(4.3) \quad v_1(x), \ldots, v_N(x) \leq \max\{x_1, \ldots, x_N\} \quad \text{for } x \in X \setminus U_2,
(4.4) \quad \max\{v_1(x), \ldots, v_N(x)\} = 1 \quad \text{for } x \in X_2,
(4.5) \quad \max\{w_1(x), \ldots, w_N(x)\} = 1 \quad \text{for } x \in U_2,
(4.6) \quad w_1(0) = \ldots = w_N(0) = 0.
\end{align}
We will show that:
\begin{enumerate}
\item Equation (4.2) has exactly one bounded solution \( \varphi : X \to [0,1] \) satisfying
\begin{align}
(4.7) \quad \varphi(0) = 0 \quad \text{and} \quad \varphi(x) = 1 \quad \text{for } x \in X_2;
\end{align}
this solution is a continuous function.
\item If the function \( \pi \) given by (3.11) fulfills
\begin{align}
(4.8) \quad \pi(x, X_2) = 1 \quad \text{for } x \in X_2,
\end{align}
then \( \pi(\cdot, X_2) \) is a continuous solution of (4.2).
\end{enumerate}
\textbf{Proof.} First we show that (D) holds. Let \( A_1 = \{ \omega_1 \}, A_2 = \{ \omega_2 \} \). We claim that
\begin{align}
(4.9) \quad f_1(U_1) \subset X_1, \quad f_2(U_2) \subset X_2.
\end{align}
If \( x \in U_1 \), then \( x_n < b \) for \( n \in I \) and according to (4.3) we have \( v_n(x) < b \) for \( n \in I \), hence by (4.1) we see that \( \alpha_n(v_n(x)) = 0 \) for \( n \in I \), i.e. \( f_1(x) = 0 \).
If \( x \in U_2 \), then (4.5) gives \( f_2(x) \in X_2 \). From this (4.9) follows, and since \( X_1 \subset U_1 \) and \( X_2 \subset U_2 \), we have (3.1) for every \( m \in \mathbb{N} \) and \( s \in \{1,2\} \).
Similarly we verify that (3.9) holds. The task is now to find a positive integer $m$ with

$$f^m_n(x) = 0 \quad \text{for } x \in X \setminus U_2.$$

Put $\alpha(t) = \max\{\alpha_1(t), \ldots, \alpha_N(t)\}$ for $t \in [0, 1]$. Clearly, $\alpha$ is a continuous nondecreasing function,

$$\alpha(t) = 0 \quad \text{for } t \in [0, b] \quad \text{and} \quad \alpha(t) < t \quad \text{for } t \in (0, 1).$$

In particular, $\lim_{m \to \infty} \alpha^m(a) = 0$. Hence $\alpha^m(a) = 0$ for some $m \in \mathbb{N}$. Fix an $x \in X \setminus U_2$. By the monotonicity of $\alpha$ and (4.3) we have

$$f_1(x) \leq (\alpha(v_1(x)), \ldots, \alpha(v_N(x))) \leq \cdots \leq (\alpha(\max\{x_1, \ldots, x_N\}), \ldots, \alpha(\max\{x_1, \ldots, x_N\})).$$

whence

$$f_1(x) \leq (\alpha(a), \ldots, \alpha(a)) \leq (a, \ldots, a).$$

In particular, $f_1(x) \in X \setminus U_2$ and since $x \in X \setminus U_2$ was arbitrarily fixed we can replace it by $f_1(x)$ to get

$$f^2(x) \leq (\alpha(\max\{f_1(x) : n \in I\}), \ldots, \alpha(\max\{f_1(x) : n \in I\})) \leq (\alpha^2(\max\{x_1, \ldots, x_N\}), \ldots, \alpha^2(\max\{x_1, \ldots, x_N\})) \leq (\alpha^2(a), \ldots, \alpha^2(a)).$$

After $m$ steps

$$f^m_n(x) \leq (\alpha^m(a), \ldots, \alpha^m(a))$$

and $f^m_n(x) = 0$. This ends the proof of (D).

Consequently Theorem 3.4(i) yields part (i) of our example.

Since $f_1(0) = f_2(0) = 0$, we conclude that for $\pi$ given by (3.11) we have $\pi(0, X_2) = 0$. The continuity of $\pi(\cdot, X_2)$ follows from (4.8) and Theorem 3.4(ii).

Consider now continuous functions $\beta_1, \ldots, \beta_N : [0, 1] \to [0, 1]$ such that

$$\beta_n(0) = 0, \quad \beta_n(t) = 1 \quad \text{for } t \in [a, 1], \ n \in I.$$

(II) The functions $v_1, \ldots, v_N, w_1, \ldots, w_N$ defined by

$$v_n(x) = \max\{x_1, \ldots, x_N\}, \quad w_n(x) = \beta_n(\min\{x_1 + \ldots + x_N, 1\}) \quad \text{for } x \in X$$

satisfy (4.3)-(4.6). By Example (I).(i) the equation

$$\varphi(x) = p_1(\varphi_1(\max\{x_1, \ldots, x_N\}), \ldots, \varphi_N(\max\{x_1, \ldots, x_N\})) + \varphi(\beta_1(\min\{x_1 + \ldots + x_N, 1\}), \ldots, \beta_N(\min\{x_1 + \ldots + x_N, 1\}))$$

has exactly one bounded solution $\varphi : X \to \mathbb{R}$ satisfying (4.7) and this solution is a continuous function. We will show that it equals to

$$x \mapsto P^\infty_n(\lim_{n \to \infty} f^n(x, \cdot) = (1, \ldots, 1)), \quad x \in X.$$
In fact, according to [8, Theorem 1 (with \( u = 1_{\{(1, \ldots, 1)\}} \)] the function (4.11) is a (bounded) solution of (4.10). If \( x \in X_2 \), then
\[
v_n(x) = 1 = \min\{x_1 + \ldots + x_N, 1\} \quad \text{for} \ n \in I,
\]
whence \( f(x, \omega_i) = (1, \ldots, 1) \in X_2 \) for \( i = 1, 2 \). Consequently
\[
f^n(x, \omega) = (1, \ldots, 1) \quad \text{for} \ n \in \mathbb{N}, \ x \in X_2 \text{ and } \omega \in \Omega^\infty,
\]
and the function (4.11) takes the value 1 on \( X_2 \). Moreover, \( f(0, \omega_i) = 0 \) for \( i = 1, 2 \), whence \( f^n(0, \omega) = 0 \) for \( n \in \mathbb{N} \) and \( \omega \in \Omega^\infty \) and, consequently, \( \pi(0, \cdot) = 0 \).

(III) Define now the functions \( v_1, \ldots, v_N, w_1, \ldots, w_N \) by
\[
v_n(x) = x_n, \quad w_n(x) = \beta_n(x_n) \quad \text{for} \ x \in X.
\]
Clearly (4.3)–(4.6) are fulfilled. Consequently the equation
\[
\varphi(x) = p_1 \varphi(\alpha_1(x_1), \ldots, \alpha_N(x_N)) + p_2 \varphi(\beta_1(x_1), \ldots, \beta_N(x_N))
\]
has exactly one bounded solution \( \varphi : X \rightarrow \mathbb{R} \) satisfying (4.7). Assume additionally (cf. [3, Example 2.1]) that \( p_2 \leq b \) and
\[
\alpha_n(t) = 0 \quad \text{for} \ t \in [0, a], \quad \alpha_n(t) \leq \frac{t - p_2}{p_1} \quad \text{for} \ t \in [a, 1],
\]
\[
\beta_n(t) = 1 \quad \text{for} \ t \in [b, 1], \quad \beta_n(t) \leq \frac{t}{p_2} \quad \text{for} \ t \in [0, b],
\]
for \( n \in I \). Then
\[
p_1 \alpha_n(t) + p_2 \beta_n(t) \leq t \quad \text{for} \ t \in [0, 1] \text{ and } n \in I,
\]
and
\[
p_1 f_1(x) + p_2 f_2(x) \leq x \quad \text{for} \ x \in X.
\]
Due to [7, Theorem 4] for every \( x \in X \) the sequence \( (f^n(x, \cdot)) \) converges a.s. to a measurable function \( \xi(x, \cdot) : \Omega^\infty \rightarrow X \). In particular, the functions (3.11) and (3.12) coincide. Since \( f_1(X_2) \subset X_2, f_2(X_2) \subset X_2 \), we have
\[
f^n(x, \omega) \in X_2 \quad \text{for} \ x \in X_2, \ \omega \in \Omega^\infty, \ n \in \mathbb{N}.
\]
This gives (4.8), because \( X_2 \) is closed. Thus \( \pi(\cdot, X_2) \) is a continuous solution of (4.12).

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