THE SPANS OF FIVE STAR-LIKE SIMPLE CLOSED CURVES

Thelma West
University of Louisiana at Lafayette, USA

Abstract. Let $X$ be a continuum, that is a compact, connected, nonempty metric space. The span of $X$ is the least upper bound of the set of real numbers $r$ which satisfy the following conditions: there exists a continuum, $C$, contained in $X \times X$ such that $d(x, y)$ is larger than or equal to $r$ for all $(x, y)$ in $C$ and $p_1(C) = p_2(C)$, where $p_1, p_2$ are the usual projection maps. The following question has been asked. If $X$ and $Y$ are two simple closed curves in the plane and $Y$ is contained in the bounded component of the plane minus $X$, then is the span of $X$ larger than the span of $Y$? We define a set of simple closed curves, which we refer to as being five star-like. We answer this question in the affirmative when $X$ is one of these simple closed curves. We calculate the spans of the simple closed curves in this collection and consider the spans of various geometric objects related to these simple closed curves.

1. Introduction

The span of a metric continuum was defined in 1964 by A. Lelek (see [3, p.209]). Variations of the span have been defined since then (cf [4, 5, 2]). In general it is difficult to evaluate the spans of a geometric object. It is of interest how these various spans, for a particular object, are related to each other. Also of interest is how the spans of related objects compare to each other. The following question by H. Cook has been particularly interesting.

If $X_1$ and $X_2$ are two simple closed curves in the plane and $X_2$ is contained in the bounded component of $X_1$, then is the span of $X_1$ larger than the span of $X_2$? ([1])

This question has not yet been answered in general. Given some specific conditions on either $X_1$ or $X_2$, the answer has been shown to be yes. We

2000 Mathematics Subject Classification. 54F20, 54F15.
Key words and phrases. Span, simple closed curve.
give a short summary of the results for these special cases that have been
determined.

In the following results we assume that:
1. \(X_2\) is a simple closed curve,
2. \(X_3\) is a continuum that is contained in the bounded component of \(\mathbb{R}^2 - X_2\),
3. \(X_1\) is a plane separating continuum that contains \(X_2\) in one of its
   bounded components, and
4. the span of \(X\) is denoted by \(\sigma(X)\).

If \(X_2\) is the boundary of a convex region in the plane and \(X_1\) is a simple
closed curve, then \(\sigma(X_2) < \sigma(X_1)\) ([7], see also [6]).

If \(X_2\) is the boundary of a convex region in the plane, then
\(\sigma(X_3) < \sigma(X_2) < \sigma(X_1)\) [11].

Also, \(\sigma(X_3) < \sigma(X_2) < \sigma(X_1)\) when \(X_2\) is either an “indented circle” ([8, 9, 13]), or a “concave upward symmetric” simple closed curve [10], or
a simple closed curve as defined in [12]. In this paper we define a set of
simple closed curves that we refer to as five star-like and show that if \(X_2\) is
a five star-like simple closed curve and \(X_1\) and \(X_3\) are as defined previously,
then \(\alpha(X_3) < \alpha(X_2) < \alpha(X_1)\), where \(\alpha\) represents the span \(\sigma\), semispan \(\sigma_0\),
surjective span \(\sigma^*\), surjective semispan \(\sigma_0^*\), symmetric span \(s\), and surjective
symmetric span \(s^*\).

2. Preliminaries

Let \(X\) be a continuum, that is a compact, connected metric space. The
span of \(X\), \(\sigma(X)\), is the least upper bound of the set of real numbers \(r\) which
satisfy the following conditions: there exist a continuum, \(C\), and continuous
functions \(f, g : C \rightarrow X\), such that
\[
d_{\min}(f, g) = \min\{d(f(c), g(c)) \mid c \in C\} \geq r
\]
and
\[
f(C) = g(C) \quad \sigma
\]
\[
\text{span}
\]
To obtain the various other spans, we replace the preceeding equation with
the following:
\[
f(C) \subseteq g(C) \quad \sigma_0
\]
\[
\text{semispan}
\]
\[
f(C) = g(C) = X \quad \sigma^*
\]
\[
\text{surjective span}
\]
\[
f(C) \subseteq g(C) = X \quad \sigma_0^*
\]
\[
\text{surjective semispan}
\]
\[
f(C) = g(C) \quad s
\]
symmetric span

and \( \forall c \in C, \exists c' \) such that \( f(c) = g(c') \) and \( f(c') = g(c) \)

\[
f(C) = g(C) = X\quad s^*
\]

surjective symmetric span

and \( \forall c \in C, \exists c' \) such that \( f(c) = g(c') \) and \( f(c') = g(c) \).

The inequalities below follow immediately from the definitions,

\[
0 \leq \sigma^*(X) \leq \sigma(X) \leq \sigma_0(X) \leq \text{diam } X,
\]

\[
0 \leq \sigma^*(X) \leq \sigma^*_0(X) \leq \sigma_0(X) \leq \text{diam } X,
\]

\[
0 \leq s(X) \leq \sigma(X),
\]

\[
0 \leq s^*(X) \leq \sigma^*(X).
\]

The following results are easy consequences of the various definitions.

1. If \( J \) is an arc, then \( \alpha(J) = 0 \), where \( \alpha = \sigma, \sigma_0, \sigma^*, \sigma^*_0, s, s^* \).
2. If \( X \) is a simple closed curve, then \( \sigma(X) = \sigma^*(X), \sigma_0(X) = \sigma^*_0(X), \) and \( s(X) = s^*(X) \).

We utilize the following theorem from [3] in the proof of Corollary 2.

Theorem L: If \( Y \) is a closed subset of the Hilbert cube \( I^d \) and \( \rho : Y \to S \) is an essential mapping of \( Y \) onto the circumference \( S \), then

\[
\inf_{s \in S} (\rho^{-1}(s), \rho^{-1}(-s)) \leq \sigma(Y).
\]

3. MAIN RESULTS

(a) Let \( Q \) be a five-sided, convex polygon with sequentially labeled vertices \( Q_i \) for \( i = 0, 1, 2, 3, 4 \), with all interior angles \( \angle Q_i \), \( i = 0, 1, 2, 3, 4 \), larger than 90°. Extend each side \( Q_iQ_{i+1} \) of the polygon.

(b) Let \( Q_{j+2} \) be the point of intersection of \( Q_jQ_{j+1} \) and \( Q_{j+2}Q_{j+3} \) for \( j = 0, 1, 2, 3, 4 \), where all indices are taken modulo 5.

(c) Let \( X \) be the star shaped simple closed curve defined by \( X = \bigcup_{j=0}^4 (Q_jQ_{j+1} \cup Q_{j+1}Q_{j+2}) \). We refer to \( X \) as a five star-like simple closed curve.

(d) Let \( t_j \) be the point on \( Q_{j-1}Q_{j-3} \) that is the closest to \( Q_j \). Note that \( t_j \in (Q_{j-2}Q_{j-1} - \{Q_{j-2}\}) \), since \( Q_j \in Q_{j-2}Q_{j-1} \) and \( \angle Q_{j-2} > 90° \).

(e) Let \( r_j \) be the point on \( Q_{j+3}Q_{j+1} \) that is closest to \( Q_j \). Note that \( r_j \in (Q_{j+1}Q_{j+2} - \{Q_{j+1}\}) \), since \( Q_j \in Q_{j+1}Q_{j+2} \) and \( \angle Q_{j+1} > 90° \).

(f) Note that \( t_j = d(Q_j, t_j) < d(Q_j, Q_{j+2}) = q_j' \) and \( r_j = d(Q_j, r_j) < d(Q_j, Q_{j+2}) = q_j' \).
(g) Suppose that \( q_1' = \min \{ q_i' \}_{i=0}^4 \). We can make this assumption since we can relabel the vertices so that this is true. We refer to the number

\[
\text{fss}(X) = \max \{ q_3', \min \{ r_0', r_1', t_1', t_2' \} \}, \min \{ r_4', t_0', t_1' \}, \min \{ r_4', t_0', q_2' \} \}
\]

as the five star-like spread of \( X \), where \( X \) is a five star-like simple closed curve.

**Theorem 3.1.** Let \( X \) be a five star-like simple closed curve. Then \( \sigma(X) = \sigma_0(X) = \sigma^*(X) = \sigma^*_0(X) = s(X) = s^*(X) = \text{fss}(X) \).

**Proof.** Let \( f, g : I \to X \) be continuous functions from \( I \), the unit interval, onto \( X \), such that their movements are always clockwise on \( X \) and one function is constantly \( Q_j \) for some \( j \), while the other function moves from \( Q_k \) to \( Q_{k+1} \), passing through the point \( Q'_k \), where \( k \neq j, j-1 \) and \( f[I] = g[I] = X \). Let \( \mathcal{P} \) be the set consisting of all pairs of functions \( (f, g) \) that satisfy these conditions. We claim that \( \alpha(X) = \max \{ \text{dmin}(f, g) \mid (f, g) \in \mathcal{P} \} = \text{fss}(X) \), where \( \alpha = \sigma, \sigma_0, \sigma^*, \sigma^*_0, s, s^* \). We consider two cases.

**Case A:** \( \max \{ \text{dmin}(f, g) \mid (f, g) \in \mathcal{P} \} = q_3' \)

Consider the pair of functions given in Table 1.

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) )</td>
<td>( Q_0 )</td>
<td>( Q_1 )</td>
<td>( Q_2 )</td>
<td>( Q_3 )</td>
<td>( Q_4 )</td>
<td>( Q_5 )</td>
<td>( Q_6 )</td>
<td>( Q_7 )</td>
<td>( Q_8 )</td>
<td>( Q_9 )</td>
<td></td>
</tr>
<tr>
<td>( g(t) )</td>
<td>( Q_3 )</td>
<td>( Q_4 )</td>
<td>( Q_5 )</td>
<td>( Q_0 )</td>
<td>( Q_1 )</td>
<td>( Q_2 )</td>
<td>( Q_3 )</td>
<td>( Q_4 )</td>
<td>( Q_5 )</td>
<td>( Q_6 )</td>
<td></td>
</tr>
</tbody>
</table>

Note that

\[
d(Q_j, Q_{j+2}) \cup Q'_{j+2} \cup Q_{j+3} = d(Q_j, Q'_{j+2}) = q_3'.
\]

Given this observation and assumption (g) in the construction of \( X \), we see that \( \text{dmin}(f, g) = q_3' \).

We observe that \( \alpha(X) \geq d(Q_3, Q_0') = q_3' \) for \( \alpha = \sigma, \sigma_0, \sigma^*, \sigma^*_0, s, s^* \). This is true since, \( \sigma(X) = \sigma^*(X) \), \( \sigma_0(X) = \sigma^*_0(X) \), \( f(I) = g(I) = X \), and for all \( t \in [0, 1] \) there is a \( t' \in [0, 1] \) such that \( f(t) = g(t') \) and \( g(t) = f(t') \).
In particular, \((f, g)\), from Table 1, is a “better” pair than the pairs \((f_1, g_1)\) and \((f_2, g_2)\) given in Table 2 and Table 3.

### Table 2. \(f_1(t)\) and \(g_1(t)\) versus \(t\)

<table>
<thead>
<tr>
<th>(t)</th>
<th>(f_1(t))</th>
<th>(g_1(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(Q_0)</td>
<td>(Q_2)</td>
</tr>
<tr>
<td>0.1</td>
<td>(Q_0)</td>
<td>(Q_4)</td>
</tr>
<tr>
<td>0.2</td>
<td>(Q_1)</td>
<td>(Q_4)</td>
</tr>
<tr>
<td>0.3</td>
<td>(Q_2)</td>
<td>(Q_4)</td>
</tr>
<tr>
<td>0.4</td>
<td>(Q_2)</td>
<td>(Q_0)</td>
</tr>
<tr>
<td>0.5</td>
<td>(Q_3)</td>
<td>(Q_0)</td>
</tr>
<tr>
<td>0.6</td>
<td>(Q_4)</td>
<td>(Q_0)</td>
</tr>
<tr>
<td>0.7</td>
<td>(Q_4)</td>
<td>(Q_1)</td>
</tr>
<tr>
<td>0.8</td>
<td>(Q_4)</td>
<td>(Q_2)</td>
</tr>
<tr>
<td>0.9</td>
<td>(Q_0)</td>
<td>(Q_2)</td>
</tr>
<tr>
<td>1.0</td>
<td>(Q_0)</td>
<td>(Q_3)</td>
</tr>
</tbody>
</table>

### Table 3. \(f_2(t)\) and \(g_2(t)\) versus \(t\)

<table>
<thead>
<tr>
<th>(t)</th>
<th>(f_2(t))</th>
<th>(g_2(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(Q_0)</td>
<td>(Q_2)</td>
</tr>
<tr>
<td>0.1</td>
<td>(Q_1)</td>
<td>(Q_2)</td>
</tr>
<tr>
<td>0.2</td>
<td>(Q_1)</td>
<td>(Q_4)</td>
</tr>
<tr>
<td>0.3</td>
<td>(Q_1)</td>
<td>(Q_4)</td>
</tr>
<tr>
<td>0.4</td>
<td>(Q_2)</td>
<td>(Q_4)</td>
</tr>
<tr>
<td>0.5</td>
<td>(Q_2)</td>
<td>(Q_0)</td>
</tr>
<tr>
<td>0.6</td>
<td>(Q_2)</td>
<td>(Q_1)</td>
</tr>
<tr>
<td>0.7</td>
<td>(Q_4)</td>
<td>(Q_1)</td>
</tr>
<tr>
<td>0.8</td>
<td>(Q_4)</td>
<td>(Q_2)</td>
</tr>
<tr>
<td>0.9</td>
<td>(Q_4)</td>
<td>(Q_2)</td>
</tr>
<tr>
<td>1.0</td>
<td>(Q_0)</td>
<td>(Q_2)</td>
</tr>
</tbody>
</table>

Consequently, since

\[
d_{\min}(f, g) \geq \max\{d_{\min}(f_1, g_1), d_{\min}(f_2, g_2)\}
\]

it must be the case that

\[
d_{\min}(f, g) = q_3' \geq \min\{r_4', t_0'\}
\]

and

\[
d_{\min}(f, g) = q_3' \geq \min\{t_2', r_1'\} \quad (*)
\]
This implies that \( \operatorname{fss}(X) = q'_3. \)

We define a continuous function \( p_j : Y \to Q_j' Q_j \cup Q_j' Q_{j+1} \) where \( j = 0, 1, 2, 3, 4, \) and \( Y \) is the closure of the bounded component of \( R^2 - X. \) First we define \( p_j \) on \( Y \cap Y_1 \) where \( Y_1 \) is the closure of the bounded component of \( R^2 - (Q_j' Q_{j+1} \cup Q_j' Q_{j+2} \cup Q_j' Q_{j+3} \cup Q_j' Q_{j+4} \cup Q_j' Q_{j+5} \cup Q_j' Q_{j+6} \cup Q_j' Q_{j+7} \). \)

We consider two cases in defining \( p_j \) on \( Y_1 \) for \( j = 0, 1, 2, 3, 4. \)

**Case** \( p_j, Y_1, 1: t'_{j+2} \leq r'_{j+1} \)

Let \( t \in T_{j+2}Q_{j+1} \) and let \( L_t \) be the line that is parallel to \( Q_{j+2} Q_{j+3} \) and passes through the point \( t. \) For each \( y \in Y \cap L_t, \) let \( p_j(y) = t. \) It may be the case \( Q_{j+1} = t_{j+2}. \) If so, then \( L_t \cap Y = \{Q_{j+1}, Q_{j+2}\} \) and \( \operatorname{diam}(p_j^{-1}(t)) = d(Q_{j+2}, t_{j+2}) = t'_{j+2}. \) Otherwise, \( Q_{j+1} t_{j+2} Q_{j+2} Q_{j+3} \) forms a right triangle and for each \( t, \) \( \operatorname{diam}(p_j^{-1}(t)) \leq t'_{j+2}. \)

Let \( h_1 : [0, 1] \to Q_j' t_{j+2} \) be the homeomorphism such that \( h_1(t) = (1 - t) t_{j+2} + t Q_j'. \)

Let \( h_2 : [0, 1] \to Q_{j+2} Q_{j+3} \) be the homeomorphism such that \( h_2(t) = (1 - t) t_{j+2} + t Q_{j+3}. \)

Let \( L_t \) be the line connecting \( h_1(t) \) and \( h_2(t) \) for each \( t \in [0, 1]. \) For each \( y \in Y \cap L_t \) let \( p_j(y) = h_1(t). \) Note that \( \operatorname{diam}(p_j^{-1}(\{h_1(t)\})) \leq \max\{t'_{j+2}, q'_{j+3}\}. \)

**Case** \( p_j, Y_1, 1: r'_{j+1} < t'_{j+2} \)

Let \( t \in T_{j+1}Q_{j+2} \) and let \( L_t \) be the line that is parallel to \( Q_{j+1} Q_{j+1} \) and passes through the point \( t. \) Let \( t' \) be the point such that \( \{t'\} = L_t \cap ((Q_{j+1} Q_{j+1} - Q_{j+1} Q_{j+1}) \cup Q_{j+1}). \) For each \( y \in Y \cap L_t \) let \( p_j(y) = t'. \) Note that for each \( t, \) \( \operatorname{diam}(p_j^{-1}(\{t'\})) \leq r'_{j+1}. \)

Let \( h_1 : [0, 1] \to Q_j' Q_{j+1} \) be the homeomorphism such that \( h_1(t) = (1 - t) Q_{j+1} + t Q_j'. \)

Let \( h_2 : [0, 1] \to t_{j+1} Q_{j+3} \) be the homeomorphism such that \( h_2(t) = (1 - t) t_{j+1} + t Q_{j+3}. \)

Let \( L_t \) be the line connecting \( h_1(t) \) and \( h_2(t). \) For each \( y \in L_t \cap Y \) let \( p_j(y) = h_1(t). \) Note that for each \( t, \) \( \operatorname{diam}(p_j^{-1}(\{h_1(t)\})) \leq \max\{r'_{j+1}, q'_{j+3}\}. \)

The definition of \( p_j \) on \( Y \cap Y_r, \) where \( Y_r \) is the closure of the bounded component of \( R^2 - (Q_j' Q_{j+1} \cup Q_j' Q_{j+2} \cup Q_j' Q_{j+3} \cup Q_j' Q_{j+4} \cup Q_j' Q_{j+5} \cup Q_j' Q_{j+6} \cup Q_j' Q_{j+7} \) \) is defined in a similar manner in the two corresponding cases.

**Case** \( p_j, Y_r, 1: r'_{j+4} \leq t'_{j} \)

In this case for \( y \in Y \cap Y_r, \) \( \operatorname{diam}(p_j^{-1}(\{p_j(y)\})) \leq \max\{r'_{j+4}, q'_{j+3}\} \)

**Case** \( p_j, Y_r, 2: t'_{j} < r'_{j+4} \)

In this case for \( y \in Y \cap Y_r, \) \( \operatorname{diam}(p_j^{-1}(\{p_j(y)\})) \leq \max\{t'_{j}, q'_{j+3}\}. \)
We see that for \( y \in Y_1 \), \( \text{diam} \left( p^{-1}_y \{(p_j(y))\} \right) \leq \max \{ \min \{ r_{j+1}', t_{j+2}' \}, q_{j+3}' \} \). For \( y \in Y_r \), \( \text{diam} \left( p^{-1}_y \{(p_j(y))\} \right) \leq \max \{ \min \{ r_{j+4}', t_j' \}, q_{j+3}' \} \). Consequently, for \( y \in Y \),
\[
\text{diam} \left( p^{-1}_y \{(p_j(y))\} \right) \leq \max \{ \min \{ r_{j+1}', t_{j+2}' \}, \min \{ r_{j+4}', t_j' \}, q_{j+3}' \}.
\]

Let \( f^*, g^* : C \to Z \) be any two continuous functions from a continuum \( C \) into a continuum \( Z \subseteq Y \), such that \( f^*(C) \subseteq g^*(C) \subseteq Z \). Consider \( p_j \circ f^*, p_j \circ g^* : C \to (Q_j, Q_j \cup Q_{j+1}) \). The image of \( p_j \circ g^*(C) \) is an arc and \( p_j \circ f^*(C) \subseteq p_j \circ g^*(C) \). Since all the spans of an arc are zero, there is a \( c \in C \) such that \( p_j \circ f^*(c) = p_j \circ g^*(c) \). Consequently, \( d(f^*(c), g^*(c)) \leq \text{diam} \left( p^{-1}_j \{p_j(g^*(c))\} \right) \leq \max \{ \min \{ r_{j+1}', t_{j+2}' \}, \min \{ r_{j+4}', t_j' \}, q_{j+3}' \} \) and
\[
\sigma_0(Z) \leq \max \{ \min \{ r_{j+1}', t_{j+2}' \}, \min \{ r_{j+4}', t_j' \}, q_{j+3}' \}.
\]

In this case (i.e., case A) when \( \text{fss}(X) = \text{dmin}(f, g) \) where \( f \) and \( g \) are defined in Table 1, we conclude that \( \sigma_0(X) \leq q_{j+3} \) by taking \( Z = X \subseteq Y \), \( j = 0 \), and using (*) . Given the inequalities relating the various spans and the fact that for each \( t \in [0, 1] \) there is a \( t' \in [0, 1] \) such that \( g(t) = f(t') \) and \( f(t) = g(t') \), we conclude that \( \sigma(X) = \sigma_0(X) = \sigma^*(X) = \sigma_0^*(X) = s(X) = s^*(X) = \text{fss}(X) = q_3 \). This completes case A.

**Case B:** \( \max \{ \text{dmin}(f, g) \mid (f, g) \in \mathcal{P} \} > q_3' \)

Let \( (f, g) \in \mathcal{P} \) such that \( \max \{ \text{dmin}(f, g) \mid (f, g) \in \mathcal{P} \} = \text{dmin}(f, g) > q_3' \).

Since \( r_3' < q_3 \) and \( t_3 < q_3' \), the pair \( (f, g) \) can not include the following (forbidden) steps as given in Table 4 and Table 5.

### Table 4. Forbidden steps

<table>
<thead>
<tr>
<th>( t )</th>
<th>( f(t) )</th>
<th>( g(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_1 )</td>
<td>( Q_4 )</td>
<td>( Q_3 )</td>
</tr>
<tr>
<td>( t_2 )</td>
<td>( Q_0 )</td>
<td>( Q_3 )</td>
</tr>
</tbody>
</table>

### Table 5. Forbidden steps

<table>
<thead>
<tr>
<th>( t )</th>
<th>( f(t) )</th>
<th>( g(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_1 )</td>
<td>( Q_3 )</td>
<td>( Q_4 )</td>
</tr>
<tr>
<td>( t_2 )</td>
<td>( Q_0 )</td>
<td>( Q_3 )</td>
</tr>
</tbody>
</table>

For any pair of functions, \( f \) and \( g \), we consider all the possible starting values of \( f \) and \( g \), that is \( f(0) \) and \( g(0) \), and determine what steps are possible for \( t > 0 \) given the restriction \( \text{dmin}(f, g) > d(Q_3, Q_4) = q_3' \). Because of the symmetry of this process, we consider either \( f(0) = Q_j \) and \( g(0) = Q_k \) where \( k \neq j \), or \( f(0) = Q_k \) and \( g(0) = Q_j \) for \( k \neq j \), but not both. All
these possibilities are summarized in the tables below. For \( n \in \mathbb{N} \), we let 
\[ 0 = t_0 < t_1 < t_2 < \cdots < t_n < 1. \]

Table 6. Left to right: Patterns I, II, III, IV.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( f(t) )</th>
<th>( g(t) )</th>
<th>( t )</th>
<th>( f(t) )</th>
<th>( g(t) )</th>
<th>( t )</th>
<th>( f(t) )</th>
<th>( g(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( Q_0 )</td>
<td>( Q_2 )</td>
<td>0</td>
<td>( Q_0 )</td>
<td>( Q_2 )</td>
<td>0</td>
<td>( Q_0 )</td>
<td>( Q_2 )</td>
</tr>
<tr>
<td>1</td>
<td>( Q_1 )</td>
<td>( Q_2 )</td>
<td>1</td>
<td>( Q_1 )</td>
<td>( Q_2 )</td>
<td>1</td>
<td>( Q_1 )</td>
<td>( Q_2 )</td>
</tr>
<tr>
<td>2</td>
<td>( Q_1 )</td>
<td>( Q_3 )</td>
<td>2</td>
<td>( Q_1 )</td>
<td>( Q_4 )</td>
<td>2</td>
<td>( Q_1 )</td>
<td>( Q_4 )</td>
</tr>
<tr>
<td>3</td>
<td>( Q_4 )</td>
<td>( Q_0 )</td>
<td>3</td>
<td>( Q_3 )</td>
<td>( Q_0 )</td>
<td>3</td>
<td>( Q_3 )</td>
<td>( Q_0 )</td>
</tr>
<tr>
<td>4</td>
<td>( Q_2 )</td>
<td>( Q_0 )</td>
<td>4</td>
<td>( Q_2 )</td>
<td>( Q_0 )</td>
<td>4</td>
<td>( Q_2 )</td>
<td>( Q_0 )</td>
</tr>
</tbody>
</table>

Table 7.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( f(t) )</th>
<th>( g(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( Q_0 )</td>
<td>( Q_1 )</td>
</tr>
<tr>
<td>1</td>
<td>( Q_0 )</td>
<td>( Q_2 )</td>
</tr>
</tbody>
</table>

Table 8. Left to right: Patterns V and VI.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( f(t) )</th>
<th>( g(t) )</th>
<th>( t )</th>
<th>( f(t) )</th>
<th>( g(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( Q_4 )</td>
<td>( Q_0 )</td>
<td>0</td>
<td>( Q_4 )</td>
<td>( Q_0 )</td>
</tr>
<tr>
<td>1</td>
<td>( Q_4 )</td>
<td>( Q_1 )</td>
<td>1</td>
<td>( Q_4 )</td>
<td>( Q_1 )</td>
</tr>
<tr>
<td>2</td>
<td>( Q_4 )</td>
<td>( Q_2 )</td>
<td>2</td>
<td>( Q_4 )</td>
<td>( Q_2 )</td>
</tr>
<tr>
<td>3</td>
<td>( Q_4 )</td>
<td>( Q_0 )</td>
<td>3</td>
<td>( Q_4 )</td>
<td>( Q_0 )</td>
</tr>
</tbody>
</table>

Table 9.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( f(t) )</th>
<th>( g(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( Q_3 )</td>
<td>( Q_0 )</td>
</tr>
<tr>
<td>1</td>
<td>( Q_4 )</td>
<td>( Q_0 )</td>
</tr>
</tbody>
</table>

We start with \( f(0) = Q_0 \) and \( g(0) = Q_2 \) in Table 6. We see that there are four possibilities: pattern I, II, III, or IV. In Table 7, where \( f(0) = Q_0 \) and \( g(0) = Q_1 \), there is only one possible second step. After this step, the functions would follow one of the patterns I, II, III, or IV in Table 6. In Table 8, we start with \( f(0) = Q_4 \) and \( g(0) = Q_0 \). We see that there are two possibilities, patterns V and VI. Each of these ends with \( f(t_3) = Q_0 \) and \( g(t_3) = Q_2 \). These functions would then follow either pattern I, II, III, or IV.
Table 10. Left to right: Patterns VII and VIII.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$f(t)$</th>
<th>$g(t)$</th>
<th>$t$</th>
<th>$f(t)$</th>
<th>$g(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$Q_4$</td>
<td>$Q_1$</td>
<td>0</td>
<td>$Q_4$</td>
<td>$Q_1$</td>
</tr>
<tr>
<td>$t_1$</td>
<td>$Q_4$</td>
<td>$Q_2$</td>
<td>$t_1$</td>
<td>$Q_0$</td>
<td>$Q_1$</td>
</tr>
<tr>
<td>$t_2$</td>
<td>$Q_0$</td>
<td>$Q_2$</td>
<td>$t_2$</td>
<td>$Q_0$</td>
<td>$Q_2$</td>
</tr>
</tbody>
</table>

Table 11.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$f(t)$</th>
<th>$g(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$Q_2$</td>
<td>$Q_1$</td>
</tr>
<tr>
<td>$t_1$</td>
<td>$Q_3$</td>
<td>$Q_1$</td>
</tr>
<tr>
<td>$t_2$</td>
<td>$Q_4$</td>
<td>$Q_1$</td>
</tr>
</tbody>
</table>

Table 12.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$f(t)$</th>
<th>$g(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$Q_3$</td>
<td>$Q_1$</td>
</tr>
<tr>
<td>$t_1$</td>
<td>$Q_4$</td>
<td>$Q_1$</td>
</tr>
</tbody>
</table>

Table 13.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$f(t)$</th>
<th>$g(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$Q_3$</td>
<td>$Q_2$</td>
</tr>
<tr>
<td>$t_1$</td>
<td>$Q_4$</td>
<td>$Q_2$</td>
</tr>
<tr>
<td>$t_2$</td>
<td>$Q_0$</td>
<td>$Q_2$</td>
</tr>
</tbody>
</table>

In Table 6. In Table 9 we have $f(0) = Q_3$ and $g(0) = Q_0$. There is only one possible second step, that is $f(t_1) = Q_4$ and $g(t_1) = Q_0$. These functions then would follow either pattern V or VI in Table 8. After this, the functions would follow one of the patterns, I, II, III, or IV in Table 6. In Table 10, we have $f(0) = Q_4$ and $g(0) = Q_1$. There are two possibilities, patterns VII or VIII. Both of these patterns end with $f(t_2) = Q_0$ and $g(t_2) = Q_2$. These functions would then follow one of the patterns I, II, III, or IV in Table 6. Similarly, the functions in Table 11 would subsequently follow one of the patterns VII or VIII in Table 10, then follow one of the patterns I, II, III, or IV in Table 6. The functions in Table 12 would follow either pattern VII or VIII in Table 10, then either pattern I, II, III, or IV in Table 6. The pairs of functions in tables 13 and 14 would each follow one of the patterns I, II, III, or IV in Table 6.

We have considered all possible combinations for $f(0)$ and $g(0)$. Note that $f(0) = Q_3$ and $g(0) = Q_4$ is not possible, since for $t_1$, $f(t_1) = Q_3$, $g(t_1) = Q_0$, and $d_{min}(f, g) \leq r_3 < d(Q_3, Q_0)$. We see that all pairs of functions $f$ and $g$, satisfying our conditions, contain one of the patterns I, II, III, or IV. So, if
there is a pair of functions \((f, g) \in \mathcal{P}\) such that \(d_{\text{min}}(f, g) > d(Q_3, Q_0^t) = q_3^t\), then \(f\) and \(g\) contain one of the patterns I, II, III, or IV. So, for any pair, \((f, g) \in \mathcal{P}\), we are now considering, we see that:

\[
d_{\text{min}}(f, g) \leq \max\{\min I, \min II, \min III, \min IV\}
\]

where

\[
d(Q_1, t_i) = t_i', \quad d(Q_1, r_i) = r_i', \quad d(Q_1, Q_{i+2}^t) = q_i^t, \quad \text{and}
\]

\[
\min I = \min\{r_0, r_1, t_1, t_2, q_1\} = \min\{r_0', r_1', t_1', t_2'\}
\]

\[
\min II = \min\{r_1', t_2', q_1, q_2, q_4\} = \min\{r_1', t_2', q_4\}
\]

\[
\min III = \min\{r_0, r_4, t_0, t_1, q_0\} = \min\{r_0', r_4', t_0', t_1'\}
\]

\[
\min IV = \min\{r_4', t_0, q_0, q_2, q_4\} = \min\{r_4', t_0, q_2\}.
\]

Note that patterns I and III are mirror images of each other, as are II and IV.

If the “best” \(f\) and \(g\) is not as given in Table 1, then since all other pairs, \(f\) and \(g\), contain one of the patterns I, II, III, or IV, then the best \(f\) and \(g\) must be one of the pairs of functions given in Table 15. Hence, the “best” pair \(f\) and \(g\) will be from either Table I, II, III, or IV in Table 15, and

\[
d_{\text{min}}(f, g) = \max\{\min I, \min II, \min III, \min IV\},
\]

where \(d_{\text{min}}(f, g) = \min I\) for Table I, \(d_{\text{min}}(f, g) = \min II\) for Table II, \(d_{\text{min}}(f, g) = \min III\) for Table III, and \(d_{\text{min}}(f, g) = \min IV\) for Table IV. Consequently, \(\alpha(X) \geq \max\{\min I, \min II, \min III, \min IV\}\), where \(\alpha = \sigma, \sigma_0, \sigma^*, \sigma_0^*, s, s^*\) since the functions \(f\) and \(g\) satisfy the requirements for all of the spans. We will show that \(\sigma_0(X) \leq \max\{\min I, \min II, \min III, \min IV\}\) and consequently that \(\alpha(X) = \max\{\min I, \min II, \min III, \min IV\}\) for \(\alpha = \sigma, \sigma_0, \sigma^*, \sigma_0^*, s, s^*\).

Since I and III are mirror images of each other as are II and IV, we need to examine only two cases,

\[
\min I = \max\{\min I, \min II, \min III, \min IV\}
\]

and

\[
\min II = \max\{\min I, \min II, \min III, \min IV\}
\]

**Case 1:** \(\min II = \max\{\min I, \min II, \min III, \min IV\} = \min\{r_1', t_2', q_4\}\).

Suppose that \(\max\{\min I, \min II, \min III, \min IV\} = \min II\), that is, the best pair of functions \(f\) and \(g\) is from Table II. So, \(d_{\text{min}}(f, g) = \min\{r_1', t_2', q_4\}\),
II = min\{r_1, t_2, q'_4\} and \(\alpha(X) \geq \min\{r_1, t_2, q'_4\}\), where \(\alpha = \sigma, \sigma_0, \sigma^*, \sigma_0^*, s,\) and \(s^*\). We will now show that \(\alpha(X) = \min\{r_1, t_2, q'_4\} = \min IV\), and \(\min IV = \min\{\sigma, \sigma_0\}\).

Observe that \(\min IV = \min\{\sigma, \sigma_0\}\), and \(\min IV = \min\{\sigma, \sigma_0, s, \sigma^*, \sigma_0^*, s, \sigma^*\}\).

<table>
<thead>
<tr>
<th>(t)</th>
<th>(f(t))</th>
<th>(g(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(Q_0)</td>
<td>(Q_2)</td>
</tr>
<tr>
<td>0.1</td>
<td>(Q_1)</td>
<td>(Q_2)</td>
</tr>
<tr>
<td>0.2</td>
<td>(Q_1)</td>
<td>(Q_3)</td>
</tr>
<tr>
<td>0.3</td>
<td>(Q_1)</td>
<td>(Q_4)</td>
</tr>
<tr>
<td>0.4</td>
<td>(Q_1)</td>
<td>(Q_0)</td>
</tr>
<tr>
<td>0.5</td>
<td>(Q_0)</td>
<td>(Q_2)</td>
</tr>
<tr>
<td>0.6</td>
<td>(Q_2)</td>
<td>(Q_1)</td>
</tr>
<tr>
<td>0.7</td>
<td>(Q_1)</td>
<td>(Q_3)</td>
</tr>
<tr>
<td>0.8</td>
<td>(Q_4)</td>
<td>(Q_1)</td>
</tr>
<tr>
<td>0.9</td>
<td>(Q_0)</td>
<td>(Q_2)</td>
</tr>
<tr>
<td>1.0</td>
<td>(Q_0)</td>
<td>(Q_2)</td>
</tr>
</tbody>
</table>

Let \(f^*, g^*: C \to Z\) be continuous functions from a continuum \(C\) into a continuum \(Z \subseteq Y\) such that \(f^*(C) \subseteq g^*(C)\). We consider three subcases.

**Case 1A:** \(\min II = r_1, r_1 \leq t_2, \) and \(r_1 \leq q'_4\).

Observe that \(\min IV = \min\{r_4, t_0, q'_2\} \leq \min II = r_1' \leq t_2' < q'_2\), so \(\min IV \neq q'_2\) and \(\min IV = \min\{r_4, t_0\}\). So, either \(r_4' \leq r_1\) or \(t_0' \leq r_1\). Since \(r_1 \leq t_2\), we define \(p_0\) on \(Y_1\) by using case \(p_0, Y_1, 1\) when \(r_1 = t_2\) and case \(p_0, Y_1, 2\) when \(r_1 < t_2\). Also, either \(t_0' \leq t_0\) or \(t_0' < r_4\).

**Case 1B:** \(\min II = t_2, t_2 < r_1, \) and \(t_2 \leq q'_4\).

Observe that \(\min IV = \min\{r_4, t_0, q'_2\} \leq \min IV \neq q'_2, \) and \(\min IV = \min\{r_4, t_0\}\). So, either \(r_4' \leq t_2' = \min IV\), and \(t_0' \leq t_2\). Since \(t_2 < r_1\), we define \(p_0\) on \(Y_1\) by using case \(p_0, Y_1, 1\). Also, either \(r_4' \leq t_0\) or \(t_0' < r_4\). We define \(p_0\) on \(Y_1\) by case \(p_0, Y_1, 1\) when \(r_4' \leq t_0\) and by case \(p_0, Y_1, 2\) when \(t_0' < r_4\). Again we conclude that there is a \(c \in C\) such that \(d(f^*(c), g^*(c)) \leq \max\{t_0, r_4, q_3, t_2\} = t_2\).

**Case 1C:** \(\min II = q_4, q_4 < r_1, \) and \(q_4 < t_2\).
Observe that \( \min I = \min \{r_0', r_1', t_1', t_2'\} \leq \min II = q_4' < r_1' \) and \( q_4' < t_2' \). \( \min I = \min \{r_0', t_1'\} \), so either \( r_0' \leq q_4' = \min(f, g) \) or \( t_1' \leq q_4' = \min(f, g) \). We define \( p_1 : Y \rightarrow \overrightarrow{Q_1Q_1} \cup \overrightarrow{Q_1Q_2} \) based on either \( r_0 \leq t_1' \) or \( t_1' < r_0' \) for \( y \in Y_r \) and either \( t_3' \leq r_2' \) or \( r_2' < t_3' \) for \( y \in Y_t \). Note that \( t_3' < q_3 < q_4' \). So we see that there is a \( c \in C \) such that \( d(f^*(c), g^*(c)) \leq \max \{r_0', t_1'\} \), \( \{\min(r_2', t_3'), q_4'\} = q_4' \).

Therefore, in case I we conclude that \( \sigma_0(X) \leq \min(f, g) = \min II \). Since \( X \) is a simple closed curve we know that \( \sigma(X) = \sigma^*(X) \) and \( \sigma_0(X) = \sigma_0^*(X) \). Also, we see that for all \( t \in [0, 1] \) there is a \( t' \in [0, 1] \) such that \( f(t') = g(t') \) and \( f(t') = g(t) \). Consequently, we see that \( \sigma(X) = \sigma_0(X) = \sigma^*(X) = \sigma_0^*(X) = s(X) = s^*(X) = \min II \).

Case 2: \( \min I = \max \{\min I, \min II, \min III, \min IV\} = \min \{r_0', r_1', t_1', t_2'\} \).

Suppose that \( \min I = \max \{\min I, \min II, \min III, \min IV\} \), that the best pair of functions \( f \) and \( g \) is from Table I. So, \( \min(f, g) = \min I = \min \{r_0', r_1', t_1', t_2'\} \) and \( \alpha(X) \geq \min \{r_0', r_1', t_1', t_2'\} \) where \( \alpha = \sigma, \sigma_0, \sigma^*, \sigma_0^*, s \) and \( s^* \). We will now show that \( \alpha(X) = \min \{r_0', r_1', t_1', t_2'\} = \min(f, g) \) where \( \alpha = \sigma, \sigma_0, \sigma^*, \sigma_0^*, s \) and \( s^* \).

Let \( f^*, g^* : C \rightarrow Z \) be continuous functions from a continuum \( C \) into a continuum \( Z \subseteq Y \) such that \( f^*(C) \subseteq g^*(C) \). We consider four subcases.

Case 2A: \( \min I = r_0', r_0' \leq t_1', r_0' \leq r_1', \) and \( r_0' \leq t_2' \).

Observe that \( \min II = \min \{r_1', t_2', q_4\} \leq \min I = r_0' \). We can assume that \( r_0' \neq r_1' \), since this case has been covered in case 1A, so \( r_0' < r_1' \). We can assume that \( r_0' \neq t_2' \), since this case has been covered in case 1B, so \( r_0' < t_2' \). So, we assume that \( \min II = q_4' \leq \min I = r_0' \). So, we have that \( q_4' \leq r_0' = \min(f, g) \). We can assume that \( q_4' \neq r_0' \), since this is covered in case 1C. So, \( q_4' < r_0' = \min(f, g) \). We define \( p_1 : Y \rightarrow \overrightarrow{Q_1Q_1} \cup \overrightarrow{Q_1Q_2} \) based on case \( p_1 : Y_r, 1 \) for \( y \in Y_r \), since \( r_0' \leq t_1' \) and based on either case \( p_1 : Y_t, 1 \) if \( t_3' \leq r_2' \) or case \( p_1 : Y_t, 2 \) if \( r_2' < t_3' \) for \( y \in Y_t \). Consider \( p_1 \circ f^*, p_1 \circ g^* : C \rightarrow \overrightarrow{Q_1Q_1} \cup \overrightarrow{Q_1Q_2} \). Since the range is an interval there is a \( c \in C \) such that \( p_1 \circ f^*(c) = p_1 \circ g^*(c) \) and \( d(f^*(c), g^*(c)) \leq \max \{t_3', r_2', q_4', r_0'\} = r_0' \).

Case 2B: \( \min I = t_1', t_1' < r_0', t_1' \leq r_1', \) and \( t_1' \leq t_2' \).

Observe that \( \min II = \min \{r_1', t_2', q_4\} \leq \min I = t_1' \). We can assume that \( t_1' \neq r_1' \), since this is already covered in case 1A, so \( t_1' < r_1' \). We can assume that \( t_1' \neq t_2' \), since this is already covered in case 1B, so \( t_1' < t_2' \). So we assume that \( \min II = q_4' \leq \min I = t_1' \). So, we have that \( q_4' \leq t_1' = \min(f, g) \). We can assume that \( q_4' \neq t_1' \), since this is covered in case 1C. So, \( q_4' < t_1' = \min(f, g) \).

We define \( p_1 : Y \rightarrow \overrightarrow{Q_1Q_1} \cup \overrightarrow{Q_1Q_2} \) based on case \( p_1 : Y_r, 2 \) for \( y \in Y_r \), since \( t_1' < r_0' \) and based on either case \( p_1 : Y_t, 1 \) if \( t_3' \leq r_2' \) or case \( p_1 : Y_t, 2 \) if \( r_2' < t_3' \) for \( y \in Y_t \). Consider \( p_1 \circ f^*, p_1 \circ g^* : C \rightarrow \overrightarrow{Q_1Q_1} \cup \overrightarrow{Q_1Q_2} \). Since the
range is an interval there is a \( c \in C \) such that \( p_1 \circ f^*(c) = p_1 \circ g^*(c) \) and
\[
d(f^*(c), g^*(c)) \leq \max\{\min\{t_3', r_2'\}, q_3', t_1'\} = t_1'.
\]

Case 2C: \( \min I = \tau_1, \tau_1 < \tau_0, \tau_1 < t_1, \) and \( \tau_1' \leq t_2' \).
This case has been covered in case 1A.

Case 2D: \( \min I = \tau_2, \tau_2' < \tau_0, \tau_2 < \tau_1, \) and \( \tau_2' < t_2' \).
This case has already been covered in case 1B.

Therefore in case 2 we conclude that \( \sigma_0(X) \leq \min I \). Also, \( \sigma(X) = \sigma^*(X) \) and \( \sigma_0(X) = \sigma_0^*(X) \) since \( X \) is a simple closed curve and for all \( t \in [0, 1] \) there is a \( t' \in [0, 1] \) such that \( f(t) = g(t) \) and \( f(t') = g(t) \).
Consequently, \( \sigma(X) = \sigma_0(X) = \sigma^*(X) = \sigma_0^*(X) = s(X) = s^*(X) = \min I \).

The other three cases can be proved in a similar manner.

Hence, \( \sigma(X) = \sigma_0(X) = \sigma^*(X) = \sigma_0^*(X) = s(X) = s^*(X) = \max\{\hat{q}_j + 3, \min I, \min II, \min III, \min IV\} = \text{fss}(X) \).

**Theorem 3.2.** Let \( Z \) be a continuum such that \( Z \subseteq Y \), where \( Y \) is the closure of the bounded component of \( R^2 - X \) and \( X \) is a five star-like simple closed curve, then \( \alpha(Z) \leq \text{fss}(X) \), where \( \alpha = \sigma, \sigma_0, \sigma^*, \sigma_0^*, s, s^* \).

**Proof.** Since each of the functions \( p_i \) for \( i = 0, 1, 2, 3, 4 \) were defined for any continuum \( Z \subseteq Y \) where \( Y \) is the closure of the bounded component of \( R^2 - X \), we see that for any such \( Z \), \( \alpha(Z) \leq \text{fss}(X) \), where \( \alpha = \sigma, \sigma_0, \sigma^*, \sigma_0^*, s, s^* \).

**Corollary 3.3.** Let \( Z \) be a simple closed curve such that \( Z \subseteq Y \) where \( Y \) is the closure of the bounded component of \( R^2 - X \) and \( X \) is a five star-like simple closed curve, then
\[
\sigma(Z) \leq \sigma(X) .
\]
So, in this situation, the question by Howard Cook is answered in the affirmative.

**Corollary 3.4.** Let \( Y \) be a plane separating continuum and \( X \) be a five star-like simple closed curve contained in the closure of a bounded component of \( R^2 - Y \), then
\[
\sigma(X) \leq \sigma(Y) .
\]

**Proof.** Let \( Y \) be a plane separating continuum such that \( X \) is contained in one of the bounded components of \( R^2 - Y \). We can assume that the origin, \( O \), is in the bounded component of \( R^2 - Q \). There is an \( r > 0 \) such that the circle with center the origin and radius \( r \), \( C(O, r) \), is such that \( Y \) is contained in the bounded component of \( R^2 - C(O, r) \). We will show that \( \sigma(Y) \geq \sigma(X) \).

We will consider the case where \( \sigma(X) = \min II \). The lines \( \overrightarrow{Q_3Q_1} \) and \( \overrightarrow{Q_3Q_4} \) separate the bounded component of \( R^2 - C(O, r) \) into four wedges. We let \( V_0 \) be the closure of the wedge that contains \( Q_4 \) and \( Q_0 \) and \( W_0 \) be the closure
of the wedge that is opposite \( V_0 \). The set \( V_0 - (\overline{Q_1Q_2} \cup \overline{Q_3Q_4} \cup \overline{Q_4Q_5} \cup \overline{Q_5Q_6}) \) has two components. Let \( W_0^r \) be the closure of the component that intersects \( C(O, r) \). For each \( z \in W_0 \cap Y \), let \( r(z) \in \overline{Q_3z} \cap W_0 \cap C(O, r) \) and for each \( z \in W_0^r \cap Y \), let \( r(z) \in \overline{Q_3z} \cap W_0^r \cap C(O, r) \). The lines \( \overline{Q_3Q_4} \) and \( \overline{Q_4Q_5} \) separate the bounded component of \( R^2 - C(O, r) \) into four wedges. Let \( V_1 \) be the closure of the wedge that contains \( Q_4^r \). Let \( W_1 \) be the closure of the wedge that is opposite \( V_1 \). The set \( \overline{Q_3Q_1} \cup \overline{Q_1Q_2} \) separates \( V_1 \) into two components. Let \( W_1^r \) be the closure of the component that intersects \( C(O, r) \). For each \( z \in W_1 \cap Y \), let \( r(z) \in \overline{Q_3z} \cap W_1 \cap C(O, r) \), and for each \( z \in W_1^r \cap Y \), let \( r(z) \in \overline{Q_3z} \cap W_1^r \cap C(O, r) \). Similarly, using the lines \( \overline{Q_1Q_4} \) and \( \overline{Q_4Q_5} \), we define the sets \( W_2 \) and \( W_2^r \). Note that \( Q_1 \in W_2 \) and \( Q_2, Q_3 \in W_2^r \). We define \( r \) for \( z \in W_2 \cap Y \) by \( r(z) \in \overline{Q_3z} \cap W_2 \cap C(O, r) \) and for \( z \in W_2^r \cap Y \) by \( r(z) \in \overline{Q_3z} \cap W_2^r \cap C(O, r) \). Observe that for either \( x \in W_0 \) and \( y \in W_0^r \cap \overline{Q_3x} \), or \( x \in W_1 \) and \( y \in W_1^r \cap \overline{Q_3x} \), \( d(x, y) \geq \min II \).

We can rotate \( X \cup Y \) in the plane about the origin so that the ray \( \overrightarrow{Ox} \) corresponds to the positive x-axis, where \( \{ x \} = W_0 \cap W_1^r \cap \overline{Q_1Q_2} \cap C(O, r) \). Let \( \theta_j \) for \( j = 0, 1, 2, 3, 4, 5, 6 \), be angles in the rotated position such that \( 0 = \theta_0 < \theta_1 < \theta_2 < \theta_3 < \theta_4 < \theta_5 < \theta_6 = 2\pi \), where \( re^{i\theta_0}, re^{i\theta_3} \in \overline{Q_1Q_2} \cap C(O, r) \), \( re^{i\theta_1}, re^{i\theta_4} \in \overline{Q_3Q_4} \cap C(O, r) \), and \( re^{i\theta_2}, re^{i\theta_5} \in \overline{Q_4Q_5} \cap C(O, r) \). Let \( f : [0, 2\pi] \rightarrow [0, 2\pi] \) be a piecewise linear, surjective function, such that \( f(\theta_j) = j(\xi) \), and for \( \theta \in [\theta_j, \theta_{j+1}] \), \( f(\theta) = j(\xi) + (\theta - \theta_j)(\xi) \) for each \( j = 0, 1, 2, 3, 4, 5 \).

Let \( h : C(O, r) \rightarrow C(O, r) \) be the homeomorphism given by \( h(re^{i\theta}) = h(re^{i\theta_j}) \).

We define \( p : Y \rightarrow C(O, r) \) by \( p(y) = h \circ r(y) = h(re^{i\theta y}) = re^{i\theta_j} \) where \( r(y) = re^{i\theta y} \). We see that \( p \) is an essential map from \( Y \) onto \( C(O, r) \). Also, for each \( y, z \in Y \) such that \( p(y) \) and \( p(z) \) are diametrically opposed to each other on \( C(O, r) \), \( d(p^{-1}(p(y)), p^{-1}(p(z))) \geq \min II \).

So in this case, by Theorem L, \( \sigma(Y) \geq \sigma(X) = \min II \). The cases where \( \sigma(X) = \min \{I_1 \} \), min I, min III, or min IV can be shown in a similar manner. We conclude that in all cases \( \sigma(Y) \geq \sigma(X) \).

When \( Y \) is a simple closed curve, the question by Howard Cook is answered in the affirmative in this situation.

References


T. West  
Department of Mathematics  
University of Louisiana at Lafayette  
LA 70504-1010  
USA  
E-mail: ThelmaWest@Yahoo.com  

Received: 23.5.2006.  