Triangles from products of sides with cevians

Zvonko Ćerin*

Abstract. We consider the problem of determining for which central points $X$ of the triangle $ABC$ the products of lengths of sides and cevians of $X$ will be sides of a triangle. We shall prove that twenty-seven out of hundred and one central points from the Kimberling’s list have this property. The algebraic method of proof for this result is also used to obtain some new examples of three segments that are sides of a triangle and are built from elements of a given triangle.

Key words: triangle, cevian, triangular, central point, isogonal conjugate

1. Introduction

One of the basic problems in triangle geometry is to decide when three given segments are sides of a triangle. The opening chapter of the book *Recent Advances in Geometric Inequalities* by Mitrinović, Pečarić, and Volenec [6] gives an extensive survey of results on this question.

The present article is looking for ways of associating to a triangle $ABC$ a point $X$ of the plane such that the products $a|AX_a|$, $b|BX_b|$, and $c|CX_c|$ of the lengths of segments $AX_a$, $BX_b$, and $CX_c$ with the lengths $a$, $b$, $c$ of the corresponding sides of

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ABC are always sides of a triangle, where \(X_a, X_b,\) and \(X_c\) are the intersections of lines \(AX, BX,\) and \(CX\) with the sidelines \(BC, CA,\) and \(AB,\) respectively. Recall [3] that the segments \(AX_a, BX_b,\) and \(CX_c\) are called the cevians of the point \(X.\)

The orthocenter \(H\) is an easy example of such a point \(X.\) Indeed, the segments \(AH_a, BH_b,\) and \(CH_c\) are the altitudes so that the products \(a|AH_a|, b|BH_b|,\) and \(c|CH_c|\) being all equal to twice the area of \(ABC\) are sides of an equilateral triangle.

Since \(H\) is just one of central points of a triangle \(ABC\) listed in Table 1 of [4], we can state a problem that we completely answer in this paper.

**Problem 1.** For what natural numbers \(i\) less than 102 will the central point \(X_i\) of the triangle \(ABC\) from the Kimberling’s list have the property that the products \(a|AX_{ia}|, b|BX_{ib}|,\) and \(c|CX_{ic}|\) of the sides with the cevians of \(X_i\) are sides of a triangle?

We shall get the solution of this problem with an entirely algebraic proof in an analytic approach. Our main result is the following theorem.

**Theorem 1.** From 101 centers \(X_i\) of the triangle \(ABC\) from the Kimberling’s Table 1, only the values 2, 4, 6, 7, 8, 10, 19, 25, 27, 28, 31, 32, 34, 39, 42, 56, 57, 58, 65, 69, 75, 76, 78, 81, 82, 83, and 89 of the index \(i\) have the property that the products \(a|AX_{ia}|, b|BX_{ib}|,\) and \(c|CX_{ic}|\) are sides of a triangle regardless of the shape of \(ABC.\)

Perhaps, some readers will be disappointed with our method which we outline below. They can rescue the old-fashioned geometry with their own more traditional proofs. However, they should give the author freedom to prove these (we hope) new results in any correct way including the present that could be easily followed with personal computers with very modest hardware requirements (Pentium 90Mhz with 32MB memory extension) and with the already standard software package Maple V (version 4). We also beg pardon to those wishing more details with the simple excuse that writing them all down would make this paper too long to be taken for publication in any journal.

With the power of computers at our disposal, we can now consider and open up new areas of research in geometry of triangles (see [1] and [7]). This paper is simply an example of such a computer aided discovery in mathematics (see [2] and [5]).

The author is thankful to Professors G. M. Gianella and V. Volenec for help during the work on this paper.

2. Preliminaries

For an expression \(f,\) let \([f]\) denote a triple \((f, \varphi(f), \psi(f)),\) where \(\varphi(f)\) and \(\psi(f)\) are cyclic permutations of \(f.\) For example, if \(f = \sin A\) and \(g = b + c,\) then

\[
[f] = (\sin A, \sin B, \sin C) \quad \text{and} \quad [g] = (b + c, c + a, a + b).
\]

Let us call a triple \([a]\) of real numbers triangular provided \(a, b,\) and \(c\) are sides of a triangle. The letter \(\Omega\) is reserved for the set of all triangular triples.

Let \(T\) denote a function that maps each triple \([a]\) of real numbers to a number

\[
2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4.
\]
Since \( T([a]) = (a + b + c)(b + c - a)(a - b + c)(a + b - c) \), it is clear that for a triple \([a]\) of positive real numbers \([a] \in \Omega\) if and only if \( T([a]) > 0\). Let \( T_i \) be a short notation for \( T([a | AX_i a]|) \), where \( X_i \) is the \( i \)-th central point of \( ABC \) and \( i = 1, \ldots, 101 \).

### 3. Discussion of the general problem

Some readers might be tempted in thinking that the general problem in the second paragraph of the introduction has an easy solution so that our task of identifying only twenty-seven strange points in the plane of the triangle \( ABC \) which have the required property might appear insignificant. In this section we shall argue that the general problem is extremely complicated and that even with computers it is impossible to describe precisely its solution so that our modest theorem is the only information and positive result available.

For a moment, assume that the vertices of \( ABC \) are at the complex numbers \( u, v, \) and \( w \) on the unit circle in the Gauss complex plane. Let \( X \) be a point whose affix is a complex number \( x \) and let \( y \) denote the complex conjugate \( \bar{x} \) of \( x \). Then the product of the lengths of the side \( BC \) and the cevian \( AX_a \) is

\[
|BC| \cdot |AX_a| = \sqrt{(v - w)^2 (w - u)^2 (u - v)^2 (u - x) (u y - 1) - u v w (u x + uv wy - u^2 - v w)^2}.
\]

Of course, the other two products \(|CA| \cdot |BX_b|\) and \(|AB| \cdot |CX_c|\) are the cyclic permutations of \(|BC| \cdot |AX_a|\). The triangle test has the form

\[
T(|BC| \cdot |AX_a|) = \frac{-(v - w)^4 (w - u)^4 (u - v)^4 (\sum_{i=0}^{10} \sum_{j=0}^{\kappa_i} \lambda_{ij} y^j) x^i)}{(u v w)^2 p_a^4 p_b^4 p_c^4},
\]

where the numbers \( \kappa_i \) are given in the Table 1, the coefficients \( \lambda_{ij} \) are polynomials in \( u, v, \) and \( w, \) and \( p_a = u x + uv wy - u^2 - v w \) is the equation of the parallel through \( A \) to \( BC \), while \( p_b \) and \( p_c \) are its cyclic permutations.

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Table 1.

When completely expanded the above polynomial in \( x \) and \( y \) has 4365 terms so that it is difficult for printing (it would take several pages!). This polynomial is the equation of the curve of order 12 which is the boundary for the region where \( T(|BC| \cdot |AX_a|) > 0 \). The following figure shows the triangle \( ABC \), its anticomplementary triangle \( A_a B_a C_a \) (whose sidelines must be excluded), and this curve. Not much could be said about its properties because of the enormous size of its equation. In particular, it is not true that all of the interior of \( ABC \) is in the above...
region as the figure might suggest.

Figure 1. $ABC$, $A_dB_dC_a$ (dashed), and the curve $T([|BC| \cdot |AX_a|]) = 0$.

4. Placement of $ABC$

We shall position the triangle $ABC$ in the following fashion with respect to the rectangular coordinate system in order to simplify our calculations. The vertex $A$ is the origin with coordinates $(0, 0)$, the vertex $B$ is on the $x$-axis and has coordinates $(rh, 0)$, and the vertex $C$ has coordinates $(gqr/k, 2fgr/k)$, where $h = f + g$, $k = fg - 1$, $p = f^2 + 1$, $q = g^2 - 1$, $s = g^2 + 1$, $t = g^2 - 1$, $u = f^4 + 1$, and $v = g^4 + 1$. The three parameters $r$, $f$, and $g$ are the inradius and the cotangents of half of angles at vertices $A$ and $B$. Without loss of generality, we can assume that both $f$ and $g$ are larger than 1 (i.e., that angles $A$ and $B$ are acute).

Nice features of this placement are that all central points from Table 1 in [4] have rational functions in $f$, $g$, and $r$ as coordinates and that we can easily switch from $f$, $g$, and $r$ to side lengths $a$, $b$, and $c$ and back with substitutions

\[
a = \frac{rs}{k}, \quad b = \frac{rg}{k}, \quad c = rh,
\]

\[
f = \frac{(b+c)^2 - a^2}{\sqrt{T([a])}}, \quad g = \frac{(a+c)^2 - b^2}{\sqrt{T([a])}}, \quad r = \frac{\sqrt{T([a])}}{2(a + b + c)}.
\]

Moreover, since we use the Cartesian coordinate system, computation of distances of points and all other formulas and techniques of analytic geometry are available.
and well-known to the widest audience. A price to pay for these conveniences is that symmetry has been lost.

The third advantage of the above position of the base triangle is that we can easily find coordinates of a point with given trilinears. More precisely, if a point $P$ with coordinates $x$ and $y$ has projections $P_a$, $P_b$, and $P_c$ onto the sidelines $BC$, $CA$, and $AB$ and $\lambda = PP_a/PP_b$ and $\mu = PP_b/PP_c$, then

$$x = \frac{gh(p\mu + q)r}{fs\lambda\mu + gpp + h}, \quad y = \frac{2fghr}{fs\lambda\mu + gpp + h}.$$

These formulas will greatly simplify our exposition because there will be no need to give explicitly coordinates of points but only its first trilinear coordinate. For example, we write $X_6[a]$ to indicate that the symmedian point $X_6$ has trilinears equal to $a:b:c$. Then we use the above formulas with $\lambda = a/b$ and $\mu = b/c$ to get the coordinates

$$\left(\frac{fqt + 2gur}{2(f^2v + f^2u + gq^2)}, \frac{fgkr}{2(f^2v + f^2u + gq^2)}\right).$$

of $X_6$ in our coordinate system.

5. The elimination of 74 central points

An easy task is to eliminate 74 central points $X_i$ by exhibiting a triangle for which $T_i \leq 0$. In fact, only eight triangles all with $r = 1$ and

$$t_1 = \{f = 10^6, g = 10\}, \quad t_2 = \{f = \frac{10^6 + 1}{10^6}, g = 10\}, \quad t_3 = \{f = 2, g = 20\},$$

$$t_4 = \{f = 2, g = 7\}, \quad t_5 = \{f = 2 + \sqrt{7}, g = \sqrt{7}\}, \quad t_6 = \{f = 10^3, g = \frac{102}{100}\},$$

$$t_7 = \{f = 30, g = 300\}, \quad t_8 = \{f = 3, g = 16\},$$

will suffice. Indeed, for $j = 1, \ldots, 8$, $T_i \leq 0$ or $T_i$ is not well-defined for the triangle $t_j$ and $i \in I_j$, where

$$I_0 = \{1, \ldots, 101\}, \quad I_2 = \{3, 22, 23, 26, 40, 52, 64, 68, 73, 77\},$$

$$I_3 = \{24, 48, 50, 74, 93, 95, 97, 98, 101\}, \quad I_4 = \{51, 67, 70, 94\},$$

$$I_5 = \{20, 30, 53, 96\}, \quad I_6 = \{5, 59, 84\}, \quad I_7 = \{41, 88\}, \quad I_8 = \{54\},$$

$$I_9 = \{2, 4, 6, 7, 8, 10, 19, 25, 27, 28, 31, 32, 34, 39, 42, 56, 57, 58, 65, 69, 75, 76, 78, 81, 82, 83, 89\},$$

and $I_1 = I_0 - I_2 - I_3 - I_4 - I_5 - I_6 - I_7 - I_8 - I_9$.

The above statement is simple to state but the reader should be aware that there is a lot of work behind it because we must know coordinates of each central point from the Kimberling’s list. Under the assumption that one believes that the above claim is true, we can proceed to show that for indices in the set $I_9$ the triangle test $T_i$ is positive regardless of the shape of $ABC$. 

6. $X_2\left[\frac{1}{a}\right]$ - the Centroid

$X_2$ is the intersection of medians which join vertices with midpoints of opposite sides. Its coordinates are $(r \left( k (h + f) + f - g \right) / (3k), 2rfg / (3k))$. Hence,

$$T_2 = \frac{4r^8 f^2 g^2 h^2 (f^2 v + f g q t + g^2 u)^2}{k^6}$$

is always positive.

7. $X_4\left[\sec A\right]$ - the orthocenter

$X_4$ is the intersection of altitudes which are perpendiculars through vertices to opposite sides. Hence,

$$T_4 = \frac{48r^8 f^4 g^4 h^4}{k^4}$$

is always positive.

8. $X_7\left[\sec^2 \frac{A}{2}\right]$ - the Gergonne point

$X_7$ is the concurrence point of lines $AA_p$, $BB_p$, and $CC_p$, where $A_p$, $B_p$, and $C_p$ are projections of the incenter $I$ onto the sidelines $BC$, $CA$, and $AB$, respectively. One can easily find that

$$T_7 = \frac{16f^2 g^2 h^2 r^8 (f^2 g^2 + f^2 + g^2 + fg)^2}{k^4}$$

is always positive.

9. $X_8\left[csc^2 \frac{A}{2}\right]$ - the Nagel point

$X_8$ is the intersection of lines $AA_{\text{ex}}$, $BB_{\text{ex}}$, and $CC_{\text{ex}}$, where $A_{\text{ex}}$, $B_{\text{ex}}$, and $C_{\text{ex}}$ are projections of excenters $A_\text{e}$, $B_\text{e}$, and $C_\text{e}$ onto sidelines $BC$, $CA$, and $AB$, respectively. One can easily find

$$T_8 = \frac{16f^4 g^4 h^4 r^8 S_8}{k^6},$$

where $S_8 = (k^2 - k + 1)h^2 - 3k^3$. Let us use the inequality $\frac{h}{2} \geq \sqrt{k + 1}$ between the arithmetic and geometric means of positive numbers $f$ and $g$ to write $h = 2\sqrt{k + 1} + \eta$ for some $\eta \geq 0$. When we substitute this value for $h$ into $S_8$ it becomes

$$(k^2 - k + 1)\eta^2 + 4\sqrt{k + 1} + k(k^2 - k + 1)\eta + 4 + k^3.$$  

Since the polynomial $k^2 - k + 1$ is always positive we conclude that $S_8$ and therefore also $T_8$ is always positive.
10. The central point $X_{69}[\cos A \csc^2 A]$

$X_{69}$ is the isogonal conjugate of the central point $X_{25}$ - the center of homothety of the orthic triangle $A_1B_1C_1$ and the tangential triangle $A_iB_iC_i$ of a given triangle $ABC$. It can also be described as the intersection of the line joining the Gergonne point $X_7$ with the Nagel point $X_8$ and the line joining the centroid $X_2$ with the Grebe-Lemoine point $X_6$.

In the same way as above we obtain

$$T_{69} = \frac{16 f^2 g^2 h^2 r^8 S_{69}}{k^6},$$

where $S_{69} = (k^2 + k + 1)(k^2) h^4 - k^2(1 + k) (2 k^2 + 11 k + 11) h^2 + k^4 (1 + k)^2$.

In order to see that $S_{69} > 0$, we shall use again the inequality $\frac{k}{k+1} \geq \sqrt{k+1}$ between the arithmetic and geometric means of positive numbers $f$ and $g$ to represent the sum as $2 \sqrt{k+1} + \eta$ for some $\eta \geq 0$. When we put this value for $h$ into $S_{69}$ this expression becomes

$$(k^2 + k + 1)^2 \eta^4 + 8 \sqrt{k+1} (k^2 + k + 1) \eta^3 + (1 + k) (22 k^4 + 37 k^3 + 61 k^2 + 48 k + 24) \eta^2 + 4 (1 + k)^{3/2} (6 k^4 + 5 k^3 + 13 k^2 + 16 k + 8) \eta + (9 k^4 - 12 k^3 + 4 k^2 + 32 k + 16) (1 + k)^2.$$

The first three terms of the first parenthesis of the trailing coefficient of this polynomial in $\eta$ is $(3k-2)^2 k^2$ so that $S_{69}$ and therefore also $T_{69}$ is obviously always positive.

11. $X_{75}[\frac{1}{a^2}]$ - the isogonal conjugate of the 2nd power point

$X_{75}$ is the isogonal conjugate of the 2nd Power Point $X_{31}[a^2]$. It can also be described as the intersection of the line joining the Gergonne point $X_7$ with the Nagel point $X_8$ and the line joining the Spieker center $X_{10}$ with the 3rd Brocard point $X_{76}$. It follows that

$$T_{75} = \frac{16 f^2 g^2 h^2 p^2 r^8 S_{75}}{k^6 (k+2)^4 (p+2k)^4 (s+2k)^4},$$

where $S_{75} = \sum_{i=0}^{5} k_i h^{2i} k^{\lambda_i}$ with $\lambda_i = 10, 8, 6, 4, 2, 0, \text{ for } i = 0, \ldots, 5$ and $k_i$ is a (product of) polynomial(s) in the variable $k$ represented as sequences $(a_0, \ldots, a_n)$ of their integer coefficients as follows: $[k_0] (2,1)^2 (1,1)^2 [k_1] -2 (1,1) (86, 232, 248, 133, 36, 4)$ $[k_2] (296, 1128, 1830, 1648, 888, 280, 45, 2)$ $[k_3] -2 (204, 1204, 3246, 5060, 4895, 2990, 1131, 245, 24)$ $[k_4] (148, 972, 2837, 4698, 815, 3152, 1293, 294, 20, -4)$ $[k_5] 2 (-1,1)^2 (1,1)^2 (1,6,6,2) (2,6,6,1)$.

It is not clear how one can argue that the polynomial $S_{75}$ is always positive. But, the following miraculous method will accomplish this goal.
Write $S_{75}$ in terms of $f$ and $g$. We get a polynomial $U_{75}$ with 138 terms. Since both $f$ and $g$ are larger than 1, we shall replace them with $1 + f'$ and $1 + g'$, where new variables $f'$ and $g'$ are positive. This substitution will give us a new polynomial $V_{75}$ with 386 terms only 17 of which have negative coefficients. If all coefficients were positive, we would be done. In order to get rid of these 17 troublesome terms, we must perform two more substitutions that reflect cases $f' \geq g'$ and $g' \geq f'$. Hence, if we replace $f'$ with $g' + \delta$ for $\delta \geq 0$, from $V_{75}$ we shall get a polynomial $P_{75}$ in $g'$ and $\delta$ with 437 terms and all coefficients positive. Similarly, if we substitute $g'$ with $f' + \varepsilon$ for $\varepsilon \geq 0$, from $V_{75}$ we shall get a polynomial $Q_{75}$ in $f'$ and $\varepsilon$ also with 437 terms and all coefficients positive. This concludes our proof that $T_{75} > 0$.

12. $X_{10}\left[\frac{b+c}{a}\right]$ - the Spieker center

$X_{10}$ is the incenter of the medial triangle $A_mB_mC_m$ whose vertices are midpoints of sides. It follows that

$$T_{10} = \frac{16 f^4 g^4 h^4 r^8 S_{10}}{k^6 (3k + 2)^3 (3p + 2k)^4 (3s + 2k)^4},$$

where $S_{10} = \sum_{i=0}^{6} k_i h^{2i} k^{\lambda_i}$ with $\lambda_i = 12, 10, 8, 4, 2, 0, 0$ for $i = 0, \ldots, 6$ and $k_i$ is a (product of) polynomial(s) in the variable $k$ represented as sequences $(a_0, \ldots, a_n)$ of their integer coefficients as follows: $[k_0] = (3, 4) (16, 48, 39, 4) [k_1] = (2, 880, 4432, 8363, 7138, 2569, 260) [k_2] = (22544, 128368, 289059, 324460, 187350, 59056, 4669) [k_3] = (36864, 319488, 1212448, 2682464, 3958282, 4310828, 3704374, 2479304, 1171308, 332880, 41760) [k_5] = (3, 4, 3) (12, 31, 17) (17, 31, 12).

The method of proof that $T_{10} > 0$ is the same as the above proof for $T_{75} > 0$. Polynomials $U_{10}, V_{10},$ and $P_{10},$ are somewhat larger having 171, 405, and 479 terms. This time, only eight terms of $V_{10}$ have negative coefficients.

13. The proofs for the other points

The proofs that $T_j > 0$ for $j = 6, 19, 25, 27, 28, 31, 32, 34, 39, 42, 56, 57, 58, 65, 76, 78, 81, 82, 83,$ and 89 use the same method and follow the same steps as the above proofs for $j = 75$ and 10. The only difference is that polynomials $S_j, U_j, V_j, P_j,$ and $Q_j$ are much larger so that in some cases even with our efficient notation it is practically impossible to write them down. The above table gives information on their sizes.
14. The new triangular triples

We can now compute the lengths of cevians of central points $X_j$ for $j = 2, 8, 7, 69, 75, 10, 6, 25, 19, 31, 32, 39, 42, 56, 57, 58, 76, 65, 81,$ and $83$ and apply the transformation formulas to get the following corollary.

**Corollary 1.** If the triple $[a]$ is triangular, then the triples

$$\left[ a \sqrt{2(b^2 + c^2) - a^2} \right], \quad \left[ \sqrt{2(b - c)^2 + a(b + c - a)} \right],$$

$$\left[ \sqrt{3b^2 + 3c^2 - 2bc} \right], \quad \left[ a \sqrt{(b^2 - bc + c^2)(b + c)^2 - a^2} \right],$$

$$\left[ \sqrt{2(b^2 + c^2)a^2 + 3(b^2 - c^2)^2 - a^4} \right], \quad \left[ \frac{a \sqrt{(b^2 - bc + c^2)(b + c)^2 - a^2}bc}{b + c} \right],$$

$$\left[ \sqrt{(2b^2 + 2c^2 - bc)(c + b)(b^2 - bc + c^2)a^2 - a^4} \right], \quad \left[ \frac{a \sqrt{2a^2(b^2 + c^2) + 3(b^2 - c^2)^2 - a^4}}{a^2(b^2 + c^2) + (b^2 - c^2)^2} \right].$$

**Table 2.**

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Table 2.
\[ \sqrt{a^5 (b+c)^2 + a^3 (b^2 - c^2)^2 - a (b+c)^7 (b-c)^4 - a^7} \]
\[ \frac{\sqrt{(b+c)^2 (b^2 - b+c + c^2) - a^2 b c}}{(b+c) (b^2 - b+c + c^2)} \]
\[ \frac{\sqrt{a^4 (b^2 + c^2) + a^2 (b^3 + 5 b^2 c^2 + c^3) + 2 b^2 c^2 (b^2 + c^2) - a^6}}{a^2 (b^2 + c^2) + 2 b^2 c^2} \]
\[ \frac{\sqrt{a^2 (2 b^2 - b+c + 2 c^2) + a (b+c)^4 - a^4 (b+c) - a^4 + b c (b+c)^2}}{a (b^2 + c^2) + b c (b+c)} \]
\[ \frac{\sqrt{a (2 b^2 - b+c + 2 c^2) - a^4 + b^3 + c^3}}{a (b^2 + c^2) + b^3 + c^3} \]
\[ \frac{\sqrt{a^2 (2 b^2 - b+c + 2 c^2) + a b c (b+c) - a^4 - (b^2 + c^2) (b-c)^4}}{a (b+c) + b^2 + c^2} \]
\[ \frac{\sqrt{a (a^2 (b^2 + b c + c^2) + a (b+c)^3 + 2 b c (b^2 + c^2) - a^3 (b+c) - a^4)}}{a (b+c) + b^2 + c^2} \]
and
\[ \frac{a \sqrt{a^4 (b^2 + c^2) + a^2 (b^4 + 5 b^2 c^2 + c^4) + 2 b^2 c^2 (b^2 + c^2) - a^6}}{2 a^2 + b^2 + c^2} \]

are also triangular.

**Corollary 2.** Products of sides with corresponding medians, symmedians, Gergonne cevians, Nagel cevians, and Spieker cevians are sides of a triangle.

**References**


