Mathematical foundations of the methods for multicriterial decision making*

TIHOMIR HUNJAK†

Abstract. *In this paper the mathematical foundations of the methods for multicriterial decision making are presented.*

Key words: *multicriterial decision making, Analytic Hierarchy Process, ELECTRE method, PROMETHEE method, TOPSIS method*

Sažetak. *U ovom radu prikazuju se matematičke osnove metoda za višekriterijalno odlučivanje.*

Ključne riječi: *višekriterijalno odlučivanje, analitički hijerarhijski proces, metoda ELECTRE, metoda PROMETHEE, metoda TOPSIS*

1. Introduction

The multicriteria decision making problem has been observed in this paper:

$$\max\{f_1(a), f_2(a), \cdots, f_k(a) : a \in A\}. \tag{1}$$

A is a set of n possible decision alternatives, f_1, f_2, ..., f_k are criteria by which the alternatives are evaluated. If all the criteria are not equally important, their weights can be marked by w_1, w_2, ..., w_k. The basic information for this kind of decision problem can be shown in the form of an evaluation table:

	$f_1(.)$	$f_2(.)$	...	$f_j(.)$	...	$f_k(.)$
	w_1	w_2	...	w_j	...	w_k
a_1	$f_1(a_1)$	$f_2(a_1)$	$\cdots$	$f_j(a_1)$	$\cdots$	$f_k(a_1)$
a_2	$f_1(a_2)$	$f_2(a_2)$	$\cdots$	$f_j(a_2)$	$\cdots$	$f_k(a_2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_i	$f_1(a_i)$	$f_2(a_i)$	$\cdots$	$f_j(a_i)$	$\cdots$	$f_k(a_i)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_n	$f_1(a_n)$	$f_2(a_n)$	$\cdots$	$f_j(a_n)$	$\cdots$	$f_k(a_n)$

Table 1. *Evaluation table*

*The lecture presented at the MATHEMATICAL COLLOQUIUM in Osijek organized by Croatian Mathematical Society – Division Osijek, April 11, 1997.

†Department of Mathematics, Faculty of Organisation and Informatics, Pavlinska 2, HR-42 000 Varaždin, Croatia, e-mail: `Tihomir.Hunjak@foi.hr`

To solve the problem (1) means to choose the best alternative or to rank all the alternatives. This kind of decision problem solving, including a discrete set of explicitly described alternatives, is different from the multicriteria optimization problem solving where the set of alternatives has been determined implicitly by constraints. To solve the second problem the vector optimization was used. The efficient solutions or Pareto optimal solutions of the vector optimization problem can be characterized in a few ways which enable their identification, but the concept is not usable to solve the problem (1). To solve the problem (1) it is necessary to use the procedure which meets the following criteria: (i) aberrations in evaluation of alternatives according to the individual criteria should be considered, (ii) the scaling effect, which occurs as a result of the use of different measuring scales used for evaluation of alternatives according to different criteria, must be eliminated and (iii) there must exist a possibility for a clear interpretation of the weight of the criteria.

It is necessary to stress that the method must enable that for each pair of alternatives $a, b \in A$ during their comparing , one of the following statements can be chosen:

- aPb or bPa - a is preferred to b or vice versa
- aIb - a and b are indifferent
- aRb - a and b are incomparable.

The relation $P \cup I \cup R$ is a partial ranking relation, and one can say that a preference structure has been introduced into the set A of alternatives together with it.

The main concepts for problem solving (1) satisfying the mentioned conditions are:

(a) multiattribute utility function (value function) for deterministic case,

(b) scalarizing of the problem (1),

(c) introduction of an outranking relation into the set A.

(a) It is necessary to assess a value function $v(\cdot)$ of the form

$$v(a) = w_1 v_1(f_1(a)) + w_2 v_2(f_2(a)) + \ldots + w_k v_k(f_k(a))$$

where $v_j(f_j(a))$ are assumed to be strictly positive, and maximise it over the set A. The Saaty method known as the Analytic Hierarchy Process (AHP) has been shown as an example of the method explicable by this theoretical approach

(b) A few methods have been developed which characterize the efficient solutions of the multicriteria programming problem by its transformation into a scalar form. The compromise programming i.e. the search of the solution closest to the ideal solution, is the method which can also be used for discrete problem solving. The example of this concept for problem solving can be recognised in the TOPSIS method.

(c) According to the information from the evaluation table, the relation which ranks alternatives partially or totally has been introduced into the set A. The methods ELECTRE (I and II) and PROMETHEE (I and II) are described in this paper as illustratory examples of this approach.

2. Analytic hierarchy process

The Analytic Hierarchy Process (AHP) developed by T. Saaty [1] is useful as a decision making methodology when multiple costs and multiple benefits are relevant for determining the priorities of the alternatives.

The basic steps in constructing and examining an AHP model are: (1) decompose the problem into a hierarhical structure, (2) perform judgments to establish priorities for the elements of the hierarchy, (3) synthesis of the model, (4) perform a sensitivity analysis.

There exist different types of AHP hierarchies, but for this paper it is enough to understand the basic AHP model which includes the goal, criteria and alternatives (*Figure 1*).

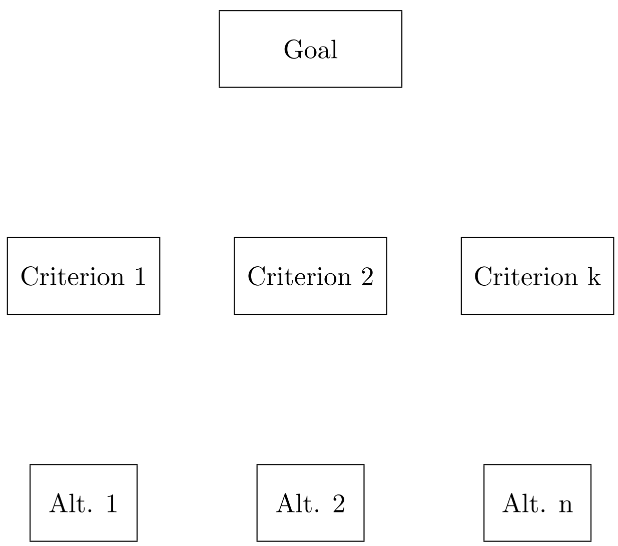

Figure 1. *Basic AHP model with goal, criteria and alternatives*

After constructing AHP hierarchy, it is necessary to perform judgments to establish priorities for the elements of the hierachy. The decision maker's judgments about the relative importance between two elements in each pair of all elements on the same level of the hierarchy are expressed by the help of the following nine-point intensity scale (The Fundamental Scale):

Definition	*Intensity of preference*
equally important	1
moderately more important	3
strongly more important	5
very strongly more important	7
extremely more important	9

Table 2. *Fundamental Scale*

Once all judgments have been performed, they are all synthesized by the help of a related mathematical model (2) which is briefly described here.

Let n be the number of criterion (or alternatives) for which we want to find the weights w_i. Let $a_{ij} = w_i/w_j$, where w_i is the weight of i-th criterion (or priority of i-th alternative) be the element of matrix A. The pairwise comparison matrix A and vector w satisfy the equation

$$Aw = nw \tag{2}$$

Because of a special form of the matrix A (each row is a constant multiple of the first row, all elements are positive and $a_{ij} = 1/a_{ji}$), rank of A is one, all eigenvalues are zero, except for one, and nonzero eigenvalue has a value of n.

If the matrix A contains inconsistencies, the vector w of the weights can be obtained using the equations

$$(A - \lambda_{max} I)w = 0$$

$$\sum w_i = 1$$

where λ_{max} is the largest eigenvalue of matrix A. Because of the characteristics of the matrix A, $\lambda_{max} \geq n$ and the difference $\lambda_{max} - n$ can be used for measuring inconsistencies. A consistency index CI has been constructing

$$CI = \frac{\lambda_{max} - n}{n - 1}$$

and consistency ratio is defined as $CR = \frac{CI}{RI}$, where RI is a random index (random index is the consistency index of many randomly generated pairwise comparison matrices of size n). If the value of CR is less or equal to 0,10, the pairwise comparisons are considered to be acceptable. Otherwise, the comparisons must be repeated in order to resolve the inconsistencies.

3. TOPSIS method

Let us denote $f_i^* = \max\limits_{a \in A} f_i(a)$. Criteria functions can be benefit functions or cost functions. In the latter the function with the opposite sign is presumed to be maximized. The vector $F^* = (f_1^*, f_2^*, \dots, f_k^*)$ is an ideal solution (the ideal point, the ideal vector) of the problem (1). If there is the alternative $a^* \in A$ so that $f_i(a^*) = f_i^*$, $\forall i \in \{1, 2, \dots, k\}$, then a^* is a perfect solution of the problem (1). Most often such a solution does not exist. In such a case it is possible to find the alternative from the set A for which the vector $F(a) = (f_1(a), f_2(a), \dots, f_k(a))$ is closest to the ideal solution. This solution is called a compromise solution. Different metrics can be chosen for measuring distances from the ideal solution and the result of the choice are the possibilities for different interpretations of compromise solutions for problem (1).

For $a \in A$ to measure the distance of the vector $F(a) = (f_1(a), f_2(a), \dots, f_k(a))$ from the ideal solution Minkowski's L_p metrics is most frequently used

$$d_p(a) = \Big(\sum_{i=1}^{k} (f_i^* - f_i(a))^p \Big)^{\frac{1}{p}}.$$

The main characteristics of this metrics are $d_1 \geq d_p \geq d_\infty$, $d_\infty = \max_i \{f_i^* - f_i(a)\}$, at which the choice of parameters p depends on the decision maker's attitude to the interpretations tied to its extreme values ($p = 1$ gives a maximal total gain, and $p = \infty$ is a minimal total deviation). One of the very important characteristics of the compromise solution is connected to a value choice of the parameter p where for $1 \leq p < \infty$ solutions are Pareto optimal or efficient, while it does not have to be the same for $p = \infty$, but in this case at least one compromise solution is at the same time an efficient one.

If functions f_i, $i \in \{1, 2, \dots, k\}$, have different weights w_i, $i \in \{1, 2, \dots, k\}$, the following function

$$d_p(a) = \Big(\sum_{i=1}^{k} w_i^p (f_i^* - f_i(a))^p \Big)^{\frac{1}{p}}.$$

is used for the decision maker who uses them as the solving criteria for the distance measuring of the vector to the ideal solution. Different from the described compromise ranking where the alternative from the set A closest to the ideal solution is being searched upon, the TOPSIS Method (Technique for Order Preference by Similarity to Ideal Solution) introduces into the problem analysis so called negative ideal solution (the ideal solution is the most wished possibility; the negative ideal solution would thus have the least desirable characteristics).

Let $f_i^- = \min_{a \in A} f_i(a)$ be a mark for the weakest criterial value for the criterium f_i, $i \in \{1, 2, \dots, k\}$. Vector $F^- = (f_1^-, f_2^-, \dots, f_k^-)$ is called the negative ideal solution of the problem (1). For the alternative $a \in A$ the distance of the vector $F(a) = (f_1(a), f_2(a), \dots, f_k(a))$ from the negative ideal solution is

$$d_p^-(a) = \left(\sum_{i=1}^{k} w_i^p (f_i^- - f_i(a))^p \right)^{\frac{1}{p}}.$$

To be able to identify in the set A the alternative closest to the ideal solution and the same time farthest from the negative ideal solution and in accordance with the TOPSIS Method it is necessary to form the following functions

$$D_p(a) = \frac{d_p^-(a)}{d_p(a) + d_p^-(a)}, \quad p = 1, 2, \infty.$$

TOPSIS solution of the problem (1) is the alternative for which the maximal value of this function has been achieved.

4. ELECTRE methods

There are four types of the ELECTRE method and all have the same base.

Concordance index

The concordance index c_{lk} is defined for each pair of alternatives $a_l, a_k \in A$ as a sum of criteria weight according to which a_l is not weaker than a_k:

$$c_{lk} = \sum_{i/f_i(a_l) \geq f_i(a_k)} w_i.$$

The values of all the concordance indexes are being written into the concordance matrix C. The concordance index is a measure of intensity of the domination of the alternative a_l over the alternative a_k.

Discordance index

This index measures the resistance of one alternative against the domination of the other. Because of different measuring scales belonging to different criteria, first it is necessary to transform all criteria values to comparable scales. It can be done in a few ways, but the authors suggest the procedure of vector normalization for the ELECTRE method.

In order to simplify the picture of the procedure for the values from the evaluation table (table 1) the following mark is introduced $x_{ij} = f_j(a_i)$.

The elements of the normalized evaluation table can be calculated as

$$\frac{x_{ij}}{\sqrt{\sum\limits_k x_{kj}^2}}\,.$$

All criteria do not have to be equally important. Because of that, every column in the normalized evaluation table is multiplied by a weight of the associated criterium and by this one gets a ponderized normalized table of values which elements are

$$x_{ij}^* = w_j \frac{x_{ij}}{\sqrt{\sum\limits_k x_{kj}^2}}\,.$$

The discordance indexes d_{kl} can be calculated using these values

$$d_{kl} = \frac{\max\limits_{j/f_j(a_l)>f_j(a_k)} |x_{kj}^* - x_{lj}^*|}{\max\limits_{j\in J} |x_{kj}^* - x_{lj}^*|}$$

(the maximum difference among valuing of alternatives according to the criteria where the alternative a_l is better than a_k is divided by the maximum difference of valuing according to all the criteria). The discordance matrix D is formed from the discordance indexes.

Let $\bar{c}$ i $\bar{d}$ be average values of the concordance indexes i.e. of the discordance indexes. According to this, the MI matrix is formed from the concordance matrix and discordance matrix. Its elements are

$$m_{ij} = \begin{cases} 1, & \text{iff } \; c_{ij} > \bar{c} \; \text{ and } \; d_{ij} < \bar{d} \\ 0, & \text{otherwise.} \end{cases}$$

If $m_{ij} = 1$, it is presumed that the alternative a_i dominates over the alternative a_j (the intensity of the domination of a_j over a_j is higher than the average and the resistance of the alternative a_j to that domination is weaker than the average.) The matrix can be considered to be a matrix of graph indexes where the alternatives are knots and only those with dominance are connected by arches. The exit knot of the arch belongs to the alternative which dominates over the one with the corresponding entrance knot.

The nondominated alternative make the kernel of the graph. The final decision is reached according to the stability analysis of the kernel taking into the account the value changes of the indexes $\bar{c}$ i $\bar{d}$ and the weight criteria.

If one wants completely to rank the alternatives in the set A, the procedure of the ELECTRE II method is used. In that case it is necessary to calculate the pure value of the concordance with the dominance $c_k = \sum_{i=1,\, i\neq k}^{n} c_{ki} - \sum_{i=1,\, i\neq k}^{n} c_{ik}$ and pure value of the discordance with the domination $d_k = \sum_{i=1, i\neq k}^{n} d_{ki} - \sum_{i=1, i\neq k}^{n} d_{ik}$. The alternatives are ranked then according to the highest average rank.

5. PROMETHEE methods

The PROMETHEE Method (Preference Ranking Organization METHod for Enrichment Evaluations) can be described in three steps:

Step 1. Enrichment of the preference structure
The generalized criteria are introduced to enable considering the span of deviation in valuing alternatives to individual criteria.

Step 2. Enrichment of the dominance relation
The outranking relation is built according to the estimate of the alternatives to all criteria. The total level of the preference with which one alternative dominates over the other is calculated for each pair of alternatives.

Step 3. Decision analysis
The PROMETHEE I method gives a partial ranking of the set A. The information on the incomparable alternatives is also given. The PROMETHEE II method gives a complete ranking of the set A.

Step 1. Generalized criteria
Let f_i be a criterion and as such it needs to be maximalized. The comparing of the alternatives from the set A in pairs and with the help of this criterion gives a simple preference structure which defines the dominance relation in the set A

$$f_i(a) > f_i(b) \;\Leftrightarrow\; aPb$$

$$f_i(a) = f_i(b) \;\Leftrightarrow\; aIb.$$

To be able to take into account different measuring scales belonging to individual criteria and different deviations within those scales the generalized criterion is associated to each criterion. The preference relation $P_i(a,b)$ is defined for each criterion. It also defines the level of preference intensity from a in relation to b according to that criterion. It is supposed that $P_i(a,b)$ is a function of the difference $d = f_i(a) - f_i(b)$ and $0 \leq Pi(a,b) \leq 1$. The $P_i(a,b)$ value interpretations in terms of preference between a and b are the following:

$P_i(a,b) = 0$ if $d \leq 0$,	no preference (or indifference)
$P_i(a,b) \approx 0$ if $d > 0$,	weak preference
$P_i(a,b) \approx 1$ if $d >> 0$,	strong preference
$P_i(a,b) = 1$ if $d >>> 0$,	strict preference.

It is obvious that the generalized criterion is a non-decreasing function of d. Six types of functions of the generalized criterion are used in the PROMETHEE Methods. It is necessary to establish the values of parameters (not more that two) for individual criteria. The clear economic interpretation must be applicable to the values of parameters. The decision maker decides upon the choice of the generalized criterium type and the values of the corresponding parameters.

Step 2: Outranking relation

When the generalized criterium is chosen $(f_i(\cdot), P_i(\cdot,\cdot))$ the multicriteria preference index of a over b is defined for every criterium $f_i(\cdot)$ taking into account all the criteria

$$\pi(a,b) = \sum_{i=1}^{k} w_i P_i(a,b), \quad \sum_{i=1}^{k} w_i = 1$$

$w_i > 0$ $(i = 1,\ldots,k)$ are weights associated to each criterion.

The properties of $\pi(a,b)$ are:

$\pi(a,a) = 0$ and $0 \leq \pi(a,b) \leq 1$, $\forall a,b \in A$

$\pi(a,b) \approx 0$ implies a weak global preference of a over b,

$\pi(a,b) \approx 1$ implies a strong global preference of a over b,

$\pi(a,b)$ expresses with which degree a is preferred to over all the criteria. The values $\pi(a,b)$ and $\pi(b,a)$ are calculated for each pair of alternatives $a,b \in A$. In this way a complete valued outranking relation is constructed on A.

Step 3: Decision analysis

In this step two outranking flows are defined:

$$\phi^+(a) = \frac{1}{n-1} \sum_{x \in A} \pi(a,x) \text{ - the positive outranking flow}$$

$$\phi^-(a) = \frac{1}{n-1} \sum_{x \in A} \pi(x,a) \text{ - the negative outranking flow}$$

The positive outranking flow $\phi^+(a)$ represents the outranking character of a - how much this alternative is outranking all the others. The negative outranking flow $\phi^-(a)$ represents the outranked character of a - how much this alternative is outranked by all the others.

The PROMETHEE I partial ranking.

With positive and negative outranking flows two rankings (S^+, I^+) and (S^-, I^-) of the alternatives are defined:

$$aS^+b \quad \text{iff} \quad \phi^+(a) > \phi^+(b), \quad aI^+b \quad \text{iff} \quad \phi^+(a) = \phi^+(b)$$

$$aS^-b \quad \text{iff} \quad \phi^-(a) < \phi^-(b), \quad aI^-b \quad \text{iff} \quad \phi^-(a) = \phi^-(b).$$

The PROMETHEE I partial ranking is the intersection of these two rankings:

aP^Ib if and only if $((aS^+b$ and $aS^-b)$ or $(aS^+b$ and $aI^-b)$ or $(aI^+b$ and $aS^-b))$

aI^Ib if and only if aI^+b and aI^-b

aRb otherwise.
P, I and R denote preference, indifference and incomparability, respectively.

It is important to point out that PROMETHEE I Method does not rank the set A completely. If a decision maker wants a complete ranking of the alternatives, he can use the PROMETHEE II Method.

The PROMETHEE II complete ranking.
The set A can be ranked completely in the way that the net outranking flow $\phi(a) = \phi^+(a) - \phi^-(a)$ is introduced for each alternative $a \in A$. It is obvious that if $\phi(a)$ is stronger, the alternative a is better.

The PROMETHEE II complete ranking is then defined:

$$aP^{II}b \quad \text{iff} \quad \phi(a) > \phi(b)$$

$$aI^{II}b \quad \text{iff} \quad \phi(a) = \phi(b).$$

In this way it is possible to compare all alternatives but it does not have to be an advantage in relation to the PROMETHEE I Method.

References

[1] T. L. SAATY, *The Analytic Hierarchy Process*, McGraw-Hill, 1980.

[2] C. L. HWANG, K. YOON, *Multiple Attribute Decision Making*, Lecture Notes in Economics and Mathematical Systems, Vol. **164**, Springer, Berlin, 1979.

[3] J. P. BRANS, B. MARESCHAL, PH. VINCKE, *How to select and how to rank projects: The PROMETHEE method*, Europ. J. Operat. Res. **24**(1986), 228-238.

[4] M. ROUBENS, P. VINCKE, *Preference modelling*, Lecture Notes in Economics and Mathematical Systems, Vol. **250**, Springer, Berlin, 1985.

[5] K. YOON, *A Reconcilation Among Discrete Compromise Solutions*, J. Opl. Res. Soc. **38**(1987), 277-286.