The maximal number of U-k – seminets of the maximal degree*

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Abstract. *Aczel (1965) investigated quasigroups,* 3*-nets and nomograms and Belousov (1971) k-nets and associated (k-1) – quasigroups. There are different 3 – seminets and k-seminets (see e.g. Havel (1967), Taylor (1971), Ušan (1977), Galić (1989), etc.) to which by some rules one can assign corresponding algebraic structures (partial quasigroups and partial groupoids). Galić (1990) defines U-k – seminets of the maximal degree and shows the existence and construction in dependence on the set P over which one constructs a* k*-seminet. In this paper it is shown how many U-k – seminets of maximal degree μ can be constructed over the set P for the given t-order.*

Key words: *U-k – seminets, k – seminets, t – order, maximal degree*

Sažetak. **Maksimalni broj U-k – polumreža maksimalnog stupnja**. *Aczel (1965) proučava kvazigrupe, 3 – mreže i nomograme, a Belousov (1971) k – mreže i pridružene (k-1) – kvazigrupe. Postoje različite 3 – polumreže i k – polumreže (vidi primjerice Havel (1967), Taylor (1971), Ušan (1977), Galić (1989) i drugi) kojima se po nekom pravilu mogu pridružiti odgovarajuće algebarske strukture (parcijalne kvazigrupe i parcijalni grupoidi). Galić (1990) definira U-k – polumreže maksimalnog stupnja te pokazuje postojanje i konstrukciju u ovisnosti o skupu P nad kojim se konstruira k – polumreža. U ovom radu pokazano je koliko se U-k – polumreža maksimalnog stupnja μ može konstruirati nad skupom P za zadani t-red.*

Ključne riječi: *U-k – polumreže, k – polumreže, t – red, maksimalni stupanj*

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Let $\mathbf{P}$ be a nonempty set and $\mathcal{B}$ family of nonempty subsets of $\mathbf{P}$, i.e. $\mathcal{B} \subset \mathcal{P}(\mathbf{P})\backslash\{\emptyset\}$. Let $\Pi = \{X_1, ..., X_k\}$, $k \geq 3$, be a partition of the set $\mathcal{B}$. Elements of the set $\mathbf{P}$, $\mathcal{B}$ and Π are called points, blocks and classes, respectively. Then the ordered triple $N = (\mathbf{P}, \mathcal{B}, \Pi)$ is called a ***U-k* − seminet** if the following conditions are satisfied:

C1. *Any two blocks from different classes have at most one point in common, that is*

$$(\forall x^i \in X_i)(\forall x^j \in X_j)(i \neq j \Rightarrow |x^i \cap x^j| \leq 1),$$

C2. *Every point of* $\mathbf{P}$ *lies in exactly one block from each class, that is*

$$(\forall p \in \mathbf{P})(\forall i \in \{1, 2, ..., k\})(\exists! x^i \in X_i : p \in x^i).$$

If in a *U-k* − seminet $N = (\mathbf{P}, \mathcal{B}, \Pi)$ we need to specify classes, then we will denote this *U-k* − seminet by $N = (\mathbf{P}, X_1, ..., X_k)$, where the number k denotes the degree of the seminet.

Number of points of a set A, i.e. the cardinal number of the set A will be denoted by $|A|$.

Number of points of a maximal block, i.e. the number:

$$d = \max\{|b| : b \in \mathcal{B}\}$$

is called **t-order** of a *U-k* − seminet.

$N = (\mathbf{P}, \mathcal{B}, \Pi)$ is said to be a *U*-μ − **seminet of maximal degree** μ over the set $\mathbf{P}$ if for every *U-k* − seminet $N' = (\mathbf{P}, \mathcal{B}', \Pi')$ over the set $\mathbf{P}$, one has $\mu \geq k$.

The existence and the construction of a *U-k* − seminet with maximal degree for some sets $\mathbf{P}$ are proved by Galić (1990). The following theorem considers that issue:

Theorem 1. *Let* $\mathbf{P}$ *be a set of points such that* $|\mathbf{P}| = t \geq 3$. *Then for*

$$2 \leq d < \frac{t+2}{2} \tag{1}$$

there exists a U-μ – seminet $N = (\mathbf{P}, \mathcal{B}, \Pi)$ *of maximal degree μ and t-order equal to d over the set* $\mathbf{P}$, *so that*

$$\mu = t - d + 2. \tag{2}$$

We will consider here maximal degree μ of a *U-k* − seminet which depends on the number of points of the set $\mathbf{P}$ ($|\mathbf{P}| = t$) and t-order equal to d.

Two sets $\mathbf{P}$ and $\mathbf{P}$' are said to be **equipotent** if $|\mathbf{P}| = |\mathbf{P}'|$.

Theorem 2. *Let $m \in \mathbb{N}$ be a natural number. Then there exists m nonequipotent sets* $\mathbf{P}$ *($|\mathbf{P}| = t$), over which we can form a* U-k *– seminet* $N = (\mathbf{P}, \mathcal{B}, \Pi)$ *of maximal degree $\mu = t - d + 2$, with t-order equal to d, so that $m = t - d$.*

Proof. Let $m \in \mathbb{N}$. Then by *Theorem 1* we can form a *U-k* − seminet $N = (\mathbf{P}, \mathcal{B}, \Pi)$ over the set $\mathbf{P}$, ($|\mathbf{P}| = t$) of maximal degree $\mu = t - d + 2$ for

$$2 \leq d < \frac{t+2}{2}.$$

If in (1) we put $m = t - d$, i.e. $d = t - m$, we obtain the following:

$$2 \leq t - m < \frac{t+2}{2},$$

which finally gives

$$m + 2 \leq t < 2m + 2. \tag{3}$$

□

These inequalities for the variable t have m integer solutions. It means that there exist m nonequipotent sets $\mathbf{P}$ such that $|\mathbf{P}| = t$, which satisfy the *Theorem* (see *Table 1*).

Theorem 3. *Let* $N = (\mathbf{P}, \mathcal{B}, \Pi)$ *be a* U-k – *seminet of maximal degree* μ *over the set* $\mathbf{P}$*, with* t*-order equal to* d*. Then for* $|\mathbf{P}| = t \geq 3$ *the following holds:*

$$a) \qquad \frac{t+2}{2} < \mu \leq t, \tag{4}$$

$$b) \qquad \mu \leq t < 2\mu - 2. \tag{5}$$

Proof. a) Since from (2) it follows that $d = t - \mu + 2$, then by substituting into (1) we obtain

$$2 \leq t - \mu + 2 < \frac{t+2}{2}.$$

Therefore, because $2 \leq t - \mu + 2$, it follows that

$$\mu \leq t \tag{6}$$

By using the inequality $t - \mu + 2 < \frac{t+2}{2}$ we get

$$\frac{t+2}{2} < \mu. \tag{7}$$

From (6) and (7) it directly follows (4) (see *Table 2.*)

b) From (7) it directly follows that

$$t < 2\mu - 2. \tag{8}$$

(6) and (8) now give (5) (see *Table 3*). □

The results of *Theorem 1, 2* and *3* corresponding to the maximal degree of a *U-k* – seminet (over the given set $\mathbf{P}$, ($|\mathbf{P}| = t$) and the given t-order equal to d) is shown in the following tables.

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Table 1. Maximal degree μ of a U-k – seminet which depends on the cardinal number t of **P** and t-order equal to d, (resp. $m = t - d$), that is $\mu = f(t, m)$.

t \ m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	3														
4		4													
5		4	5												
6			5	6											
7			5	6	7										
8				6	7	8									
9				6	7	8	9								
10					7	8	9	10							
11					7	8	9	10	11						
12						8	9	10	11	12					
13						8	9	10	11	12	13				
14							9	10	11	12	13	14			
15							9	10	11	12	13	14	15		
16								10	11	12	13	14	15	16	
17								10	11	12	13	14	15	16	17
18									11	12	13	14	15	16	17
19									11	12	13	14	15	16	17
20										12	13	14	15	16	17
21										12	13	14	15	16	17
22											13	14	15	16	17
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25												14	15	16	17
26													15	16	17
27													15	16	17
28														16	17
29														16	17
30															17
31															17

Table 2. Maximal degree μ of a U-k – seminet for the given set **P** ($|\mathbf{P}| = t$) and t-order equal to d, i.e. $\mu = f(t, d)$.

t	3	4	5	6	7	8	9	10
$\left[\frac{t+2}{2}\right]$	2	3	3	4	4	5	5	6
$\frac{t+2}{2} < \mu \le t$	3	4	4,5	5,6	5,6,7	6,7,8	6,7,8,9	7,8,9,10

Table 3. The cardinal number t of the set **P** for the given maximal degree μ of a U-k – seminet of the t-order equal to d, i.e. $t = f(\mu, d)$.

μ	3	4	5	6	7	8
$2\mu - 2$	4	6	8	10	12	14
$\mu \le t < 2\mu - 2$	3	4,5	5,6,7	6,7,8,9	7,8,9,10,11	8,9,10,11,12,13