On a refinement of the Chebyshev and Popoviciu inequalities*

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Abstract. *We establish a refinement of the discrete Chebyshev inequality and an analogous one for the Popoviciu inequality.*

Key words: *discrete Chebyshev inequality, Popoviciu inequality*

Sažetak Poboljšanje Čebiševljeve i Popoviciu-ove nejednakosti. *U radu je pokazano poboljšanje diskretne Čebiševljeve i jedna analogija Popovicu-ove nejednakosti.*

Ključne riječi: *diskretna Čebiševljeva nejednakost, Popoviciu-ova nejednakost*

1. Introduction

A fundamental inequality in probability is the discrete Chebyshev inequality, which states the following.

Theorem A. *Suppose a and b are n–tuples of real numbers, both nondecreasing or both nonincreasing, and p is an n–tuple of positive numbers. Then*

$$T_n(a,b;p) := \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j a_j b_j - \sum_{i=1}^{n} p_i a_i \sum_{j=1}^{n} p_j b_j \geq 0. \tag{1}$$

Recently an improvement has been derived by Alzer [1].

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*Received November 12, 1996.

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Theorem B. *If a, b and p are defined as above, then*

$$T_n(a,b;p) \geq \min_{2\leq i,j\leq n}[(a_i - a_{i-1})(b_j - b_{i-1})] \cdot T_n(e,e;p), \tag{2}$$

where $e = (1,2,\ldots,n)$. Equality holds if and only if

$$a_i = a_1 + (i-1)\alpha \quad \textit{and} \quad b_i = b_1 + (i-1)\beta \quad (i = 1,\ldots,n), \tag{3}$$

where α and β are positive or negative real numbers accoding as a and b are both nondecreasing or nonincreasing n–tuples.

In fact it is possible to give a corresponding upper bound for $T_n(a,b;p)$. Set

$$m(a) = \min_{1\leq i<n}(a_{i+1} - a_i), \quad M(a) = \max_{1\leq i<n}(a_{i+1} - a_i).$$

Lupaş [2] has shown that with the same condition for a, b and p,

$$m(a)m(b) \leq \frac{T(a,b;p)}{T(e,e;p)} \leq M(a)M(b).$$

We note that the first inequality is equivalent to (2).

The condition that p is a positive n–tuple can be weakened to the condition

$$0 \leq P_n \leq P_k \quad (k = 1,2,\ldots,n-1), \tag{4}$$

where $P_k := \sum_{i=1}^{k} p_i$ $(k = 1,2,\ldots,n)$ (see [4]).

The result was established *via* an Abel–type identity. This appears to be of a more general applicability, and in this article we shall employ it to derive two new results: a refinement for the Chebyshev inequality and one for Popoviciu's inequality.

Since the identity is not proved in [4], we present a proof in Section 2. An interesting feature is that although this generalizes Abel's identity, it can be established by repeated use of the basic Abel identity. The latter therefore appears to hold a key role in connection with the cluster of results mentioned above. In Section 3 we prove our new refinements of the Chebyshev and Popoviciu results.

2. An Abel–type identity

Proposition 1 below is a useful consequence of repeated use of Abel's identity

$$\sum_{j=1}^{n} p_j c_j \ = \ P_n c_n - \sum_{j=1}^{n-1} P_j \Delta c_j,$$

where $\Delta c_j := c_{j+1} - c_j$ and P_j is defined as in the introduction.

It will be useful to introduce also a variant. Put $\bar{P}_j = \sum_{i=j}^n p_i$ $(j = 1, \dots, n)$. On substituting for the definitions of P_j, $\bar{P}_j$ and interchanging the order of summation, we derive

$$\sum_{j=1}^n p_j c_j = c_i P_n - \sum_{j=1}^{i-1} P_j \Delta c_j + \sum_{j=i+1}^n \bar{P}_j \Delta c_{j-1} \qquad (1 \le i \le n),$$

which is an extension of Abel's identity.

Proposition 1. *Suppose $a = (a_i)_1^n$, $b = (b_i)_1^n$, $p = (p_i)_1^n$ are real n–tuples and $T_n(a, b; p)$ is defined by the left–hand relation in (1). Then*

$$T(a, b; p) = \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i-1} \bar{P}_{i+1} P_j \Delta b_j + \sum_{j=i+1}^n P_i \bar{P}_j \Delta b_{j-1} \right) \Delta a_i.$$

Proof. From its definition, we have

$$T(a, b; p) = \sum_{i=1}^n p_i a_i \left(\sum_{j=1}^n p_j (b_i - b_j) \right) = \sum_{i=1}^n p_i h_i a_i,$$

where

$$h_i := \sum_{j=1}^n p_j (b_i - b_j). \tag{5}$$

Accordingly, by Abel's identity,

$$T(a, b; p) = \left(\sum_{i=1}^n p_i h_i \right) a_n - \sum_{i=1}^{n-1} \left(\sum_{j=1}^i p_i h_i \right) \Delta a_i,$$

and since

$$\sum_{i=1}^n p_i h_i = \sum_{i=1}^n p_i \sum_{j=1}^n p_j (b_i - b_j) = 0,$$

we thus have

$$T(a, b; p) = - \sum_{i=1}^{n-1} \left(\sum_{j=1}^i p_j h_j \right) \Delta a_i. \tag{6}$$

Again by Abel's identity,

$$\sum_{j=1}^i p_j h_j = h_i P_i - \sum_{j=1}^{i-1} P_j \Delta h_j = h_i P_i - \sum_{j=1}^{i-1} P_j P_n \Delta b_j. \tag{7}$$

Further, from (5) and our extension of Abel's identity,

$$h_i = \sum_{j=1}^{i-1} P_j \Delta b_j - \sum_{j=i+1}^{n} \bar{P}_j \Delta b_{j-1}, \tag{8}$$

and so (6) yields

$$T(a,b;p) = \sum_{i=1}^{n-1} \left(h_i P_i - \sum_{j=1}^{i-1} P_j P_n \Delta b_j \right) \Delta a_i \quad [by\ (7)]$$

$$= -\sum_{i=1}^{n-1} \left[P_k \left(\sum_{j=1}^{i-1} P_j \Delta b_j - \sum_{j=i+1}^{n} \bar{P}_j \Delta b_{j-1} \right) - \sum_{j=1}^{i-1} P_j P_n \Delta b_j \right] \Delta a_k \quad [by\ (8)]$$

$$= \sum_{i=1}^{n-1} \left(\bar{P}_{i+1} \sum_{j=1}^{i-1} P_j \Delta b_j + P_i \sum_{j=i+1}^{n} \bar{P}_j \Delta b_{j-1} \right) \Delta a_i,$$

and we are done.

3. Applications

We now proceed to an application of *Proposition 1* to give a refinement of Chebyshev's inequality. With the notation

$$|a| = (|a_1|, \ldots, |a_n|),$$

we have the following result.

Theorem 1. *Let a and b be n-tuples of real numbers, both nondecreasing or both nonincreasing, and p a real n–tuple satisfying (4). Then*

$$T_n(a;b;p) \geq |T_n(|a|, |b|, p)| \geq 0.$$

Proof. For a nondecreasing n–tuple we have

$$\Delta a_i = a_{i+1} - a_i = |a_{i+1} - a_i| \geq ||a_{i+1}| - |a_i|| = |\Delta |a_i|| \,,$$

so that by *Proposition 1*

$$
\begin{aligned}
T(a,b;p) &= \sum_{k=1}^{n-1}\left(\bar{P}_{k+1}\sum_{j=1}^{k-1}P_j\Delta b_j + P_k\sum_{j=k+1}^{n}\bar{P}_j\Delta b_{j-1}\right)\Delta a_k \\
&\geq \sum_{k=1}^{n-1}\left(\bar{P}_{k+1}\sum_{j=1}^{k-1}P_j\left|\Delta|b_j|\right| + P_k\sum_{j=k+1}^{n}\bar{P}_j\left|\Delta|b_{j-1}|\right|\right)\left|\Delta|a_k|\right| \\
&\geq \left|\sum_{k=1}^{n-1}\left(\bar{P}_{k+1}\sum_{j=1}^{k-1}P_j\Delta|b_j| + P_k\sum_{j=k+1}^{n}\bar{P}_j\Delta|b_{j-1}|\right)\Delta|a_k|\right| \\
&= \left|T_n(|a|,|b|;p)\right|,
\end{aligned}
$$

giving the required result.

We conclude by considering Popoviciu's inequality [6], which states the following.

Theorem C. *Suppose*

$$F(a,b;x) = \sum_{i=1}^{n}\sum_{j=1}^{m}x_{i,j}a_ib_j,$$

where all the quantities involved are real numbers. Then

$$F(a,b;x) \geq 0 \tag{9}$$

for all sequences $a=(a_1,\ldots,a_n)$ and $b=(b_1,\ldots,b_m)$ which are monotonic is the same sense if and only if

$$
\begin{array}{ll}
X_{r,s} \geq 0 & (r=2,\ldots,n;\ s=2,\ldots,m), \\
X_{r,1} = 0 & (r=1,\ldots,n), \\
X_{1,s} = 0 & (s=2,\ldots,m),
\end{array}
\tag{10}
$$

where $X_{r,s} = \sum_{i=r}^{n}\sum_{j=s}^{m}x_{i,j}$.

Remark 1. For the case $m=n$, we recover Chebyshev's inequality under condition (4) with the choice

$$
x_{i,j} = \begin{cases} p_i(P_n-p_i) & \text{for } i=j \\ -p_ip_j & \text{for } i\neq j. \end{cases}
$$

Relation (9) is a simple consequence of the identity

$$\begin{aligned} F(a,b;x) &= a_1 b_1 X_{1,1} + a_1 \sum_{s=2}^{m} X_{1,s} \Delta b_{s-1} \\ &\quad + b_1 \sum_{r=2}^{n} X_{r,1} \Delta a_{r-1} + \sum_{r=2}^{n} \sum_{s=2}^{m} X_{r,s} \Delta a_{r-1} \Delta b_{s-1} \end{aligned} \tag{11}$$

(see [5] and also [3, p. 341]).

Interpolations of (9) which contain (2) and (3) are obtained in [4].

Finally, we derive an analogue of *Theorem 1* for F.

Theorem 2. *Suppose* $x_{i,j}$ *(*$1 \le i \le n$, $1 \le j \le m$*) are real numbers satisfying (10). If the sequences* a *and* b *are monotone in the same sense, then*

$$F(a,b;x) \ge |F(|a|,|b|;x)| \ge 0.$$

Proof. By (10) F reduces to the last term in (11), so

$$\begin{aligned} F(a,b;x) &= \sum_{r=2}^{n} \sum_{s=2}^{m} X_{r,s} \Delta a_{r-1} \Delta b_{s-1} \\ &\ge \sum_{r=2}^{n} \sum_{s=2}^{m} X_{r,s} \left|\Delta |a_{r-1}|\right| \times \left|\Delta |b_{s-1}|\right| \\ &= \sum_{r=2}^{n} \sum_{s=2}^{m} X_{r,s} \left|\Delta |a_{r-1}| \times \Delta |b_{s-1}|\right| \\ &\ge \left| \sum_{r=2}^{n} \sum_{s=2}^{m} X_{r,s} \Delta |a_{r-1}| \times \Delta |b_{s-1}| \right| \\ &= \left| F(|a|,|b|;x) \right|. \end{aligned}$$

Remark 2. As in *Remark 1* we can obtain *Theorem 1* from *Theorem 2.*

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