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INEQUALITIES VIA (p, r)-CONVEX FUNCTIONS

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Abstract. The main aim of this paper is to introduce a new class of convex functions, which is called (p, r)-convex functions. We derive some new Hermite-Hadamard type integral inequalities via (p, r)-convex functions. Some special cases are also discussed. These results can be considered as significant improvement of the known results. The technique and ideas of this paper may motivate further research.

1. Introduction

Convexity theory has become a rich source of inspiration in pure and applied sciences. This theory had not only stimulated new and deep results in many branches of mathematical and engineering sciences, but also provided us a unified and general framework for studying a wide class of unrelated problems. For recent applications, generalizations and other aspects of convex functions and their variant forms, see [1-17] and the references therein.

Pearce et. al [13] generalized the Hermite-Hadamard inequality to a r-convex function f which is defined on an interval $[a, b]$. Let r be a fixed real number. We say that $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ is r-convex, if f is positive and $\forall x, y \in I$ and $t \in [0, 1]$, we have

$$f((1-t)a+tb) \leq \begin{cases} [(1-t)[f(a)]^r + t[f(b)]^r]^{\frac{1}{r}}, & r \neq 0 \\ (f(a))^{1-t}(f(b))^t, & r = 0. \end{cases} \tag{1.1}$$

It is clear that 0-convex functions are log-convex functions and 1-convex functions are ordinary convex functions.

Ngoc et. al [6] obtained the following Hermite-Hadamard inequality for r-convex function.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \left(\frac{r}{r+1}\right)^{\frac{1}{r}} [[f(a)]^r + [f(b)]^r]^{\frac{1}{r}}, \quad r \neq 0.$$

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Zhang and Wan [18] introduced and studied a new class of convex functions, which is called p-convex.

Definition 1.1 ([18]). *A set $I = [a, b] \subseteq \mathbb{R} \setminus \{0\}$ is said to be p-convex set, if*

$$[(1-t)x^p + ty^p]^{\frac{1}{p}} \in I, \qquad \forall x, y \in I, t \in [0, 1].$$

Some special cases of the p-convex sets are:

I. If $p = 1$, then p-convex set is a convex set.
II. If $p = -1$, then p-convex set becomes a harmonic convex set.
III. If $p = 0$, then p-convex set collapses to the geometrically convex set.

This shows that the concept of p-convex sets is quite general and unifying one.

Definition 1.2 ([18]). *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be p-convex, where $p \neq 0$, if*

$$f([(1-t)x^p + ty^p]^{\frac{1}{p}}) \leq (1-t)f(x) + tf(y), \qquad \forall x, y \in I, t \in [0, 1]. \tag{1.2}$$

We note that, if $p = 0$, then p-convex functions reduce to geometrically convex functions [7], that is,

$$f(x^{1-t}y^t) \leq (1-t)f(x) + tf(y), \qquad \forall x, y \in I, t \in [0, 1].$$

For different and appropriate choices of p, one can show that the p-convex functions include the convex functions, harmonic convex functions and geometrically convex functions as special cases.

Noor et. al [8] have obtained the integral inequality for p-convex functions, which may be regarded as a refinement of the concept of convexity. In particular, it has been shown that f is a p-convex function, if and only if, it satisfies

$$\begin{aligned} f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) &\leq \frac{1}{2}\left[f\left(\left[\frac{3a^p+b^p}{4}\right]^{\frac{1}{p}}\right) + f\left(\left[\frac{a^p+3b^p}{4}\right]^{\frac{1}{p}}\right)\right] \\ &\leq \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}\mathrm{d}x \\ &\leq \frac{1}{2}\left[f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) + \frac{f(a)+f(b)}{2}\right] \\ &\leq \frac{1}{2}[f(a)+f(b)]. \end{aligned} \tag{1.3}$$

The inequality (1.3) holds in reversed direction if f is a p-concave function.

For recent developments and properties of p-convex functions and their variant forms, see [10, 12].

It is clear from (1.1) and (1.2) that the classes of r-convex functions and p-convex functions are two distinct generalizations of the convex functions and do not contain each other. It is natural to consider a new class of convex functions which unifies these two concepts. Motivated and inspired by this fact, we introduce new class of convex functions, which is called (p, r)-convex function. This class is more general and contains several known and new classes of p-convex and r-convex functions as special cases. We discuss some properties of (p, r)-convex functions. We derive several new Hermite-Hadamard inequalities for (p, r)-convex functions. We also discuss some special cases which can be obtained from the main results. Results proved in this paper can be viewed as significant contributions in the field of integral inequalities.

Throughout this paper, we take $\mathbb{R}_+ = (0, \infty)$, unless otherwise specified.

We now consider a new class of convex functions.

Definition 1.3. *Let r be a real number. We say that $f : I = [a, b] \subseteq \mathbb{R}_+ \to \mathbb{R}$ is (p, r)-convex function, or f belongs to the class (p, r, I), if f is positive and we have*

$$f([(1-t)x^p + ty^p]^{\frac{1}{p}}) \leq \begin{cases} [(1-t)[f(x)]^r + t[f(y)]^r]^{\frac{1}{r}}, & r \neq 0, \quad p \neq 0 \\ (f(x))^{1-t}(f(y))^t, & r = 0, \quad p \neq 0. \end{cases} \tag{1.4}$$

The function f is said to be (p, r)-concave function, if and only if the function $-f$ is (p, r)-convex function.

For $t = \frac{1}{2}$ in (1.4), the function f satisfies

$$f\left(\frac{x^p + y^p}{2}\right)^{\frac{1}{p}} \leq \begin{cases} \left(\frac{[f(x)]^r + [f(y)]^r}{2}\right)^{\frac{1}{r}}, & r \neq 0, \quad p \neq 0 \\ \sqrt{(f(x))(f(y))}, & r = 0, \quad p \neq 0. \end{cases} \tag{1.5}$$

which is called Jensen (p, r)-convex function.

Remark 1.4. For different and appropriate choices of p, one can show that the (p, r)-convex functions include the r-convex functions [2], harmonic r-convex functions [11] and geometrically r-convex functions, as special cases.

Definition 1.5 ([16]). *Two functions f, g are said to be similarly ordered (f is g-monotone), if and only if,*

$$\langle f(x) - f(y), g(x) - g(y) \rangle \geq 0, \qquad \forall x, y \in \mathbb{R}^n.$$

The Euler Beta function is a special function defined by

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \mathrm{d}t, \quad \forall x, y > 0.$$

We now prove that the product of two similarly ordered (p, r)-convex functions is again a (p, r)-convex function.

Lemma 1.6. *Let f and g be two (p, r)-convex functions. Then, for $r > 0$, the product of f and g is (p, r)-convex, if f and g are similarly ordered.*

Proof. Let f and g be two similarly ordered (p, r)-convex functions. Then

$$\begin{aligned}
&f([(1-t)x^p+ty^p]^{\frac{1}{p}})g([(1-t)x^p+ty^p]^{\frac{1}{p}}) \\
&\le [(1-t)[f(x)]^r+t[f(y)]^r]^{\frac{1}{r}}[(1-t)[g(x)]^r+t[g(y)]^r]^{\frac{1}{r}} \\
&= [(1-t)^2[f(x)g(x)]^r+(1-t)t([f(x)g(y)]^r+[f(y)g(x)]^r) \\
&\quad +t^2[f(y)g(y)]^r]^{\frac{1}{r}} \\
&\le [(1-t)^2[f(x)g(x)]^r+(1-t)t([f(x)g(x)]^r+[f(y)g(y)]^r) \\
&\quad +t^2[f(y)g(y)]^r]^{\frac{1}{r}} \\
&= [(1-t)[f(x)g(x)]^r+t[f(y)g(y)]^r]^{\frac{1}{r}}. \qquad (1.6)
\end{aligned}$$

This shows, product of two similarly ordered (p, r)-convex functions is again a (p, r)-convex function.

The converse of this result is not true. □

Remark 1.7. The product of two similarly ordered $(0, r)$-convex functions is again a $(0, r)$-convex function.

2. Main results

In this section, we obtain new Hermite-Hadamard inequality for (p, r)-convex functions.

Theorem 2.1. *Let $f : I = [a, b] \subset \mathbb{R}_+ \to \mathbb{R}$ be (p, r)-convex function with $a < b$. If $f \in L[a, b]$, then*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}\mathrm{d}x \le ([f(a)]^r+[f(b)]^r)^{\frac{1}{r}}\left(\frac{r}{1+r}\right). \qquad (2.1)$$

Proof. Let f be (p, r)-convex function. Then, taking $x = [(1-t)a^p + tb^p]^{\frac{1}{p}}$ and $y = [ta^p + (1-t)b^p]^{\frac{1}{p}}$ in (1.5), we have

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \le \left[\frac{[f([(1-t)a^p+tb^p]^{\frac{1}{p}})]^r+[f([ta^p+(1-t)b^p]^{\frac{1}{p}})]^r}{2}\right]^{\frac{1}{r}}.$$

Now integrating the above inequality and using Minkowski's inequality, we get

$$\begin{aligned}
&f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \\
&\quad \le \left(\frac{1}{2}\right)^{\frac{1}{r}}\left[\int_0^1 [[f([(1-t)a^p+tb^p]^{\frac{1}{p}})]^r+[f([ta^p+(1-t)b^p]^{\frac{1}{p}})]^r]^{\frac{1}{r}}\mathrm{d}t\right]
\end{aligned}$$

$$\leq \left(\frac{1}{2}\right)^{\frac{1}{r}} \left[\left(\int_0^1 f([(1-t)a^p + tb^p]^{\frac{1}{p}}) dt \right)^r + \left(\int_0^1 f([ta^p + (1-t)b^p]^{\frac{1}{p}}) dt \right)^r \right]^{\frac{1}{r}}$$
$$= \left[\left(\frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} \mathrm{d}x \right)^r \right]^{\frac{1}{r}}.$$

This implies

$$\begin{aligned}
\left(f\left(\left[\frac{a^p + b^p}{2} \right]^{\frac{1}{p}} \right) \right)^r &\leq \left(\frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} \mathrm{d}x \right)^r \\
&= \left(\int_0^1 f([ta^p + (1-t)b^p]^{\frac{1}{p}}) dt \right)^r \\
&\leq \left(\int_0^1 [(1-t)[f(a)]^r + t[f(b)]^r]^{\frac{1}{r}} dt \right)^r \\
&\leq \left(\int_0^1 (1-t)^{\frac{1}{r}} [f(a)] dt \right)^r + \left(\int_0^1 t^{\frac{1}{r}} [f(b)] dt \right)^r \\
&= [[f(a)]^r + [f(b)]^r] \left(\frac{r}{1+r} \right)^r,
\end{aligned}$$

which is the required result. □

We now discuss some special cases.

Special Cases:

I). If $p = 1$ and under the assumptions of Theorem 2.1, then (2.1) reduces to the following know result [6]:

$$f\left(\frac{a+b}{2}\right) \quad \leq \quad \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x \leq ([f(a)]^r + [f(b)]^r)^{\frac{1}{r}} \left(\frac{r}{r+1}\right).$$

II). If $p = -1$ and under the assumptions of Theorem 2.1, then (2.1) reduces to the following new result:

$$f\left(\frac{2ab}{a+b}\right) \quad \leq \quad \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x \leq ([f(a)]^r + [f(b)]^r)^{\frac{1}{r}} \left(\frac{r}{r+1}\right).$$

III). If $p = 0$ and under the assumptions of Theorem 2.1, then (2.1) reduces to the following new result:

$$f(\sqrt{ab}) \quad \leq \quad \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x}\,\mathrm{d}x \leq ([f(a)]^r + [f(b)]^r)^{\frac{1}{r}} \left(\frac{r}{r+1}\right).$$

Theorem 2.2. *Let $f, g \in (p, r, I)$ with $a < b$. Then, for $r > 0$,*

$$\begin{aligned}
(2.2) \qquad & \left(\frac{p}{b^p - a^p} \int_a^b \frac{f(x)g(x)}{x^{1-p}} \mathrm{d}x \right)^r \\
& \leq M(a,b) \left(\frac{r}{2+r} \right)^r + N(a,b) \left(\beta\left(\frac{1}{r} + 1, \frac{1}{r} + 1 \right) \right)^r,
\end{aligned}$$

where

$$M(a,b)=[f(a)g(a)]^r+[f(b)g(b)]^r, \tag{2.3}$$

$$N(a,b)=[f(a)g(b)]^r+[f(b)g(a)]^r, \tag{2.4}$$

and $\beta(\cdot,\cdot)$ *is the Beta function.*

Proof. Let $f\in(p,r,I)$ and $g\in(p,r,I)$ with $a<b$, where $r>0$, we have

$$\begin{aligned}f([(1-t)a^p+tb^p]^{\frac{1}{p}})&\leq[(1-t)[f(a)]^r+t[f(b)]^r]^{\frac{1}{r}},\\ g([(1-t)a^p+tb^p]^{\frac{1}{p}})&\leq[(1-t)[g(a)]^r+t[g(b)]^r]^{\frac{1}{r}}.\end{aligned}$$

Using Minkowski's inequality, we have

$$\begin{aligned}&\left(\frac{p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{1-p}}\mathrm{d}x\right)^r\\ &=\left(\int_0^1 f([(1-t)a^p+tb^p]^{\frac{1}{p}})g([(1-t)a^p+tb^p]^{\frac{1}{p}})\mathrm{d}t\right)^r\\ &\leq\left(\int_0^1[(1-t)^2[f(a)g(a)]^r+t(1-t)[f(b)g(a)]^r+(1-t)t[f(a)g(b)]^r\right.\\ &\qquad\left.+t^2[f(b)g(b)]^r]^{\frac{1}{r}}\mathrm{d}t\right)^r\\ &\leq[f(a)g(a)]^r\left(\int_0^1[(1-t)]^{\frac{2}{r}}\mathrm{d}t\right)^r+[f(a)g(b)]^r\left(\int_0^1[(1-t)t]^{\frac{1}{r}}\mathrm{d}t\right)^r\\ &\qquad+[f(b)g(a)]^r\left(\int_0^1[t(1-t)]^{\frac{1}{r}}\mathrm{d}t\right)^r+[f(b)g(b)]^r\left(\int_0^1[t]^{\frac{2}{r}}\mathrm{d}t\right)^r\\ &=\Big([f(a)g(a)]^r+[f(b)g(b)]^r\Big)\left(\int_0^1[t]^{\frac{2}{r}}\mathrm{d}t\right)^r\\ &\qquad+\Big([f(a)g(b)]^r+[f(b)g(a)]^r\Big)\left(\int_0^1[t(1-t)]^{\frac{1}{r}}\mathrm{d}t\right)^r\\ &=M(a,b)\left(\frac{r}{2+r}\right)^r+N(a,b)\left(\beta\left(\frac{1}{r}+1,\frac{1}{r}+1\right)\right)^r,\end{aligned}$$

which is the required result. □

Special Cases:

I). If $p=1$ and under the assumptions of Theorem 2.2, then (2.2) reduces to the following known result [6]:

$$\left(\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x\right)^r\leq M(a,b)\left(\frac{r}{2+r}\right)^r+N(a,b)\left(\beta\left(\frac{1}{r}+1,\frac{1}{r}+1\right)\right)^r.$$

II). If $p = -1$ and under the assumptions of Theorem 2.2, then (2.2) reduces to the following new result:

$$\left(\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x\right)^r$$
$$\leq M(a,b)\left(\frac{r}{2+r}\right)^r + N(a,b)\left(\beta\left(\frac{1}{r}+1, \frac{1}{r}+1\right)\right)^r.$$

III). If $p = 0$ and under the assumptions of Theorem 2.2, then (2.2) reduces to the following new result:

$$\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)g(x)}{x}\mathrm{d}x\right)^r$$
$$\leq M(a,b)\left(\frac{r}{2+r}\right)^r + N(a,b)\left(\beta\left(\frac{1}{r}+1, \frac{1}{r}+1\right)\right)^r.$$

THEOREM 2.3. *Let $f, g \in (p, r, I)$ be similarly ordered with $a < b$. Then, for $r > 0$,*

$$\frac{p}{b^p - a^p}\int_a^b \frac{f(x)g(x)}{x^{1-p}}\mathrm{d}x \leq [[f(a)g(a)]^r + [f(b)g(b)]^r]^{\frac{1}{r}}\left(\frac{r}{1+r}\right).$$

PROOF. Let $f, g \in (p, r, I)$ with $a < b$, where $r > 0$. Then

$$f([(1-t)a^p + tb^p]^{\frac{1}{p}}) \leq [(1-t)[f(a)]^r + t[f(b)]^r]^{\frac{1}{r}},$$
$$g([(1-t)a^p + tb^p]^{\frac{1}{p}}) \leq [(1-t)[g(a)]^r + t[g(b)]^r]^{\frac{1}{r}}.$$

From (1.6), we have

$$f([(1-t)a^p + tb^p]^{\frac{1}{p}})g([(1-t)a^p + tb^p]^{\frac{1}{p}}) \leq [(1-t)[f(a)g(a)]^r + t[f(b)g(b)]^r]^{\frac{1}{r}}.$$

Integrating the above inequality over $[0, 1]$ and using Minkowski's inequality, we have

$$\frac{p}{b^p - a^p}\int_a^b \frac{f(x)g(x)}{x^{1-p}}\mathrm{d}x$$
$$\leq \left(\int_0^1 [(1-t)[f(a)g(a)]^r + t[f(b)g(b)]^r]^{\frac{1}{r}}\mathrm{d}t\right)^r$$
$$\leq [f(a)g(a)]^r\left(\int_0^1 [(1-t)]^{\frac{1}{r}}\mathrm{d}t\right)^r + [f(a)g(b)]^r\left(\int_0^1 t^{\frac{1}{r}}\mathrm{d}t\right)^r$$
$$= [[f(a)g(a)]^r + [f(b)g(b)]^r]^{\frac{1}{r}}\left(\frac{r}{1+r}\right),$$

which is the required result. □

THEOREM 2.4. *Let $f, g \in (p, r, I)$ with $a < b$. Then, for $r > 0$,*

$$\left(\frac{p}{b^p - a^p}\int_a^b \frac{f(x)g([a^p + b^p - x^p]^{\frac{1}{p}})}{x^{1-p}}\mathrm{d}x\right)^r$$
$$\leq M(a,b)\left(\beta\left(\frac{1}{r}+1, \frac{1}{r}+1\right)\right)^r + N(a,b)\left(\frac{r}{2+r}\right)^r,$$

where $M(a,b)$ and $N(a,b)$ are given by (2.3) *and* (2.4), *respectively and $\beta(\cdot,\cdot)$ is the beta function.*

PROOF. Let $f, g \in (p, r, I)$ with $a < b$, where $r > 0$. Then

$$f([(1-t)a^p + tb^p]^{\frac{1}{p}}) \leq [(1-t)[f(a)]^r + t[f(b)]^r]^{\frac{1}{r}},$$
$$g([ta^p + (1-t)b^p]^{\frac{1}{p}}) \leq [t[g(a)]^r + (1-t)[g(b)]^r]^{\frac{1}{r}}.$$

Integrating and using Minkowski's inequality, we have

$$\begin{aligned}
&\left(\frac{p}{b^p - a^p}\int_a^b \frac{f(x)g([a^p + b^p - x^p]^{\frac{1}{p}})}{x^{1-p}}\mathrm{d}x\right)^r \\
&= \left(\int_0^1 f([(1-t)a^p + tb^p]^{\frac{1}{p}})g([ta^p + (1-t)b^p]^{\frac{1}{p}})\mathrm{d}t\right)^r \\
&\leq [f(a)g(a)]^r\left(\int_0^1 [t(1-t)]^{\frac{1}{r}}\mathrm{d}t\right)^r + [f(a)g(b)]^r\left(\int_0^1 [(1-t)]^{\frac{2}{r}}\mathrm{d}t\right)^r \\
&\quad + [f(b)g(a)]^r\left(\int_0^1 t^{\frac{2}{r}}\mathrm{d}t\right)^r + [f(b)g(b)]^r\left(\int_0^1 [t(1-t)]^{\frac{1}{r}}\mathrm{d}t\right)^r \\
&= \left([f(a)g(a)]^r + [f(b)g(b)]^r\right)\left(\int_0^1 [t(1-t)]^{\frac{1}{r}}\mathrm{d}t\right)^r \\
&\quad + \left([f(a)g(b)]^r + [f(b)g(a)]^r\right)\left(\int_0^1 t^{\frac{2}{r}}\mathrm{d}t\right)^r \\
&= M(a,b)\left(\beta\left(\frac{1}{r}+1, \frac{1}{r}+1\right)\right)^r + N(a,b)\left(\frac{r}{2+r}\right)^r,
\end{aligned}$$

which is the required result. ∎

THEOREM 2.5. *Let $f \in (p, r_1, I)$ and $g \in (p, r_2, I)$ with $a < b$. Then, for $r_1 > 0$, and $r_2 > 0$, we have*

$$\begin{aligned}
\frac{p}{b^p - a^p}\int_a^b \frac{f(x)g(x)}{x^{1-p}}\mathrm{d}x &\leq \frac{1}{2}\left(\frac{r_1}{2+r_1}\right)\left[[f(a)]^{r_1} + [f(b)]^{r_1}\right]^{\frac{2}{r_1}} \\
&\quad + \frac{1}{2}\left(\frac{r_2}{2+r_2}\right)\left[[g(a)]^{r_2} + [g(b)]^{r_2}\right]^{\frac{2}{r_2}}.
\end{aligned} \tag{2.5}$$

PROOF. Let $f \in (p, r_1, I)$ and $g \in (p, r_2, I)$ with $(r_1 > 0,\ r_2 > 0)$. Then

$$f([(1-t)a^p + tb^p]^{\frac{1}{p}}) \le [(1-t)[f(a)]^{r_1} + t[f(b)]^{r_1}]^{\frac{1}{r_1}},$$
$$g([(1-t)a^p + tb^p]^{\frac{1}{p}}) \le [(1-t)[g(a)]^{r_2} + t[g(b)]^{r_2}]^{\frac{1}{r_2}}.$$

Now multiplying the above inequalities and integrating over $[0, 1]$, we have

$$\frac{p}{b^p - a^p}\int_a^b \frac{f(x)g(x)}{x^{1-p}}\mathrm{d}x = \int_0^1 f([(1-t)a^p + tb^p]^{\frac{1}{p}})g([(1-t)a^p + tb^p]^{\frac{1}{p}})\mathrm{d}t$$
$$\le \int_0^1 [(1-t)[f(a)]^{r_1} + t[f(b)]^{r_1}]^{\frac{1}{r_1}}[(1-t)[g(a)]^{r_2} + t[g(b)]^{r_2}]^{\frac{1}{r_2}}\mathrm{d}t$$
$$\le \frac{1}{2}\int_0^1 [(1-t)[f(a)]^{r_1} + t[f(b)]^{r_1}]^{\frac{2}{r_1}}\mathrm{d}t + \frac{1}{2}\int_0^1 [(1-t)[g(a)]^{r_2} + t[g(b)]^{r_2}]^{\frac{2}{r_2}}\mathrm{d}t.$$

Using Minkowski's inequality, we have

$$\begin{aligned}
&\int_0^1 [(1-t)[f(a)]^{r_1} + t[f(b)]^{r_1}]^{\frac{2}{r_1}}\mathrm{d}t \\
&\le \left[\left(\int_0^1 (1-t)^{\frac{2}{r_1}}[f(a)]^2\mathrm{d}t\right)^{\frac{r_1}{2}} + \left(\int_0^1 t^{\frac{2}{r_1}}[f(b)]^2\mathrm{d}t\right)^{\frac{r_1}{2}}\right]^{\frac{2}{r_1}} \\
&= \left[[f(a)]^{r_1}\left(\int_0^1 (1-t)^{\frac{2}{r_1}}\mathrm{d}t\right)^{\frac{r_1}{2}} + [f(b)]^{r_1}\left(\int_0^1 t^{\frac{2}{r_1}}\mathrm{d}t\right)^{\frac{r_1}{2}}\right]^{\frac{2}{r_1}} \\
&= \left[[f(a)]^{r_1} + [f(b)]^{r_1}\right]^{\frac{2}{r_1}}\int_0^1 [t]^{\frac{2}{r_1}}\mathrm{d}t \\
&= \left[[f(a)]^{r_1} + [f(b)]^{r_1}\right]^{\frac{2}{r_1}}\left(\frac{r_1}{2+r_1}\right).
\end{aligned}$$

It is enough to note that a similar result holds for the function g. Thus

$$\begin{aligned}
\frac{p}{b^p - a^p}\int_a^b \frac{f(x)g(x)}{x^{1-p}}\mathrm{d}x \le{}& \frac{1}{2}\left(\frac{r_1}{2+r_1}\right)\left[[f(a)]^{r_1} + [f(b)]^{r_1}\right]^{\frac{2}{r_1}} \\
&+ \frac{1}{2}\left(\frac{r_2}{2+r_2}\right)\left[[g(a)]^{r_2} + [g(b)]^{r_2}\right]^{\frac{2}{r_2}},
\end{aligned}$$

which is the required result. □

Special Cases:

I). If $p = 1$ and under the assumptions of Theorem 2.5, then (2.5) reduces to the following know result [6]:

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \le{}& \frac{1}{2}\left(\frac{r_1}{2+r_1}\right)\left[[f(a)]^{r_1} + [f(b)]^{r_1}\right]^{\frac{2}{r_1}} \\
&+ \frac{1}{2}\left(\frac{r_2}{2+r_2}\right)\left[[g(a)]^{r_2} + [g(b)]^{r_2}\right]^{\frac{2}{r_2}}.
\end{aligned}$$

II). If $r_1 = r_2 = 2$ and under the assumptions of Theorem 2.5, then (2.5) reduces to the following new result:

$$\frac{p}{b^p - a^p}\int_a^b \frac{f(x)g(x)}{x^{1-p}}\mathrm{d}x \leq \frac{1}{4}\Bigg[[f(a)]^2 + [f(b)]^2 + [g(a)]^2 + [g(b)]^2\Bigg].$$

III). If $r_1 = r_2 = 2,\ f(x) = g(x)$ and under the assumptions of Theorem 2.5, then (2.5) reduces to the following new result:

$$\frac{p}{b^p - a^p}\int_a^b \frac{[f(x)]^2}{x^{1-p}}\mathrm{d}x \leq \frac{1}{2}\Bigg[[f(a)]^2 + [f(b)]^2\Bigg].$$

IV). If $p = -1$ and under the assumptions of Theorem 2.5, then (2.5) reduces to the following known result [9]:

$$\begin{aligned}\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x &\leq \frac{1}{2}\left(\frac{r_1}{r_1+2}\right)\left[[f(a)]^{r_1} + [f(b)]^{r_1}\right]^{\frac{2}{r_1}}\\ &+ \frac{1}{2}\left(\frac{r_2}{r_2+2}\right)\left[[g(a)]^{r_2} + [g(b)]^{r_2}\right]^{\frac{2}{r_2}}.\end{aligned}$$

V). If $p = 0$ and under the assumptions of Theorem 2.5, then (2.5) reduces to the following known result [9]:

$$\begin{aligned}\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)g(x)}{x}\mathrm{d}x &\leq \frac{1}{2}\left(\frac{r_1}{r_1+2}\right)\left[[f(a)]^{r_1} + [f(b)]^{r_1}\right]^{\frac{2}{r_1}}\\ &+ \frac{1}{2}\left(\frac{r_2}{r_2+2}\right)\left[[g(a)]^{r_2} + [g(b)]^{r_2}\right]^{\frac{2}{r_2}}.\end{aligned}$$

THEOREM 2.6. *Let $f \in (p, r_1, I)$ and $g \in (p, r_2, I)$ with $a < b$. Then for $r_1 > 0$, $r_2 > 0$ and $\frac{1}{r_1} + \frac{1}{r_2} = 1$, we have*

$$\frac{p}{b^p - a^p}\int_a^b \frac{f(x)g(x)}{x^{1-p}}\mathrm{d}x \quad\leq\quad \frac{\left([f(a)]^{r_1} + [f(b)]^{r_1}\right)^{\frac{1}{r_1}}\left([g(a)]^{r_2} + [g(b)]^{r_2}\right)^{\frac{1}{r_2}}}{2}.$$

PROOF. Let $f \in (p, r_1, I)$ and $g \in (p, r_2, I)$ with $(r_1 > 0,\ r_2 > 0)$. Then

$$\begin{aligned}f([(1-t)a^p + tb^p]^{\frac{1}{p}}) &\leq [(1-t)[f(a)]^{r_1} + t[f(b)]^{r_1}]^{\frac{1}{r_1}},\\ g([(1-t)a^p + tb^p]^{\frac{1}{p}}) &\leq [(1-t)[g(a)]^{r_2} + t[g(b)]^{r_2}]^{\frac{1}{r_2}}.\end{aligned}$$

for all $t \in [0,1]$. Now multiplying the above inequalities, using Hölder's inequality and integrating over $[0,1]$, we have

$$\frac{p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{1-p}}\mathrm{d}x = \int_0^1 f([(1-t)a^p+tb^p]^{\frac{1}{p}})g([(1-t)a^p+tb^p]^{\frac{1}{p}})\mathrm{d}t$$

$$\leq \int_0^1 [(1-t)[f(a)]^{r_1}+t[f(b)]^{r_1}]^{\frac{1}{r_1}}[(1-t)[g(a)]^{r_2}+t[g(b)]^{r_2}]^{\frac{1}{r_2}}\mathrm{d}t$$

$$\leq \left(\int_0^1 [(1-t)[f(a)]^{r_1}+t[f(b)]^{r_1}]\mathrm{d}t\right)^{\frac{1}{r_1}}\left(\int_0^1 [(1-t)^s[g(a)]^{r_2}+t[g(b)]^{r_2}]\mathrm{d}t\right)^{\frac{1}{r_2}}$$

$$= \left([f(a)]^{r_1}\int_0^1(1-t)\mathrm{d}t+[f(b)]^{r_1}\int_0^1 t\mathrm{d}t\right)^{\frac{1}{r_1}}$$

$$\times\left([g(a)]^{r_2}\int_0^1(1-t)\mathrm{d}t+[g(b)]^{r_2}\int_0^1 t\mathrm{d}t\right)^{\frac{1}{r_2}}$$

$$= \frac{\left([f(a)]^{r_1}+[f(b)]^{r_1}\right)^{\frac{1}{r_1}}\left([g(a)]^{r_2}+[g(b)]^{r_2}\right)^{\frac{1}{r_2}}}{2}.$$

Thus

$$\frac{p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{1-p}}\mathrm{d}x \quad\leq\quad \frac{\left([f(a)]^{r_1}+[f(b)]^{r_1}\right)^{\frac{1}{r_1}}\left([g(a)]^{r_2}+[g(b)]^{r_2}\right)^{\frac{1}{r_2}}}{2},$$

which is the required result. □

For suitable and appropriate choice of r_1, r_2 and p, one can obtain several new integral inequalities for various classes of convex functions.

THEOREM 2.7. *Let $f,g \in (p,r,I)$ with $a<b$. Then, for $r>0$, we have*

$$2\left[f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right]^r - \left(\frac{p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{1-p}}\mathrm{d}x\right)^r$$

$$\leq M(a,b)\left(\beta\left(\frac{1}{r}+1,\frac{1}{r}+1\right)\right)^r + N(a,b)\left(\frac{r}{2+r}\right)^r, \tag{2.6}$$

where $M(a,b)$ and $N(a,b)$ are given by (2.3) *and* (2.4), *respectively and $\beta(\cdot,\cdot)$ is the beta function.*

PROOF. Let $f,g \in (p,r,I)$ with $t=\frac{1}{2}$. Then

$$f\left(\left[\frac{x^p+y^p}{2}\right]^{\frac{1}{p}}\right) \leq \left[\frac{[f(x)]^r+[f(y)]^r}{2}\right]^{\frac{1}{r}}, \qquad \forall x,y\in I.$$

$$g\left(\left[\frac{x^p+y^p}{2}\right]^{\frac{1}{p}}\right) \leq \left[\frac{[g(x)]^r+[g(y)]^r}{2}\right]^{\frac{1}{r}}, \qquad \forall x,y\in I.$$

Let $x=[(1-t)a^p+tb^p]^{\frac{1}{p}}$, and $y=[ta^p+(1-t)b^p]^{\frac{1}{p}}$. Then we have

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leq \left(\frac{1}{2}\right)^{\frac{1}{r}} \left(\left[f([(1-t)a^p+tb^p]^{\frac{1}{p}})\right]^r+\left[f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right]^r\right)^{\frac{1}{r}}.$$

$$g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leq \left(\frac{1}{2}\right)^{\frac{1}{r}} \left(\left[g([(1-t)a^p+tb^p]^{\frac{1}{p}})\right]^r+\left[g\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right]^r\right)^{\frac{1}{r}}.$$

Let $\bar{a}=[(1-t)a^p+tb^p]^{\frac{1}{p}}$, and $\bar{b}=[ta^p+(1-t)b^p]^{\frac{1}{p}}$. Then

$$\begin{aligned}
&f\big([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\big)g\big([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\big)\\
&\leq \big(\frac{1}{4}\big)^{\frac{1}{r}}\big([f(\bar{a})]^r[g(\bar{a})]^r+[f(\bar{b})]^r[g(\bar{b})]^r+[f(\bar{a})]^r[g(\bar{b})]^r+[f(\bar{b})]^r[g(\bar{a})]^r\big)^{\frac{1}{r}}\\
&=\big(\frac{1}{4}\big)^{\frac{1}{r}}\int_0^1 \big([f(\bar{a})]^r[g(\bar{a})]^r+[f(\bar{b})]^r[g(\bar{b})]^r+[f(\bar{a})]^r[g(\bar{b})]^r+[f(\bar{b})]^r[g(\bar{a})]^r\big)^{\frac{1}{r}}\mathrm{d}t.
\end{aligned}$$

Using Minkowski's inequality, we have

$$\begin{aligned}
&\left(\int_0^1 \big([f(\bar{a})]^r[g(\bar{a})]^r+[f(\bar{b})]^r[g(\bar{b})]^r+[f(\bar{a})]^r[g(\bar{b})]^r+[f(\bar{b})]^r[g(\bar{a})]^r\big)^{\frac{1}{r}}\right)^r\mathrm{d}t\\
&\leq \left(\int_0^1[f(\bar{a})][g(\bar{a})]\mathrm{d}t\right)^r+\left(\int_0^1[f(\bar{b})][g(\bar{b})]\mathrm{d}t\right)^r\\
&\quad+\left(\int_0^1[f(\bar{a})][g(\bar{b})]\mathrm{d}t\right)^r+\left(\int_0^1[f(\bar{b})][g(\bar{a})]\mathrm{d}t\right)^r\\
&\leq \left(\int_0^1[f(\bar{a})][g(\bar{a})]\mathrm{d}t\right)^r+\left(\int_0^1[f(\bar{a})][g(\bar{b})]\mathrm{d}t\right)^r\\
&\quad+\left(\int_0^1((1-t)[f(a)]^r+t[f(b)]^r)^{\frac{1}{r}}(t[g(a)]^r+(1-t)[g(b)]^r)^{\frac{1}{r}}\mathrm{d}t\right)^r\\
&\quad+\left(\int_0^1(t[f(a)]^r+(1-t)[f(b)]^r)^{\frac{1}{r}}((1-t)[g(a)]^r+t[g(b)]^r)^{\frac{1}{r}}\mathrm{d}t\right)^r\\
&\leq 2\Bigg[\left(\frac{p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{1-p}}\mathrm{d}x\right)^r\\
&\quad+M(a,b)\left(\beta\left(\frac{1}{r}+1,\frac{1}{r}+1\right)\right)^r+N(a,b)\left(\frac{r}{2+r}\right)^r\Bigg].
\end{aligned}$$

This implies

$$\begin{aligned}
&2\left[f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right]^r-\left(\frac{p}{b^p-a^p}\int_a^b\frac{f(x)g(x)}{x^{1-p}}\mathrm{d}x\right)^r\\
&\qquad\leq M(a,b)\left(\beta\left(\frac{1}{r}+1,\frac{1}{r}+1\right)\right)^r+N(a,b)\left(\frac{r}{2+r}\right)^r,
\end{aligned}$$

which is the required result. $\square$

Special Cases:

I). If $p = 1$ and under the assumptions of Theorem 2.7, then (2.7) reduces to the following know result [6]:

$$2\big[f\big(\frac{a+b}{2}\big)g\big(\frac{a+b}{2}\big)\big]^r - \big(\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x\big)^r$$
$$\leq M(a,b)\big(\beta\big(\frac{1}{r}+1,\frac{1}{r}+1\big)\big)^r + N(a,b)\big(\frac{r}{r+2}\big)^r.$$

II). If $p = -1$ and under the assumptions of Theorem 2.7, then (2.7) reduces to the following know result [9]:

$$2\big[f\big(\frac{2ab}{a+b}\big)g\big(\frac{2ab}{a+b}\big)\big]^r - \big(\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x\big)^r$$
$$\leq M(a,b)\big(\beta\big(\frac{1}{r}+1,\frac{1}{r}+1\big)\big)^r + N(a,b)\big(\frac{r}{r+2}\big)^r.$$

Conclusion

In this paper, we have introduced a new class of convex functions, which can be viewed as unifying one. We have derived several integral inequalities via this class. Results obtained in this paper can be viewed as significant contribution in this field. For appropriate and suitable choice of p and r, one can obtain new results for various known and new classes of convex functions. Ideas and techniques of this paper may stimulate further research.

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References

[1] G. Cristescu and L. Lupsa, Non-connected Convexities and Applications, Kluwer Academic Publisher, Dordrechet, 2002.

[2] P. M. Gill, C. E. M. Pearce and J. Pečarić, *Hadamard's inequality for r-convex functions*, J. Math. Anal. Appl. **215** (1997), 461–470.

[3] J. Hadamard, *Etude sur les proprietes des fonctions entieres et en particulier dune fonction consideree par Riemann*, J. Math. Pure Appl. **58** (1893), 171–215.

[4] L. V. Hap and N. V. Vinh, *On some Hadamard-type inequalities for (h,r)-convex functions*, Int. J. Math. Anal. (Ruse) **7(42)** (2013), 2067–2075.

[5] I. Iscan, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacettepe J. Math. Stats. **43** (2014), 935–942.

[6] N. P. N. Ngoc, N. V. Vinh and P. T. T. Hien, *Integral inequalities of Hadamard type for r-convex functions*, Int. Math. Forum **4(35)** (2009), 1723–1728.

[7] C. P. Niculescu and L. E. Persson, Convex Functions and Their Applications, Springer-Verlag, New York, 2006.

[8] M. A. Noor, K. I. Noor and S. Iftikhar, *Nonconvex functions and integral inequalities*, Punjab Univ. J. Math. (Lahore) **47** (2015), 19–27.
[9] M. A. Noor, K. I. Noor and U. Awan, *Some new estimates of Hermite-Hadamard inequalities via harmonically r-convex functions*, Matematiche (Catania) 71 (2016), 117–127.
[10] M. A. Noor, K. I. Noor and S. Iftikhar, *Hermite-Hadamard inequalities for harmonic nonconvex functions*, MAGNT Research Report **4** (2016), 24–40.
[11] M. A. Noor, K. I. Noor and S. Iftikhar, *On harmonic (h, r)-convex functions*, Proc. Jangjeon Math. Soc. **21** (2018), 239–251.
[12] M. A. Noor, K. I. Noor and S. Iftikhar, *Integral inequalities for differentiable p-harmonic convex functions*, Filomat **31** (2017), 6575-6584.
[13] C. E. M. Pearce, J. Pečarić and V. Šimić, *Stolarsky means and Hadamard's inequality*, J. Math. Anal. Appl. **220** (1998), 99–109.
[14] C. E. M. Pearce, J. Pečarić and V. Šimić, *On weighted generalized logarithmic means*, Houston J. Math. **24** (1998), 459–465.
[15] J. Park, *On the Hermite-Hadamard-like type inequalities for co-ordinated (s, r)-convex mappings*, Int. J. Pure Appl. Math. **74(2)** (2012), 251-263.
[16] J. Pečarić, F. Proschan and Y. L. Tong, Convex Functions, Partial Orderings and Statistical Applications, Academic Press, New York, 1992.
[17] M. Z. Sarikaya, H. Yaldiz and H. Bozkurt, *On the Hadamard type integral inequalities involving several differentiable $(\phi - r)$-convex functions*, 2012, arXiv:1203.2278.
[18] K. S. Zhang and J. P. Wan, *p-convex functions and their properties*, Pure Appl. Math. (Xi'an) **23** (2007), 130–133.

Nejednakosti za (p, r)-konveksne funkcije

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Sažetak. Glavni cilj ovog članka je uvesti novu klasu konveksnih funkcija, koje nazivamo (p, r)-konveksne funkcije. Izvodimo neke nove integralne nejednakosti Hermite-Hadamardovog tipa za (p, r)-konveksne funkcije. Diskutiramo i neke specijalne slučajeve. Ovi rezultati se mogu smatrati značajnim poboljšanjem prije poznatih rezultata. Tehnike i ideje uvedene u ovom članku mogle bi motivirati daljnja istraživanja.

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