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BUCKLING ANALYSIS OF A LAMINATE PLATE

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Preliminary Note – Prethodno priopćenje

The paper deals with a modeling of laminate plates and with their buckling analysis. To predict the inception of buckling for plates in plane resultant forces must be included. The buckling analysis is made by the help of finite element method in program COSMOS/M. For rectangular laminate plate consisting of 4 layers with symmetric and antisymmetric stacking sequence a buckling analysis is carried out. In the illustrative example there are depicted buckling modes for symmetric laminates $[30/\text{-}30]_S$, $[45/\text{-}45]_S$, $[60/\text{-}60]_S$, $[90/\text{-}90]_S$ and results of the buckling analysis for the symmetric and antisymmetric laminates.

Key words: laminated 2-D structures, classical laminate theory, buckling analysis, finite element method

Analiza deformacije laminatne ploče. Rad se bavi modeliranjem laminatnih pliča i analizom njihove deformacije. Da bi se predvidio početak deformacije za ploče u ravnini moraju se uključiti izlazne sile. Analiza deformacije izvršena je pomoću metode konačnih elemenata u programu COSMOS/M. Za pravokutnu laminatnu ploču koja se sastoji od 4 sloja sa simetričnim i antisimetričnim nizom slaganja laminate $[30/\text{-}30]_S$, $[45/\text{-}45]_S$, $[90/\text{-}90]_S$ i rezultati analize deformacije za simetrične laminate.

Ključne riječi: laminatne dvodimenzionalne (2D) strukture, klasična laminatna teorija, analiza deformacije, metoda konačnih elemenata

INTRODUCTION

Laminate structures are typical lightweight elements with expanding application in civil and mechanical engineering. For the modeling and analysis of laminates we used the classical laminate theory (CLT). The CLT is an extension of Kirchhoff´s classical plate theory for homogeneous isotropic plates to laminated composite plates with a high width-to-thickness ratio.

The assumptions for macro-mechanical modeling of laminate are [1]:

1. All layers are in a state of plane stress, i. e.:
$$\sigma_z = \tau_{xz} = \tau_{yz} = 0. \tag{1}$$
2. Normal distances from the middle surface remain constant.
3. The transverse shear strains γ_{xz}, γ_{yz} are negligible.

In-plane strains can be noted as:

$$\varepsilon(x, y, z) = \bar{\varepsilon}(x, y) + z\kappa, \tag{2}$$

where:
$\bar{\varepsilon}$ is the vector of the in-plane or membrane strains,
κ is the vector of curvature subjected to bending and twisting.

The stress resultant force vectors are:

$$N = \int_{-h/2}^{+h/2} E(z)dz\bar{\varepsilon} + \int_{-h/2}^{+h/2} E(z)zdz\kappa,$$
$$M = \int_{-h/2}^{+h/2} E(z)zdz\bar{\varepsilon} + \int_{-h/2}^{+h/2} E(z)z^2dz\kappa, \tag{3}$$

where:
N is the in-plane stress resultant force vector,
M is the resultant moment vector,
E is matrix of elasticity.

EQUILIBRIUM EQUATIONS

The equilibrium equations are formulated for a plate element (Figure 1) and yield three force and two moment equations:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} + p_1 = 0, \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + p_2 = 0,$$
$$\frac{\partial V_{xz}}{\partial x} + \frac{\partial V_{yz}}{\partial y} + p_3 = 0, \frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} = V_{xz},$$
$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = V_{yz}. \tag{4}$$

p_1, p_2, p_3 are plate loads in the x, y, z direction respectively

For a coupling of in-plane loads and lateral deflection, the equilibrium Eqs (4) will be formulated for the deformed plate element and are modified to:

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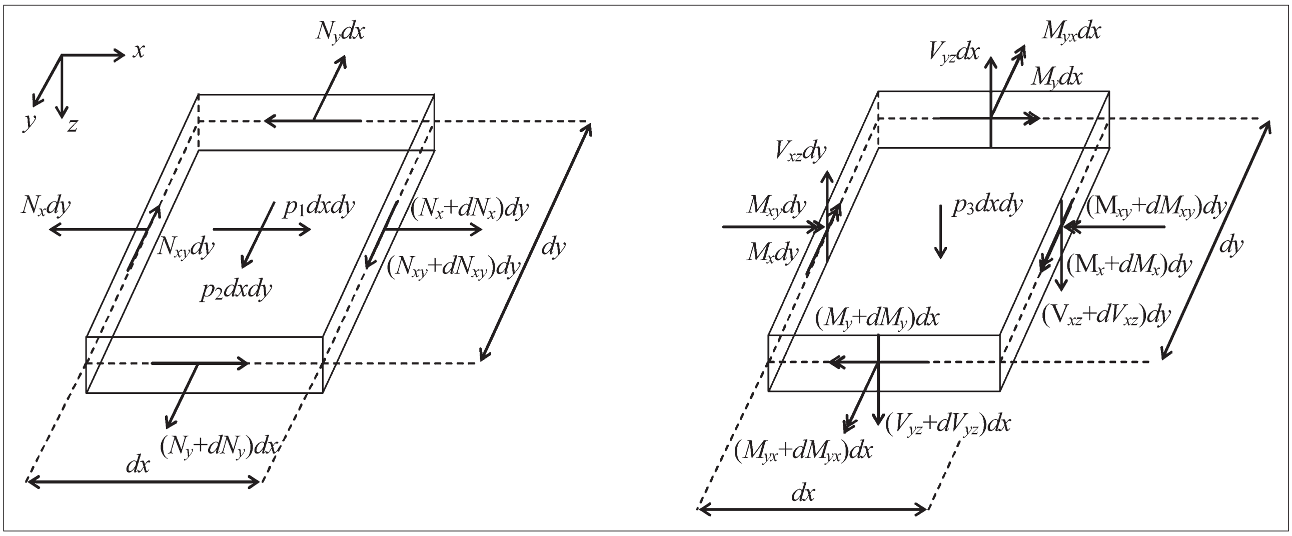


Figure 1. Stress resultants applied to a plate element

$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}=p_3+N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}$$
$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{yx}}{\partial y}=0,\ \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=0. \qquad (5)$$

In the general case of a symmetric laminate with $p_3 = 0$, the plate equation can be expressed by:

$$D_{11}\frac{\partial^4 w}{\partial x^4}+4D_{16}\frac{\partial^4 w}{\partial x^3\partial y}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+$$
$$+4D_{26}\frac{\partial^4 w}{\partial x\partial y^3}+D_{22}\frac{\partial^4 w}{\partial y^4}=N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y} \qquad (6)$$

where:
D is bending stiffness matrix.

The elements of the matrix $\boldsymbol{D}$ are:

$$D=\int_{-h/2}^{+h/2}E(z)z^2dz=\sum_{n=1}^{N}\int_{n-1_z}^{n_z}{}^nEz^2dz=\sum_{n-1}^{N}{}^nE\frac{{}^nz^3-{}^{n-1}z^3}{3} \qquad (7)$$

The buckling load is like natural vibration independent of the laterial load and p_3 is taken to be zero [2].

FINITE ELEMENT ANALYSIS

The basic idea of the FEM is a discretisation of the continuous structure. The discretisation is defined by finite element mesh make up of elements nodes. The starting point for elastostatic problems is the total potential energy. In accordance with the Ritz method the approximation is used for displacement field vector u:

$$\tilde{u}(x)=N(x)v, \qquad (8)$$

where N is the matrix of the shape functions, that are functions of the position vector $x = (x, y, z)$ and v is the element displacement vector.

For the stresses and strains we obtain from Eq. (8) the Eq. (9):

$$\sigma(x)=E\varepsilon(x)=EDN(x)v,$$
$$\varepsilon(x)=Du(x)=DN(x)v=B(x)v \qquad (9)$$

E is the elasticity matrix obtained with suitable transformations in two stages, firstly from the principal material directions to the element local directions and secondly to the global directions. B is the strain matrix,

With the approximation (Eq. 8) the total potential energy is a function of all the nodal displacement components arranged in the element displacement vector v.

The variation of the total potential energy:

$$\partial\Pi=\partial v^T\left(\int_V B^TEBvdV-\int_V N^TpdV-\int_{O_q}N^TqdO\right) \qquad (10)$$

leads to:

$$\partial v^T(Kv-f_p-f_q)=0, \qquad (11)$$

where:
K is the symmetric stiffness matrix:

$$K=\int_V B^TEBdV, \qquad (12)$$

and $\boldsymbol{f}_p$ and $\boldsymbol{f}_q$ are the vectors of the volume forces and the surface forces:

$$f_p=\int_V N^TpdV,\ f_q=\int_{O_q}N^TqdO. \qquad (13)$$

If the components of δv are independent of each other, we obtain from Eq. (11) the system of linear equations:

$$Kv=f,\ f=f_p+f_q. \qquad (14)$$

All equations considered above are valid for a single finite element and they should have an additional index E. We have the inner element energy:

$$U_E=\frac{1}{2}v_E^T\int_{V_E}B^TEBdVv_E=\frac{1}{2}v_E^TK_Ev_E, \qquad (15)$$

with the element stiffness matrix:

$$K_E=\int_{V_E}B^TEBdV,\ E=\sum_{n=1}^{N}{}^nE,$$
$${}^nE=\overline{T}^T({}^n\beta)^{(n)}E_L\overline{T}({}^n\beta), \qquad (16)$$

where:

T is the transformation matrix with:

$$T(\beta) = (T^T(\beta))^{-1}. \qquad (17)$$

Because the energy is a scalar quantity, the potential energy of the whole structure can be obtained by summing the energies of the single elements. By a Boolean matrix L_E the correct position of each single element is determined. The element displacement vector v_E is positioned into the system displacement vector by the equation:

$$v_E = L_E v, \qquad (18)$$

then we obtain the system equation by summing over all elements:

$$\left(\sum_i L_{iE}^T K_{iE} L_{iE}\right) v = \left[\sum_i L_{iE}(f_{iEp} + f_{iEq})\right]. \qquad (19)$$

The system stiffness matrix is also symmetric, but it is a singular matrix. After consideration of the boundary conditions of the whole system, K becomes a positive definite matrix and the system equations can be solved.

The finite element method (FEM) is the effective method for the numerical solution of problem formulated in partial differential equation. For buckling analysis we used FEM in program COSMOS/M. We made discretization of the structure into a number of finite elements SHELL4L [3]. SHELL4L is 4-node multi-layer composite quadrilateral plate and shell element with membrane and bending capabilities. Each layer is associated with identical orthotropic material properties. The element SHELL4L is 4-node multi-layer element with membrane and bending capabilities. Each node has 6 degrees of freedom and than we obtain the element stiffness matrix with size [24x24] [4].

EXAMPLE

For a rectangular laminate plate consisting of 4 layers with the given material constants a buckling analysis is carried out. The plate is simply supported at all boundaries and loaded by a uniaxial uniform load (Figure 2). Material constants are listed in Table 1 [5].

For the stacking structure two cases shall be considered, a) symmetric and b) antisymmetric laminate structure (Figure 3a and b).

The fibre angle is to vary: $\alpha = 0^o, 30^o, 45^o, 60^o, 90^o$. For the buckling analysis in COSMOS/M a unit pressure

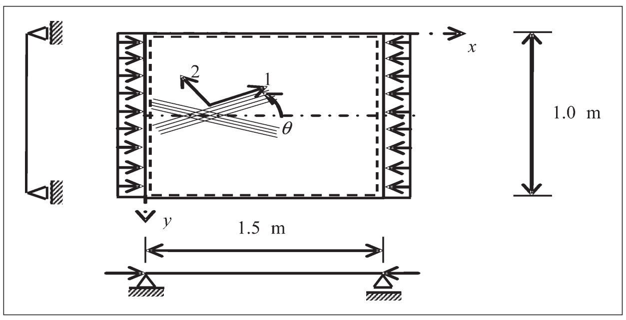
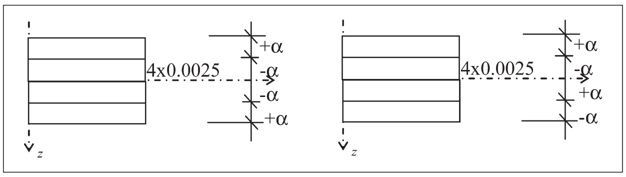


Figure 2. Rectangular laminate plate

E_1 / [GPa]	E_2 / [GPa]	G_{12} / [GPa]	ν_{12}
210	21.7	5.4	0.17

Table 1. **The material properties of each layer**



a) symmetric b) antisymmetric

Figure 3. Stacking sequence of layers

loading must be created, and the program calculates a factor to multiply the unit loading for obtaining the buckling load. At the Figure 4 are buckling modes for symmetric laminates $\alpha = 30^o, 45^o, 60^o, 90^o$, respectively. The results for the buckling factors are shown in a diagram in Figure 5.

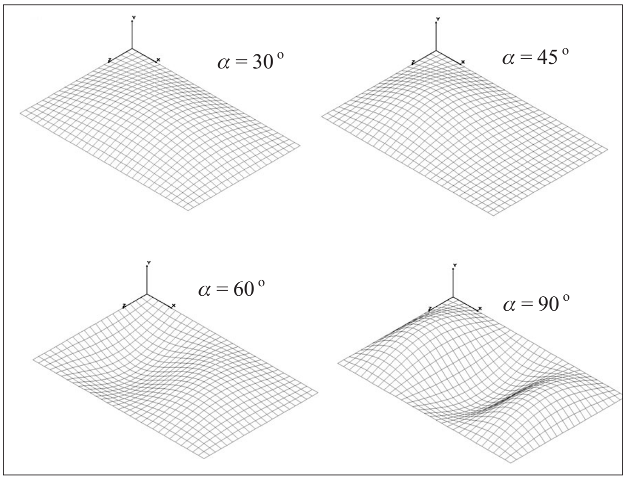


Figure 4. Buckling modes for symmetric laminates $\alpha = 30^o, 45^o, 60^o, 90^o$, respectively

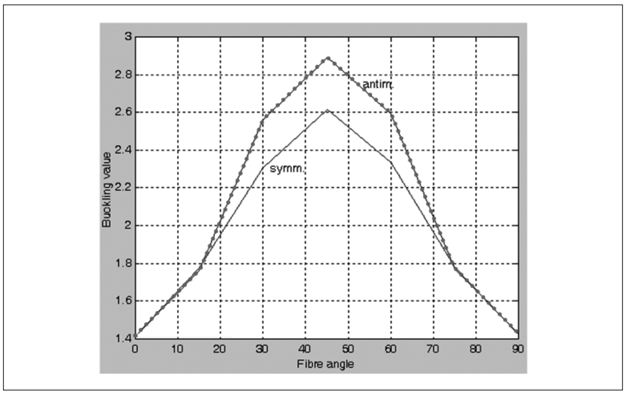


Figure 5. Results of the buckling analysis

CONCLUSION

The paper deals with a modeling of buckling analysis of laminate plates. To predict the inception of buckling for plates in-plane resultant forces must be included. The buckling modes are symmetric to the symmetric axis in loading direction. For the symmetric laminates the buckling modes for $\alpha = 0^o - 30^o$ are nearly the same.

For fibre angles α = 30 °, 45 °, 60 °, 90 ° the buckling modes have different shapes, they are shown in the Figure 4. The buckling modes for the antisymmetric laminate are very similar but not identical to the buckling modes of the symmetric laminate. A fibre angle near 45 ° leads to the highest buckling load for a quadratic plate (Figure 5). It shall be noted that the antisymmetric stacking sequence of the laminate improved the buckling stability.

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Note: The responsible translator for English language is the Author E. Kormaníková.

List of symbols:

(σ_z, τ_{xz}, τ_{yz}) - transverse normal and shear stresses
γ_{xz}, γ_{yz}, - transverse shear strains
ε - vector of in-plane strains
$\bar{\varepsilon}$ - vector of midplane strains
κ - vector of curvature subjected to bending and twisting
N - in-plane stress resultant force vector
M - resultant moment vector
E - matrix of elasticity
V - transverse shear force vector
p_1, p_2, p_3 - plate loads in the x, y, z direction
$u = (u, v, w)$ - displacement field vector
D - bending stiffness matrix
N - matrix of the shape functions
$x = (x,y,z)$ - position vector
v - element displacement vector.
B - strain matrix
σ - normal stress vector
K - symmetric stiffness matrix
f_p , f_q - vectors of the volume forces and the surface forces
U_E - inner element energy
T - transformation matrix
α - fibre angle
L_E - Boolean matrix