Affine Geometry of Minkowski Minimal Surfaces in $\mathbb{R}^3$

Dedicated to Professor Hellmuth Stachel on occasion of his 65th birthday

There are many results on affine geometry of Euclidean minimal surfaces in $\mathbb{R}^3$. Several results concern the focal surfaces of Minkowski minimal surfaces. Denoting $K_e$ the Gauss-curvature, in case of minimal surfaces the principal curvatures (eigenvalues of the shape operator) are $\varphi(-K_e)^{1/2}$ ($\varphi = \pm 1$).

**Theorem 1** (P. FRANCK [5]). Let $f : U \subseteq \mathbb{R}^2 \to f(U) =: \Phi \subseteq \mathbb{R}^3$ be a minimal immersion with $K_e \neq 0$ in $U$. The two sets of focal points $\Psi_1$ and $\Psi_{-1}$ of $f(U) = \Phi$ are parametrized by $z = f + \varphi(-K_e)^{-1/2}n_\varphi$, where $n_\varphi$ denotes the normal vector and $\Psi_1$ and $\Psi_{-1}$ correspond to $\varphi = 1$ and $\varphi = -1$ respectively. Then the following holds for $\varphi \in \{1, -1\}$:

(a) If $\Psi_\varphi$ is a regular surface, then $K_e(\Psi_\varphi) = -1/4K_e$, where $K_e(\Psi_\varphi)$ denotes the Gauss-curvature of $\Psi_\varphi$.

(b) If $\Psi_\varphi$ is a regular surface the affine normal of $\Psi_\varphi$ intersects the affine normal of $\Phi$ orthogonally.

(c) If $\Psi_\varphi$ is a regular surface then it is an affine minimal surface.

(d) If $\Psi_1$ and $\Psi_{-1}$ are both regular surfaces, then $K_a(\Psi_1) : K_a(\Psi_{-1}) = H_e(\Psi_1)^4 : H_e(\Psi_{-1})^4$, where $K_a(\Psi_1)$, $K_a(\Psi_{-1})$ and $H_e(\Psi_1)$, $H_e(\Psi_{-1})$ are the affine Gauss-curvature and the Euclidean mean curvature of $\Psi_1$ and $\Psi_{-1}$ respectively.

Results concerning the behavior of affine quantities by association are contained in

**Theorem 2** (P. FRANCK [5], F. MANHART [9]). Let $\Phi$ be a regular minimal surface with $K_e \neq 0$ and $\Psi_1$ and $\Psi_{-1}$ the sets of focal points as above. Denoting $(\lambda)\Phi$ the pencil of associated minimal surfaces to $\Phi$ the following holds:

(a) The affine normal vector $n_a(\Psi_1)$ of $\Phi$ is invariant: $n_a((\lambda)\Phi) = n_a(\Phi)$.

(b) The affine Gauss-curvature of $\Phi$ is invariant: $K_a((\lambda)\Phi) = K_a(\Phi)$.

(c) If $\Psi_1$ and $\Psi_{-1}$ are regular surfaces then denoting by $n_a(\Psi_1)$ and $n_a(\Psi_{-1})$ the affine normal vector of $\Psi_1$ and $\Psi_{-1}$ respectively, the figure of the three affine normals spanned by $n_a(\Phi)$, $n_a(\Psi_1)$, $n_a(\Psi_{-1})$ is invariant by translation along the orbit (ellipse).

In the present paper we will prove analogous results for minimal surfaces in Minkowski space and give some examples.
1 Preliminaries

A Minkowski (or Lorentz) 3-space $\mathbb{R}^3$ is $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$, where $\langle x, y \rangle$ is the scalar product

$$\langle x, y \rangle := x_1 y_1 + x_2 y_2 - x_3 y_3, x = (x_1, x_2, x_3).$$

(1)

A vector $x \in \mathbb{R}^3$ is called

- spacelike $\iff \langle x, x \rangle > 0$,
- timelike $\iff \langle x, x \rangle < 0$,
- isotropic (lightlike) $\iff \langle x, x \rangle = 0, x \neq 0$.

The (Minkowski-) length of a vector $x$ is defined by

$$\| x \| := \sqrt{\langle x, x \rangle} \geq 0.$$

(2)

The (Minkowski-) crossproduct is

$$x \times y \text{ with } \langle x \times y, z \rangle = \det(x, y, z).$$

(3)

A surface $\Phi$ in $\mathbb{R}^3$ is locally parametrized by $f : U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3$. The local coordinates are denoted by $(u, v)$ or $(u^1 := u, u^2 := v)$. Partial derivatives of a function $b : \mathbb{R}^2 \rightarrow \mathbb{R}$ or $b : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ are denoted by

$$b_{,j} := \frac{\partial}{\partial u^j} b, b_{,jk} := \frac{\partial^2}{\partial u^j \partial u^k} b.$$

The scalar product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^3$ induces a (pseudo-)Riemannian metric on $U$, the first fundamental form (I) with components

$$g_{jk} := \langle f_{,j}, f_{,k} \rangle : U \rightarrow \mathbb{R}.$$

(4)

Denoting $\Delta := \det(g_{jk})$ a surface $\Phi = f(U)$ is called

- spacelike $\iff \Delta > 0$ in $U$,
- timelike $\iff \Delta < 0$ in $U$.

Points with $\Delta = 0$ are excluded. For $T \subset U$ a Jordan measurable set the Minkowski surface area is

$$O(f(T)) := \int_T \sqrt{|\Delta|} du dv.$$

(5)

The normal vector is

$$n := \frac{f_{,1} \times f_{,2}}{\| f_{,1} \times f_{,2} \|} = \frac{f_{,1} \times f_{,2}}{\sqrt{|\Delta|}}.$$

(6)

Because of $\Delta \neq 0$, $n$ is a well defined non null vector in $U$. In the following we denote $\varepsilon := \langle n, n \rangle$. So in case of spacelike and timelike surfaces we have $\varepsilon = -1$ and $\varepsilon = 1$ and the spherical image $n(U)$ is part of the two-sheet hyperboloid $\langle x, x \rangle = -1$ and the one-sheet hyperboloid $\langle x, x \rangle = 1$, respectively. The second fundamental form (II) and the shape operator $S$ are related by $(II)(f_{,j}, f_{,k}) = \varepsilon \langle S(f_{,j}), f_{,k} \rangle$. The components of (II) and $S$ are

$$h_{jk} := (II)(f_{,j}, f_{,k}) = \varepsilon \langle n, f_{,jk} \rangle = \frac{\varepsilon}{\sqrt{|\Delta|}} \det(f_{,1}, f_{,2}, f_{,jk}),$$

(7)

$$S(f_{,j}) := -n_{,j} =: h^s_{,j} f_{,s}, \quad \text{with } h^s_{,j} = \varepsilon h_{jk} g^{ks}.$$  

Mean curvature and Gauss-curvature are

$$H := \frac{1}{2} \det S = \frac{\varepsilon}{2\Delta} (h_{11} g_{22} - 2h_{12} g_{12} + h_{22} g_{11}),$$

(9)

$$K := \varepsilon \det S = \frac{\det(h_{jk})}{\Delta}.$$  

(10)

In case of $K$ we use the sign convention used in [10], [4] and [8]. The eigenvalues of $S$ (principal curvatures of $\Phi$) are

$$k_{1,2} = H \pm \sqrt{H^2 - \varepsilon K}.$$  

(11)

The integrability conditions of Codazzi and the Theorema egregium read as usual:

$$Co : \quad h_{j,k} - h_{k,j} = \Gamma^p_{jk} h_{p,s} - \Gamma^p_{js} h_{pk},$$

(12)

$$Ga : \quad R_{p,j,k,s} = R^s_{jkps} = h_{jk} h_{sp} - h_{js} h_{kp}.$$  

(13)

**Remark 1** From (7) and (10) an easy calculation gives $\Delta^2 K = -\Delta^2 \Delta_s$ where $\Delta_s$ is the determinant of the components of the Euclidean metric. Thus Euclidean and Minkowski Gauss-curvature have different sign.

We need some basics from affine differential geometry. For details see for instance [13], [2]. From affine point of view a surface $f(U)$ is nondegenerate if

$$D := \det(D_{jk}) \neq 0, \quad D_{jk} := \det(f_{,1}, f_{,2}, f_{,jk}).$$

(14)

By (7) and (10) we have $D \neq 0 \iff K \neq 0$. Assuming $f$ to be regular and nondegenerate, it is said to be an equiaffine immersion. Then the components of the affine metric of $f(U)$ are

$$G_{jk} := |D|^{-1/4} D_{jk},$$

(15)

and the affine normal vector of $\Phi = f(U)$ is

$$n_a := (1/2) \Delta G f,$$

(16)

where $\Delta G$ is the Laplacian with respect to the affine metric. The affine shape operator $B$ defined by $B(f_{,j}) = -n_{a,j}$ has components $B_{jk}^a$ defined by

$$n_{a,j} := -B_{jk}^a f_{,k}.$$  

(17)
Affine curvature $K_a$ and affine mean curvature $H_a$ are defined by
\[ K_a := \det(B), \quad H_a := (1/2)\text{tr}(B). \tag{18} \]
In case of a constant affine normal vector the surface $\Phi$ is called an improper affine sphere and we have $K_a = H_a = 0$.

2 Minimal Surfaces and their focal surfaces

A regular surface $\Phi = f(U) \subset \mathbb{R}_3^3$ is called a (Minkowski-) minimal surface iff $H = 0$ in $U$. There are many investigations on these surfaces, for instance \cite{1}, \cite{3}, \cite{4}, \cite{6}, \cite{7}, \cite{10}, \cite{11}, \cite{12}, \cite{14}, \cite{15}. Although a spacelike surface locally maximizes the surface area defined by (5), as E.CALABI proved in \cite{3}, and timelike minimal surfaces neither maximize nor minimize surface area (see \cite{11}), we speak of minimal surfaces.

As we want to study properties of affine geometry too, we exclude points with $K = 0$ on $\Phi$. Denoting by $k_{1,2}$ the eigenvalues of the shape operator $S$ of $\Phi$, the focal surfaces of $\Phi = f(U)$ are parametrized by $z = f + n(1/k_{1,2})n$. So the focal points are real iff $k_{1,2}$ are real. From (11) the eigenvalues are in case of $H = 0$
\[ k_{1,2} = \pm \sqrt{-\varepsilon K}. \tag{19} \]

In case of a spacelike minimal surface ($\varepsilon = -1$) the Gauss-curvature is positive (cf. \cite[p. 298]{7}, \cite[p. 518]{4}), so we have two different real eigenvalues $\pm \sqrt{K}$. A timelike minimal surface ($\varepsilon = 1$) has real focal surfaces iff $K < 0$ that means by Remark 1 $\Phi$ is locally strongly convex. So in both cases locally we can take the lines of curvature as parametric lines.

**Lemma 1** (T. WEINSTEIN \cite[p. 160]{14}). Let $\Phi$ be a minimal surface in $\mathbb{R}_3^3$ with $K \neq 0$ and real focal surfaces. Then locally there is a parametrization $f: U \rightarrow \mathbb{R}_3^3$, $f(U) = \Phi$, so that
\[
\begin{align*}
  g := g_{11} > 0, g_{22} = -\varepsilon g, g_{12} = 0, \\
  h_{11} = 1, h_{22} = \varepsilon, h_{12} = 0,
\end{align*}
\]
where $\varepsilon = -1$ and $\varepsilon = 1$ refers to spacelike and timelike surfaces respectively.

The coordinate functions of (I) and (II) in Lemma 1 fulfill the Codazzi condition $Co$ (12). The Theorema egregium (13) reads
\[ g(g'' - \varepsilon g' + 2) = g'^2 - \varepsilon g^2, \tag{20} \]
where here as in the following the derivatives of $g: U \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ are denoted by
\[ g := \frac{dg}{du}, \quad g' := \frac{dg}{dv}, \quad g'' := \frac{d^2g}{dv^2}, \quad g'' := \frac{d^2g}{du dv}. \]

**Theorem 3** Let $f(U) = \Phi \subset \mathbb{R}_3^3$ be a regular minimal surface with $K \neq 0$ in $U$. The two sets of focal points $\Psi_1$ and $\Psi_{-1}$ of $\Phi = f(U)$ are parametrized by $z = f + \nu(-\varepsilon K)^{-1/2}n$, where $n$ denotes the normal vector and $\Psi_1$ and $\Psi_{-1}$ correspond to $\nu = 1$ and $\nu = -1$ respectively. Then the following holds for $\nu \in \{1, -1\}$:

(a) If $\Psi_\nu$ is a regular surface, then $K(\Psi_\nu) = -1/4K$, where $K(\Psi_\nu)$ denotes the Gauss-curvature of $\Psi_\nu$.

(b) If $\Psi_\nu$ is a regular surface, then it is non degenerate and the affine normal of $\Psi_\nu$ intersects the affine normal of $\Phi$ orthogonally.

(c) If $\Psi_\nu$ is a regular surface then it is an affine minimal surface.

(d) If $\Psi_1$ and $\Psi_{-1}$ are both regular surfaces, then
\[ K_a(\Psi_1) \cdot K_a(\Psi_{-1}) = (-\varepsilon)H(\Psi_1)^4 \cdot H(\Psi_{-1})^4, \]
where $K_a(\Psi_\nu)$ and $H(\Psi_\nu)$ are the affine Gauss-curvature and the Minkowski mean curvature of $\Psi_\nu$ respectively.

**Proof**

(a) Using the parameters of Lemma 1 we calculate the Gauss-curvature
\[ K = \frac{-\varepsilon}{g^2}, \tag{21} \]
and the Gauss equations
\[
\begin{align*}
  f_{,11} &= \gamma_1 f_{,1} + \gamma_2 f_{,2}, \\
  f_{,12} &= \gamma_2 f_{,1} + \gamma_1 f_{,2}, \\
  f_{,22} &= \varepsilon \gamma_1 f_{,1} + \gamma_2 f_{,2} + \varepsilon n,
\end{align*} \tag{22}
\]
where
\[
\begin{align*}
  \gamma_1 := \frac{g'}{2g}, \quad \gamma_2 := \frac{g'}{2g}.
\end{align*}
\]
The Weingarten equations are
\[ n_{,1} = -\frac{\varepsilon}{g} f_{,1}, \quad n_{,2} = \frac{\varepsilon}{g} f_{,2}. \tag{23} \]
The parametrizations of the focal surfaces (sets of focal points) are
\[ z = f + \phi g n, \quad \phi \in \{1, -1\}. \tag{24} \]
Expressing the derivatives of $z$ by the derivatives of $f$ and $n$ we get

\[ z_{11} = (1 - \varepsilon \varphi)f_{1,1} + \varphi gn, \quad \text{(25)} \]
\[ z_{12} = (1 + \varepsilon \varphi)f_{1,2} + \varphi g'n, \quad \text{(26)} \]
\[ z_{111} = (1 - 3\varepsilon \varphi)\gamma_{1,1} f_{1,1} + (\varepsilon - \varphi)\gamma_{1,2} f_{1,2} + (1 - \varepsilon \varphi + \varphi g) n, \]
\[ z_{112} = (1 - \varepsilon \varphi)\gamma_{1,2} f_{1,1} + (1 + \varepsilon \varphi)\gamma_{2,1} f_{1,2} + \varphi g'n, \]
\[ z_{22} = (\varepsilon + \varphi)\gamma_{1,1} f_{1,1} + (1 + 3\varepsilon \varphi)\gamma_{2,2} f_{1,2} + (\varepsilon + \varphi + \varphi g'' n). \]

Using (25) we calculate the metric $g_{jk} = \langle z_{1j}, z_{1k} \rangle$ of the focal surfaces

\[ g_{11}^1 = 2g(1 - \varepsilon \varphi) + \varepsilon g^2, \]
\[ g_{12}^1 = \varepsilon g, \]
\[ g_{22}^1 = -2\varepsilon(1 + \varepsilon \varphi) + \varepsilon g^2, \]
\[ \Delta^* = \det(g_{jk}^1) = 2g[\varepsilon g^2(1 - \varepsilon \varphi) - \varepsilon^2(1 + \varepsilon \varphi)]. \]

The determinants $D_{jk}^* := \det(z_{1j}, z_{1k})$ (according to (14)) for the focal surfaces are

\[ D_{11}^* = (1 + \varepsilon \varphi)g^2 + (\varepsilon - \varphi)g^2, \]
\[ D_{12}^* = 0, \]
\[ D_{22}^* = (\varepsilon + \varphi - 1)g^2 - (\varepsilon + \varphi)g^2, \]
\[ D^* = \det(D_{jk}^*) = -2(\varepsilon + \varphi)g^2 - 2(\varepsilon - \varphi)g^2. \]

From (26) and (27) a focal surface is non degenerate if it is regular. In case of $\varphi = 1$ or $\varphi = -1$ this requires $\hat{g} \neq 0$ or $g' \neq 0$ respectively. From (7),(10) and (14) the Gauss-curvature of a focal surface $K(\Psi_\varphi)$ is

\[ K(\Psi_\varphi) = \varepsilon^* \frac{\det(h_{jk}^*)}{\Delta^*} = \varepsilon^* \frac{D^*}{\Delta^*} = -\frac{\varepsilon^* D^*}{\Delta^*^2}, \]

because of $\text{sgn}(\Delta^*) = -\langle n^*, n^* \rangle = -\varepsilon^*$, where $n^*$ is the normal vector of the focal surface. Calculating $K(\Psi_\varphi)$ from (28) using (26) and (27) gives $K(\Psi_\varphi) = -1/4K$ from (21).

(b) From (15) the components of the affine metric of $\Phi$ are

\[ G_{11} = \varepsilon \sqrt{g}, G_{12} = 0, G_{22} = \sqrt{g}. \]

Using (29) the affine normal vector of $\Phi$ is from (16)

\[ n_\varphi(\Phi) = \frac{1}{2g^{3/2}}(\varepsilon \hat{g} f_{1,1} + g' f_{1,2} + 2\varepsilon g n). \]

In case of spacelike surfaces ($\varepsilon = -1$) we calculate the affine metric of the focal surfaces $\Psi_\varphi$

\[ \varphi = 1 : \quad G_{11}^* = G_{22}^* = -\sqrt{2}|g|, \quad G_{12}^* = 0, \]
\[ \varphi = -1 : \quad G_{11}^* = G_{22}^* = \sqrt{2}|g|, \quad G_{12}^* = 0 \]

and the affine normal

\[ \varphi = 1 : \quad n_\varphi(\Psi_1) = \frac{1}{|g|}(-\frac{1}{\sqrt{2}}|g|f_{1,1} - g' f_{1,2} + \frac{1}{2}(g^2 + g'^2)n), \]
\[ \varphi = -1 : \quad n_\varphi(\Psi_{-1}) = \frac{1}{|g|}(-\frac{1}{\sqrt{2}}|g|f_{1,1} - g' f_{1,2} + \frac{1}{2}(g^2 + g'^2)n). \]

In case of timelike surfaces ($\varepsilon = 1$) we get

\[ \varphi = 1 : \quad G_{11}^* = \sqrt{2}|g|, \quad G_{12}^* = 0, \quad G_{22}^* = -\sqrt{2}|g|, \]
\[ \varphi = -1 : \quad G_{11}^* = \sqrt{2}|g'|, \quad G_{12}^* = 0, \quad G_{22}^* = -\sqrt{2}|g'| \]

and the affine normal

\[ \varphi = 1 : \quad n_\varphi(\Psi_1) = \frac{1}{|g|}(-\frac{1}{\sqrt{2}}|g|f_{1,1} + g' f_{1,2} - \frac{1}{2}(g^2 - g'^2)n), \]
\[ \varphi = -1 : \quad n_\varphi(\Psi_{-1}) = \frac{1}{|g|}(-\frac{1}{\sqrt{2}}|g|f_{1,1} + g' f_{1,2} - \frac{1}{2}(g^2 - g'^2)n). \]

In both cases we have used (20). From (31) and (32) the affine normals of $\Psi_1$ and $\Psi_{-1}$ are parallel and using (30),(31) and (32) it follows $\langle n_\varphi(\Phi), n_\varphi(\Psi_\varphi) \rangle = 0$. So (b) is proved.

(c) From (31) and (32), using (22) and (23) a straightforward calculation gives the components $B_{jk}$ of the affine shape operator $B^*$ of the focal surfaces $\Psi_\varphi$.

In case of spacelike surfaces ($\varepsilon = -1$) we get

\[ \varphi = 1 : \quad \left( B_{jk}^* \right) = \frac{sgn(g)}{2\sqrt{g}} \begin{pmatrix} \hat{P} & P' & -\hat{P} \\ P' & P'' & -\hat{P} \end{pmatrix}, \]
\[ P := \frac{\hat{g}}{g'} (g' \neq 0), \]
\[ \varphi = -1 : \quad \left( B_{jk}^* \right) = \frac{sgn(g)}{2\sqrt{g}} \begin{pmatrix} Q' & -\hat{Q} & -Q' \\ -\hat{Q} & -Q'' & -\hat{Q} \end{pmatrix}, \]
\[ Q := 1/P, (\hat{g} \neq 0). \]

In case of timelike surfaces ($\varepsilon = 1$) we get with $P, Q$ as above

\[ \varphi = 1 : \quad \left( B_{jk}^* \right) = \frac{sgn(g)}{2\sqrt{g}} \begin{pmatrix} Q' & -\hat{Q} & -Q' \\ -\hat{Q} & -Q'' & -\hat{Q} \end{pmatrix}, \]
\[ \varphi = -1 : \quad \left( B_{jk}^* \right) = \frac{sgn(g')}{-2\sqrt{g}} \begin{pmatrix} Q' & -\hat{Q} & -Q' \\ -\hat{Q} & -Q'' & -\hat{Q} \end{pmatrix}. \]

From (33)-(36) the focal surfaces $\Psi_\varphi$ are affine minimal surfaces.

(d) From (33)-(36) the affine Gauss-curvatures of the focal surfaces are related by

\[ \varepsilon = -1 : \quad K_\varphi(\Psi_1) = \left( \frac{\hat{g}}{g'} \right)^4 K_\varphi(\Psi_{-1}) \]
\[ \varepsilon = 1 : \quad K_\varphi(\Psi_1) = \left( \frac{g'}{\hat{g}} \right)^4 K_\varphi(\Psi_{-1}). \]
The components of the second fundamental form of the focal surfaces are
\[ \varepsilon = -1, \varphi = 1 : \ h_{11}^* = h_{22}^* = \frac{|g'|}{\sqrt{g}}, \ h_{12}^* = 0, \] (39)
\[ \varepsilon = -1, \varphi = -1 : \ h_{11}^* = h_{22}^* = \frac{|g'|}{\sqrt{g}}, \ h_{12}^* = 0, \] (40)
\[ \varepsilon = 1, \varphi = 1 : \ h_{11}^* = -h_{22}^* = \frac{|g'|}{\sqrt{g}}, \ h_{12}^* = 0, \] (41)
\[ \varepsilon = 1, \varphi = -1 : \ h_{11}^* = -h_{22}^* = \frac{|g'|}{\sqrt{g}}, \ h_{12}^* = 0. \] (42)
Calculating the mean curvature of the focal surfaces from (39)-(42) using (26) together with (37) and (38) proves (d).

\[ \square \]

3 Associated spacelike minimal surfaces

On a spacelike minimal surface in \( \Phi \subset R^3 \) there are always global (1)-isothermal coordinates (see [14, p.184]), for instance the normalized ones of Lemma 1:
\[ g := g_{11} = g_{22} > 0, g_{12} = 0, \]
\[ h_{11} = 1, h_{22} = -1, h_{12} = 0. \]
Consequently we have
\[ H = 0 \iff h_{11} + h_{22} = 0 \iff f_{1,1} + f_{2,2} = 0 \iff \Delta f = 0, \] (43)
so the coordinate functions \( f^\alpha : U \subseteq R^2 \rightarrow R, (\alpha = 1, 2, 3) \) are harmonic. Then the conjugate harmonic functions are
\[ f^{*\alpha} : U \subseteq R^2 \rightarrow R, (\alpha = 1, 2, 3), \]
related to \( f^\alpha \) by
\[ f^{\alpha}_{1} = f^{*\alpha}_{2}, \ f^{\alpha}_{2} = -f^{*\alpha}_{1} (\alpha = 1, 2, 3). \] (44)
Then the one parameter family \( (\lambda) \Phi \) of associated minimal surfaces is parametrized by
\[ (\lambda) f(u, v) := \cos \lambda f(u, v) - \sin \lambda \bar{f}(u, v), \lambda \in R. \] (45)
If \( |\lambda_2 - \lambda_1| = \pi/2 \) the surfaces \( (\lambda_1) \Phi \) and \( (\lambda_2) \Phi \) are called adjoined. It is well known that the surfaces of the pencil \( (\lambda) \Phi \) share metric (1), normal vector and Gauss-curvature as in the Euclidean situation (see [14, p.184]).

**Theorem 4** Let \( \Phi \) be a spacelike minimal surface in \( R^3 \) with \( K \neq 0 \) and \( (\lambda) \Phi \) the family of the associated minimal surfaces. Then the following holds.

(a) The affine normal vector \( n_a(\Phi) \) of \( \Phi \) is invariant: \( n_a(\lambda) \Phi = n_a(\Phi) \).

(b) The affine Gauss-curvature of \( \Phi \) is invariant: \( K_a(\lambda) \Phi = K_a(\Phi) \).

(c) Assuming the focal surfaces \( \Psi_1 \) and \( \Psi_{-1} \) of \( \Phi \) not to be degenerate and denoting by \( n_a(\Psi_1) \) and \( n_a(\Psi_{-1}) \) the affine normal vectors of \( \Psi_1 \) and \( \Psi_{-1} \) respectively, the figure of the three affine normals spanned by \( n_a(\Phi) \), \( n_a(\Psi_1) \), \( n_a(\Psi_{-1}) \) is invariant by translation along the orbit.

**Proof**
(a) We use the coordinates according to Lemma 1. From (45) we calculate the derivatives of \( (\lambda) f \) and from this
\[ (\lambda) g_{jk} = g_{jk}, \]
\[ (\lambda) D_{11} = -g \cos \lambda, (\lambda) D_{12} = g \sin \lambda, (\lambda) D_{22} = g \cos \lambda, \]
where (46) expresses the well-known isometry. Further we get the components of the affine metric
\[ (\lambda) G_{11} = -\sqrt{g} \cos \lambda, (\lambda) G_{12} = \sqrt{g} \sin \lambda, (\lambda) G_{22} = \sqrt{g} \cos \lambda. \]
(47)

Using this the affine normal is
\[ n_a(\lambda) \Phi = \frac{1}{2g^{1/2}}(-gf_{1} + g'f_{2} - 2gn). \] (49)
Because of \( \varepsilon = -1 \), comparison with (30) proves (a).

(b) Denoting the components of the affine shape operator of \( f \) and \( (\lambda) f \) by \( B^k \) and \( (\lambda) B^k \) respectively, we get
\[ (\lambda) B^k_{j} = \left( \begin{array}{cc} B^1_{1} \cos \lambda + B^2_{1} \sin \lambda & B^1_{2} \cos \lambda + B^2_{2} \sin \lambda \\ -B^1_{1} \sin \lambda + B^2_{1} \cos \lambda & B^2_{2} \sin \lambda - B^2_{1} \cos \lambda \end{array} \right). \] (50)
From this we get
\[ K_a(\lambda) \Phi = \det (\lambda) B^k_{j} = K_a(\Phi). \]
(c) From (46) we have \( (\lambda) K = K \). Together with the invariance of the normal \( n(\Phi) \) and the affine normals \( n_a(\Phi) \) and \( n_a(\Psi_1), n_a(\Psi_{-1}) \) this gives the invariance of the figure of the three affine normals. \( \square \)
In figure 1 the dark surfaces are associated minimal surfaces (to the so called elliptic catenoid (62)). Starting with the elliptic catenoid (left figure) the adjoined surface is the spacelike portion of a right helicoid (right figure). It is well known, that the surfaces are screw surfaces in \( \mathbb{R}^3 \) (see [12]). The red surfaces are the focal surfaces. The focal surface on the left is the surface (69), which is an improper affine sphere. Because of Theorem 4 the focal surfaces of every surface of the pencil is an improper affine sphere.

### 4 Associated timelike minimal surfaces

A timelike minimal surface in \( \Phi \subset \mathbb{R}^3 \) admits locally a representation in isotropic coordinates

\[
f(u, v) = g(u) + h(v), \quad g : I \in \mathbb{R} \rightarrow \mathbb{R}^3, \quad h : J \in \mathbb{R} \rightarrow \mathbb{R}^3,
\]

where \( g(I) \) and \( h(J) \) are isotropic curves

\[
(\dot{g}, \dot{g}) = (h', h') = 0; \quad \dot{g} := \frac{dg}{du}, \quad \dot{h} := \frac{dh}{dv}.
\]

(see [14, p.184] or [6, p.338]). So it is

\[
g_{11} = g_{22} = 0, \quad g_{12} = (\dot{g}, \dot{h}) \neq 0.
\]

This means a timelike minimal surface is locally a surface of translation with isotropic generating curves. The conjugate minimal surface \( \Phi \) is locally parameterized by

\[
\tilde{f}(u, v) = g(u) - h(v),
\]

and the family of associated minimal surfaces \( ^{(\lambda)}\Phi \) is given by

\[
^{(\lambda)}f(u, v) = \cosh f(u, v) + \sinh \tilde{f}(u, v), \quad \lambda \in \mathbb{R}.
\]

**Remark 2**: It is \(^{(0)}f = f\). Obviously the surface \( \tilde{f}(U) \) does not to belong to the family of associated surfaces ([16, p.338]).

Analogous to Theorem 4 we have

**Theorem 5** Let \( \Phi \) be a timelike minimal surface in \( \mathbb{R}^3 \) with \( K \neq 0 \) and \( ^{(\lambda)}\Phi \) the family of the associated minimal surfaces. Then the following holds.

1. The affine normal vector \( n_a(\Phi) \) of \( \Phi \) is invariant: \( n_a(\Phi) = n_a(\Phi) \).
2. The affine Gauss-curvature of \( \Phi \) is invariant: \( K_a(\Phi) = K_a(\Phi) \).
3. Assuming the focal surfaces \( \Psi_1 \) and \( \Psi_{-1} \) of \( \Phi \) not to be degenerate and denoting by \( n_a(\Psi_1) \) and \( n_a(\Psi_{-1}) \) the affine normal vectors of \( \Psi_1 \) and \( \Psi_{-1} \) respectively, the figure of the three affine normals spanned by \( n_a(\Phi), n_a(\Psi_1), n_a(\Psi_{-1}) \) is invariant by translation along the orbit.

**Proof**

(a) We use the isotropic coordinates from above. From (54) we calculate the derivatives of \(^{(\lambda)}f\) and from this

\[
^{(\lambda)}g_{jk} = g_{jk},
\]

\[
^{(\lambda)}D_{11} = (\cosh \lambda + \sinh \lambda)D_{11},
\]

\[
^{(\lambda)}D_{12} = 0,
\]

\[
^{(\lambda)}D_{22} = (\cosh \lambda - \sinh \lambda)D_{22},
\]
where (55) expresses the well known isometry. The components of the affine metric are

\[
^{(\lambda)}G_{11} = (\cosh \lambda + \sinh \lambda)G_{11}, \\
^{(\lambda)}G_{12} = 0, \\
^{(\lambda)}G_{22} = (\cosh \lambda - \sinh \lambda)G_{22}.
\]

Calculating the affine normal by (16) yields \(n_a(^{(\lambda)}\Phi) = n_a(\Phi)\).

(b) Denoting the components of the affine shape operator of \(f\) and \(^{(\lambda)}f\) by \(B^k_j\) and \(^{(\lambda)}B^k_j\) respectively, we get

\[
^{(\lambda)}B^k_j = \begin{pmatrix} e^{-\lambda}B^1_1 & e^{-\lambda}B^1_2 \\ e^{\lambda}B^2_1 & e^{\lambda}B^2_2 \end{pmatrix}.
\]

Thus it is

\[
K_a(^{(\lambda)}\Phi) = \det \left(^{(\lambda)}B^j_k\right) = K_a(\Phi).
\]

(c) The argument is as in the proof of Theorem 4.

5 Further Examples

A well known class of minimal surfaces in Minkowski space is that of rotation surfaces, that means surfaces admitting a one parameter family of isometries in \(\mathbb{R}^3\) fixing the points of a straight line. If the axis is timelike \((x^3 - axis)\) or spacelike \((x^1 - axis)\) or isotropic \((x^1 = x^3, x^2 = 0)\) representations of this rotations are

\[
\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \mapsto \begin{pmatrix} \cosh v & -\sinh v & 0 \\ \sinh v & \cosh v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix},
\]

\[
\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sinh v & \cosh v \\ 0 & -\sinh v & \cosh v \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix},
\]

\[
\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \mapsto \begin{pmatrix} 1 - \frac{2}{v^2} & 0 & 0 \\ -v & 1 & v \\ -\frac{2}{v^2} & -v & 1 + \frac{2}{v^2} \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}.
\]

There are seven types of minimal surfaces with rotational symmetry (see [1], [6], [7], [10], [15]).

The following surfaces are spacelike \((\varepsilon = -1)\)

\[
f(u, v) = (\sinh u \cos v, \sinh u \sin v, u),
\]

\[
f(u, v) = (u, \sinh u \sinh v, \sinh u \sin v),
\]

\[
f(u, v) = (u, \pm \cosh u \cosh v, \sinh u \sinh v),
\]

\[
f(u, v) = (-u^3 + u - uv^2, -2uv, -u^3 - u - uv^2).
\]

where the rotation axis is timelike, spacelike or isotropic respectively.

In case of timelike surfaces we have

\[
f(u, v) = (\sinh u \cos v, \sinh u \sin v, u),
\]

\[
f(u, v) = (u, \sinh u \sinh v, \sinh u \sin v),
\]

\[
f(u, v) = (u, \pm \cosh u \cosh v, \cosh u \sinh v),
\]

\[
f(u, v) = (-u^3 + u - uv^2, -2uv, -u^3 - u - uv^2).
\]

where in case of surfaces (65) and (68) the rotation axis is timelike and isotropic respectively. Surfaces (66) and (67) have a spacelike rotation axis. Clearly we have to exclude discrete values of \(u\) in order to have regular surfaces.
Calculation of the focal surfaces gives in case of surfaces (62), (63) and (64)

\[ f(u, v) = (2\sinh u \cos v, 2\sinh u \sin v, u + \sinh u \cosh u), \] (69)

\[ f(u, v) = (u + \cos u \sin v, 2\sin u \sin v, 2\sin u \cosh v), \] (70)

\[ f(u, v) = (-2u^3 + 2u - 2uv^2, -4uv, -2u^3 - 2u - 2uv^2). \] (71)

and in case of surfaces (65), (66), (67) and (68)

\[ f(u, v) = (2\sin u \cos v, 2\sin u \sin v, u + \cos u \sin u), \] (72)

\[ f(u, v) = (u + \cosh u \sinh v, 2\sinh u \sinh v, 2\sinh u \cosh v), \] (73)

\[ f(u, v) = (u - \cosh u \sinh v, \pm 2\cosh u \cosh v, 2\cosh u \sinh v), \] (74)

\[ f(u, v) = (2u^3 + 2u - 2uv^2, -4uv, 2u^3 - 2u - 2uv^2). \] (75)

According to Theorem 3 the focal surfaces (69)–(75) are affine minimal surfaces, in fact we have improper affine spheres the affine normals of which are parallel to the axis of rotation.

**Remark 3** In case of surfaces (64) and (68) the focal surfaces (71) and (75) respectively, coincide (up to a scaling factor 2) with the surfaces (68) and (64) respectively. That means: The (non degenerate) focal surface of a spacelike minimal surface of rotation with isotropic axis is a timelike minimal surface of rotation with isotropic axis and vice versa (figure 3).

According to Theorem 3 the focal surfaces (69)-(75) are affine minimal surfaces, in fact we have improper affine spheres the affine normals of which are parallel to the axis of rotation.

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Figure 3. *Surfaces of rotation with isotropic axis: Timelike (spacelike) surface as focal surface (in red color) of a spacelike (timelike) surface.*

**References**


