Cyclical Surfaces Created by a Conical Helix

ABSTRACT

The paper describes cyclical surfaces created by revolution of a circle about an edge of the trihedron of a conical helix that is moving evenly along the helix. This Euclidean metric transformation is composed from revolution about one of the coordinate axes and transformation of the right-handed coordinate system to the right-handed system of the moving trihedron in every point of the conical helix. This transformation is analytically represented by a functional matrix of 4th order. These surfaces are determined at particular parameter values which have influence on the surface shape. The vector equation of surfaces and some illustrations of this group of surfaces are presented in the paper. The surfaces are illustrated and modelled in the programme Maple.

Key words: cyclical surface, conical helix, trihedron

MSC 2000: 15A04, 53A05, 14J26

1 The vector equation of the cyclical surface created by a conical helix

Let the conical helix $s$ in right-handed coordinate system $(O;x,y,z)$ be located on the circular conical surface with the meridian defined analytically by the vector function $(r_1 + u(r_2 - r_1), 0, uv, 1), u \in \mathbb{R}, (r_1 \neq 0, r_2 \neq 0, v \neq 0)$ (Figure 1).

The helix has the parametrical equations (1), where $l$ is a number of helix screws and the parameter $q = +1$ for right-turned helix and $q = -1$ for left-turned one (Figure 2) [3].

\[
\begin{align*}
    x &= \left( r_1 + \frac{\varepsilon(r_2 - r_1)}{2l\pi} \right) \cos \varepsilon \\
    y &= q \left( r_1 + \frac{\varepsilon(r_2 - r_1)}{2l\pi} \right) \sin \varepsilon, \varepsilon \in (0, 2\pi). \\
    z &= \frac{ve}{2l\pi}
\end{align*}
\] (1)

Figure 2: Conical helix right-turned and left-turned

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The trihedron \((R; t, n, b)\) in the coordinate system \((O; x, y, z)\) is given in every point \(R \in s\) with coordinates determined in equation (2), where \(t\) is the tangent, \(b\) is the binormal and \(n\) is the principal normal of the conical helix \(s\) (Figure 3). The unit vectors of the trihedron are expressed in equations (3) and (4) [1].

\[
\begin{align*}
R(x_R, y_R, z_R) &= \left( r_1 + \frac{\varepsilon(r_2 - r_1)}{2\pi} \cos \varepsilon, r_1 + \frac{\varepsilon(r_2 - r_1)}{2\pi} \sin \varepsilon, \frac{v \varepsilon}{2\pi} \right). \\
\end{align*}
\]

(2)

\[
\begin{align*}
t &= (a_t, b_t, c_t), \\
a_t &= -\frac{1}{h} \left( (r_2 - r_1) \cos \varepsilon - (2r_1 \pi + \varepsilon(r_2 - r_1)) \sin \varepsilon \right), \\
b_t &= -\frac{1}{h} \left( q(r_2 - r_1) \sin \varepsilon + (2r_1 \pi + \varepsilon(r_2 - r_1)) \cos \varepsilon \right), \\
c_t &= -\frac{1}{h} (v), \text{ where} \\
h &= \sqrt{(r_2 - r_1)^2 + (2r_1 \pi + \varepsilon(r_2 - r_1))^2 + v^2}.
\end{align*}
\]

(3)

\[
\begin{align*}
n &= (a_n, b_n, c_n) = \frac{tp \times t}{|tp \times t|}, \\
b &= (a_b, b_b, c_b) = \frac{t \times n}{|t \times n|}.
\end{align*}
\]

(4)

\[
\begin{align*}
\text{tp} &= (r_2 - r_1) \cos \varepsilon, q(r_2 - r_1) \sin \varepsilon, v) \text{ in equation (4) is the direction vector of the generator of the conical surface passing through the point } R.
\end{align*}
\]

We will revolve the circle \(k_0 = (S, r)\) represented by the vector function \(\mathbf{r}(u) = (x(u), y(u), z(u), 1)\), \(u \in (0, 2\pi)\) through the angle \(\varepsilon' = m \varepsilon\) about the local coordinate axis \(x\) (resp. \(y\), resp. \(z\)). The corresponding rotation is represented by the matrix function \(T_x(\varepsilon'(|\varepsilon|))\) (resp. \(T_y(\varepsilon'(|\varepsilon|))\), resp. \(T_z(\varepsilon'(|\varepsilon|))\)) in equation (5) [2]. The parameter \(m\) is a multiple of the radian velocity of the point \(R\) moving on the helix. Right-handed rotation of the circle about the corresponding axis \(x\), \(y\), or \(z\) is represented by the parameter \(q' = +1\) and left-handed rotation by the parameter \(q' = -1\).

\[
\begin{align*}
T_x(\varepsilon'(|\varepsilon|)) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & \cos \varepsilon'(|\varepsilon|) & q' \sin \varepsilon'(|\varepsilon|) & 0 \\
0 & -q' \sin \varepsilon'(|\varepsilon|) & \cos \varepsilon'(|\varepsilon|) & 0 \\
0 & 0 & 0 & 1 \end{pmatrix}, \\
T_y(\varepsilon'(|\varepsilon|)) &= \begin{pmatrix} \cos \varepsilon'(|\varepsilon|) & 0 & q' \sin \varepsilon'(|\varepsilon|) & 0 \\
0 & 1 & 0 & 0 \\
-\varepsilon'(|\varepsilon|) & 0 & \cos \varepsilon'(|\varepsilon|) & 0 \\
0 & 0 & 0 & 1 \end{pmatrix}, \\
T_z(\varepsilon'(|\varepsilon|)) &= \begin{pmatrix} \cos \varepsilon'(|\varepsilon|) & -q' \sin \varepsilon'(|\varepsilon|) & 0 & 0 \\
q' \sin \varepsilon'(|\varepsilon|) & \cos \varepsilon'(|\varepsilon|) & 0 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix}.
\end{align*}
\]

(5)

Then we will transform the rotated circle \(k'\) given in the local coordinate system \((O; x, y, z)\) into the circle \(k\) in the coordinate system \((R; t, n, b)\) where are expressed in (4) (Figure 4) by the transformation given by \(M(\varepsilon)\).

\[
\begin{align*}
\mathbf{M}(\varepsilon) &= \begin{pmatrix} a_t & b_t & c_t & 0 \\
a_b & b_b & c_b & 0 \\
a_n & b_n & c_n & 0 \\
x_R & y_R & z_R & 1 \end{pmatrix}.
\end{align*}
\]

(6)

The vector equation of the cyclical surface created by the conical helix is expressed as a product of three matrices for \(i = x, y, z\).

\[
\begin{align*}
\mathbf{p}(u, \varepsilon) &= \mathbf{r}(u) \cdot T_i(\varepsilon'(|\varepsilon|)) \cdot \mathbf{M}(\varepsilon).
\end{align*}
\]

(7)
2 Classification of the cyclical surfaces created by a conical helix

The three basic types of cyclical surfaces described above can be characterised with respect to the axis of revolution of the moved circle:

I the circle rotates about the tangent \( t \),
II the circle rotates about the binormal \( b \),
III the circle rotates about the principal normal \( n \)

in the point \( R \) of the conical helix \( s \).

The circle \( k_0 \) in dependence on the axis of revolution \( o \) = \( x \) (resp. \( o = y \), resp. \( o = z \)) is in:

A meridian plane,
B normal plane. (Figure 5)

![Figure 5: The circle \( k_0 \) in the meridian or normal plane](image)

The distance \( d = |S, o| \) between the centre \( S \) of the circle \( k_0 = (S, r) \) and the axis of revolution \( o \) can be:

a \( d > r \) the circle \( k_0 \) does not intersect the axis \( o \),
b \( d = r \) circle \( k_0 \) is tangent to the axis \( o \),
c \( d = 0 \) the centre of the circle \( k_0 \) is on the axis \( o \).

3 Display of the cyclical surfaces created by a conical helix

In this chapter we will display only these cyclical surfaces created by the conical helix, which are applicable in technical practice. Hence we will display cyclical surfaces of types IA, IB, IIA, IIIA. The surfaces of types IIB and IIIB are inapplicable in technical practice due to their forms.

In Figure 6 the cyclical surfaces of the named types are shown by rotating the circle \( k_0 \) about the corresponding coordinate axis in \((R; t, n, b)\) of the right-turned conical helix with \( l = 1 \) screw then transformed into the coordinate system \((O; x, y, z)\).

![Figure 6: Types of the cyclical surfaces created by a conical helix](image)
In Figure 7 a cyclical surfaces of type IA is shown, where the circle \( k_0 \) is in the normal plane to the axis of revolution \( o = x \). All surfaces shown in Figure 7 have the same parameters \( l = 3, m = 1 \) but different parameter \( d \). If \( d > r \) the helix \( s \) is not on the surface, if \( d = r \) the helix is on the surface and if \( d = 0 \), then the centre \( S \) of the revolving circle is on the helix \( s \). The parameter \( q' = +1 \) for first three surfaces, \( q' = -1 \) for forth and fifth surface.

We may express the vector equation of this cyclical surface of type IA from equation (7), where the vector function of the circle \( k_0 \) in the normal plane to the axis \( o = x \) is
\[
p(u, \varepsilon) = (0, r \cos t + d, r \sin t, 0, 1)
\]
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \cos \varepsilon' & q'\sin \varepsilon' & 0 & 0 \\
0 & -q'\sin \varepsilon' & \cos \varepsilon' & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]
\[
\begin{pmatrix}
a_t & b_t & c_t & 0 \\
a_b & b_b & c_b & 0 \\
a_n & b_n & c_n & 0 \\
x_R & y_R & z_R & 1
\end{pmatrix}
\]

Figure 7: Cyclical surfaces of type IA

We may express the vector equation of this cyclical surface of type IB similarly as for the surface of type IA, where the vector function of the circle \( k_0 \) lying in the meridian plane of the axis \( o = x \) is
\[
r(u) = (r \cos u + d, r \sin u, 0, 1), \quad u \in (0, 2\pi)
\]
and we get the corresponding matrix for \( i = x \) from equations (5), where \( a_t, b_t, \ldots, c_n \) are expressed in equations (3), (4), \( x_R, y_R, z_R \) are the coordinates of the point \( R \) in the equation (2).

In Figure 8 cyclical surfaces of type IB are shown, where the circle \( k_0 \) is in the meridian plane of the axis of revolution \( o = x \). The first two surfaces displayed in Figure 8 have the same parameters \( l = 3, m = 10, d > r, \) and \( q = +1 \) or \( q = -1 \). The helix \( s \) is not on these surfaces. Third and fourth surfaces have parameters \( l = 3, m = 15, q = +1 \) or \( q = -1 \). These surfaces are applicable only with parameter \( m > 1 \) in technical practice.
In Figure 9 cyclical surfaces of types IIB and IIIB are shown, where the circle is in the meridian plane of the axis of revolution $o = y$ or $o = z$. First three surfaces are created by a circle rotating about the binormal and other two about the principal normal of the conical helix $s$. All surfaces are displayed with the helix $s$ with parameters $l = 3$, $q = +1$. 

Figure 8: *Cyclical surfaces of type IB*

Figure 9: *Cyclical surfaces of type IIB and IIIB*
4 Conclusion

Many of these surfaces may be used in design practice as constructive or ornamental structural components. In Figure 10 there are displayed some illustrations of combinations of these cyclical surfaces with interesting and beautiful aesthetic forms.

Figure 10: Combinations of cyclical surfaces of different types

References

