Factors for generalized absolute Cesàro summability

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Abstract. *In this paper, a result of Bor [2] dealing with* $\mid C,\alpha;\beta\mid_k$ *summability factors has been generalized for* $\mid C,\alpha,\gamma;\beta\mid_k$ *summability factors.*

Key words: *absolute summability factors, infinite series*

AMS subject classifications: 40D15, 40F05, 40G05

Received July 22, 2007 Accepted November 21, 2007

1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence c_n and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Obviously every increasing sequence is almost increasing but the converse need not be true as can be seen from the example $b_n = ne^{(-1)^n}$. Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by t_n^α n-th Cesàro means of order α, with $\alpha > -1$, of the sequence (na_n), i.e.,

$$t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1}$$

where

$$A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1 \quad and \quad A_{-n}^\alpha = 0 \quad for \quad n > 0. \tag{2}$$

The series $\sum a_n$ is said to be summable $\mid C,\alpha\mid_k$, $k \geq 1$ and $\alpha > -1$, if (see [4])

$$\sum_{n=1}^{\infty} \frac{1}{n} \mid t_n^\alpha \mid^k < \infty, \tag{3}$$

and it is said to be summable $\mid C,\alpha;\beta\mid_k$, $k\geq 1$, $\alpha > -1$ and $\beta \geq 0$, if (see [5])

$$\sum_{n=1}^{\infty} n^{\beta k-1} \mid t_n^\alpha \mid^k < \infty. \tag{4}$$

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A series $\sum a_n$ is said to be summable $\mid C,\alpha,\gamma;\beta\mid_k$, $k\geq 1$, $\beta\geq 0$, $\alpha>-1$ and γ is a real number, if (see [8])

$$\sum_{n=1}^{\infty} n^{\gamma(\beta k+k-1)-k}\mid t_n^{\alpha}\mid^k<\infty. \tag{5}$$

If we take $\gamma=1$, then $\mid C,\alpha,\gamma;\beta\mid_k$ summability reduces to $\mid C,\alpha;\beta\mid_k$ summability.

Bor [2] has proved the following theorem for $\mid C,\alpha;\beta\mid_k$ summability factors of infinite series.

Theorem A. *Let (X_n) be an almost increasing sequence and let there be sequences (β_n) and (λ_n) such that*

$$\mid \Delta\lambda_n\mid\leq\beta_n \tag{6}$$

$$\beta_n\to 0 \quad as \quad n\to\infty \tag{7}$$

$$\sum_{n=1}^{\infty} n\mid\Delta\beta_n\mid X_n<\infty \tag{8}$$

$$\mid\lambda_n\mid X_n=O(1) \quad as \quad n\to\infty. \tag{9}$$

If the sequence (u_n^{α}), defined by (see [7])

$$u_n^{\alpha}=\begin{cases}\mid t_n^{\alpha}\mid, & \alpha=1\\ \max_{1\leq v\leq n}\mid t_v^{\alpha}\mid, & 0<\alpha<1\end{cases} \tag{10}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{\beta k-1}(u_n^{\alpha})^k=O(X_m) \quad as \quad m\to\infty, \tag{11}$$

then the series $\sum a_n\lambda_n$ is summable $\mid C,\alpha;\beta\mid_k$, $k\geq 1$ and $0\leq\beta<\alpha\leq 1$.

The aim of this paper is to generalize *Theorem A* for $\mid C,\alpha,\gamma;\beta\mid_k$ summability. Now, we shall prove the following theorem.

Theorem. *Let (X_n) be an almost increasing sequence and the sequences (β_n) and (λ_n) such that conditions (6)-(9) of* Theorem A *are satisfied. If the sequence (u_n^{α}), defined by (10) satisfies the condition*

$$\sum_{n=1}^{m} n^{\gamma(\beta k+k-1)-k}(u_n^{\alpha})^k=O(X_m) \quad as \quad m\to\infty, \tag{12}$$

then the series $\sum a_n\lambda_n$ is summable $\mid C,\alpha,\gamma;\beta\mid_k$, $k\geq 1$, $0\leq\beta<\alpha\leq 1$ and γ is a real number such that $k+\alpha k-\gamma(\beta k+k-1)>1$.

We need the following lemmas for the proof of our theorem.

Lemma 1 ([3]). *If $0<\alpha\leq 1$ and $1\leq v\leq n$, then*

$$\mid\sum_{p=0}^{v} A_{n-p}^{\alpha-1}a_p\mid\leq\max_{1\leq m\leq v}\mid\sum_{p=0}^{m} A_{m-p}^{\alpha-1}a_p\mid. \tag{13}$$

Lemma 2 ([2]). *Under the conditions on (X_n), (β_n) and (λ_n) as taken in the statement of the* Theorem, *the following conditions hold, when (8) is satisfied:*

$$n\beta_n X_n = O(1) \quad as \quad n \to \infty, \tag{14}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{15}$$

Proof of the Theorem. Let (T_n^α) be the n-th (C,α), with $0 < \alpha \leq 1$, means of the sequence $(na_n\lambda_n)$. Then by (1), we have

$$T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v. \tag{16}$$

Applying Abel's transformation, we get that

$$T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v,$$

so that making use of *Lemma 1*, we have

$$\begin{aligned} \mid T_n^\alpha \mid &\leq \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid\mid \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p \mid + \frac{\mid \lambda_n \mid}{A_n^\alpha} \mid \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \mid \\ &\leq \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} A_v^\alpha u_v^\alpha \mid \Delta\lambda_v \mid + \mid \lambda_n \mid u_n^\alpha \\ &= T_{n,1}^\alpha + T_{n,2}^\alpha, \quad say. \end{aligned}$$

Since

$$\mid T_{n,1}^\alpha + T_{n,2}^\alpha \mid^k \leq 2^k (\mid T_{n,1}^\alpha \mid^k + \mid T_{n,2}^\alpha \mid^k),$$

to complete the proof of the *Theorem*, it is enough to show that

$$\sum_{n=1}^{\infty} n^{\gamma(\beta k + k - 1) - k} \mid T_{n,r}^\alpha \mid^k < \infty \quad for \quad r = 1, 2, \quad by \quad (5).$$

Now, when $k > 1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$
\begin{aligned}
\sum_{n=2}^{m+1} n^{\gamma(\beta k+k-1)-k} \mid T_{n,1}^{\alpha} \mid^k &\leq \sum_{n=2}^{m+1} n^{\gamma(\beta k+k-1)-k} (A_n^{\alpha})^{-k} \{\sum_{v=1}^{n-1} (A_v^{\alpha})^k (u_v^{\alpha})^k \beta_v\} \times \{\sum_{v=1}^{n-1} \beta_v\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} n^{\gamma(\beta k+k-1)-k-\alpha k} \{\sum_{v=1}^{n-1} v^{\alpha k} (u_v^{\alpha})^k \beta_v\} \\
&= O(1) \sum_{v=1}^{m} v^{\alpha k} (u_v^{\alpha})^k \beta_v \sum_{n=v+1}^{m+1} \frac{1}{n^{k+\alpha k-\gamma(\beta k+k-1)}} \\
&= O(1) \sum_{v=1}^{m} v^{\alpha k} (u_v^{\alpha})^k \beta_v \int_v^{\infty} \frac{dx}{x^{k+\alpha k-\gamma(\beta k+k-1)}} \\
&= = O(1) \sum_{v=1}^{m} v \beta_v v^{\gamma(\beta k+k-1)-k} (u_v^{\alpha})^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} r^{\gamma(\beta k+k-1)-k} (u_r^{\alpha})^k \\
&+ O(1) m\beta_m \sum_{v=1}^{m} v^{\gamma(\beta k+k-1)-k} (u_v^{\alpha})^k \\
&= O(1) \sum_{v=1}^{m-1} \mid \Delta(v\beta_v) \mid X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} \mid (v+1)\Delta\beta_v - \beta_v \mid X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1) \sum_{v=1}^{m-1} \mid \beta_v \mid X_v + O(1) m\beta_m X_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of the *Theorem* and *Lemma 2*.

Finally, since $\mid \lambda_n \mid = O(1)$ by (9), we have that

$$
\begin{aligned}
\sum_{n=1}^{m} n^{\gamma(\beta k+k-1)-k} \mid T_{n,2}^{\alpha} \mid^k &= \sum_{n=1}^{m} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid n^{\gamma(\beta k+k-1)-k} (u_n^{\alpha})^k \\
&= O(1) \sum_{n=1}^{m} \mid \lambda_n \mid n^{\gamma(\beta k+k-1)-k} (u_n^{\alpha})^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} v^{\gamma(\beta k+k-1)-k} (u_v^{\alpha})^k \\
&+ O(1) \mid \lambda_m \mid \sum_{n=1}^{m} n^{\gamma(\beta k+k-1)-k} (u_n^{\alpha})^k
\end{aligned}
$$

$$= O(1) \sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m$$
$$= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) \mid \lambda_m \mid X_m = O(1) \quad as \quad m \to \infty,$$

by virtue of the hypotheses of the *Theorem* and *Lemma 2*.

Therefore, we get that

$$\sum_{n=1}^{m} n^{\gamma(\beta k+k-1)-k} \mid T_{n,r}^{\alpha} \mid^k = O(1) \quad as \quad m \to \infty, \quad for \quad r = 1, 2.$$

This completes the proof of the *Theorem*.

If we take $\gamma = 1$, then we get *Theorem A*. Also if we take $\gamma = 1$, $\beta = 0$ and $\alpha = 1$, then we obtain a result of Mishra and Srivastava [6]under weaker conditions for $\mid C, 1 \mid_k$ summability factors.

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