Any polynomial $D(4)$-quadruple is regular

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Abstract. *In this paper we prove that if $\{a, b, c, d\}$ is a set of four non-zero polynomials with integer coefficients, not all constant, such that the product of any two of its distinct elements increased by 4 is a square of a polynomial with integer coefficients, then $(a+b-c-d)^2 = (ab+4)(cd+4)$.*

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1. Introduction

Let n be an integer. A set of m positive integers is called a Diophantine m-tuple with the property $D(n)$ or simply a $D(n)$-m-tuple, if the product of any two of them increased by n is a perfect square. The first $D(1)$-quadruple, the set $\{1, 3, 8, 120\}$, was found by Fermat. The conjecture is that there does not exist a $D(1)$-quintuple. In 1969, Baker and Davenport [1] proved that the Fermat's set cannot be extended to a $D(1)$-quintuple. Recently, Dujella [3] proved that there does not exist a $D(1)$-sextuple and there are only finitely many $D(1)$-quintuples.

In the case $n = 4$ the conjecture is that there does not exist a $D(4)$-quintuple. Actually there is a stronger version of the conjecture (see [8]).

Conjecture 1. *There does not exist a $D(4)$-quintuple. Moreover, if $\{a, b, c, d\}$ is a $D(4)$-quadruple such that $a < b < c < d$, then*

$$d = a + b + c + \frac{1}{2}(abc + rst),$$

where r, s, t are positive integers defined by

$$ab + 4 = r^2,\ ac + 4 = s^2,\ bc + 4 = t^2.$$

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The first non-extendibility result for $D(4)$-m-tuples was proven by Mohanty and Ramasamy in [16]. There they proved that the $D(4)$-quadruple $\{1, 5, 12, 96\}$ cannot be extended to a $D(4)$-quintuple. Later Kedlaya [15] proved if $\{1, 5, 12, d\}$ is a $D(4)$-quadruple, then $d = 96$.

There are various generalizations of this result. One was given by Dujella and Ramasamy in [8], where they proved Conjecture 1 for a parametric family of $D(4)$-quadruples. More precisely, they proved if k and d are positive integers and

$$\{F_{2k}, 5F_{2k}, 4F_{2k+2}, d\}$$

is a $D(4)$-quadruple, then $d = 4L_{2k}F_{4k+2}$, where F_k and L_k are the k-th Fibonacci and Lucas numbers, respectively. The second generalization was given by the second author in [12]. There he proved if $k \geq 3$ is an integer and $\{k-2, k+2, 4k, d\}$ is a $D(4)$-quadruple, then $d = 4k^3 - 4k$. All these results support Conjecture 1.

The first author has recently proved that there does not exist a $D(4)$-sextuple (see [10, 11]).

A polynomial variant of the above problems was first studied by Jones [13, 14], and it was for the case $n = 1$.

Definition 1. *Let n be an integer. A set $\{a_1, a_2, \dots, a_m\}$ of m non-zero polynomials with integer coefficients, which are not all constant, is called a polynomial $D(n)$-m-tuple if for all $1 \leq i < j \leq m$ the following holds: $a_i \cdot a_j + n = b_{ij}^2$, where $b_{ij} \in \mathbb{Z}[x]$.*

Let us mention some results for general n. For some n, not every $D(n)$-triple is extensible to a quadruple. For example, for $n = 2$, there does not even exist a $D(4)$-quadruple (see [2]). And for some n, e.g. $n = 16$, the extension to a $D(n)$-quadruple is not unique (see [9]), although there the first author proved unique extensions of some polynomial $D(16)$-triples. Recently Dujella et al. [6] found an upper bound for the size of polynomial $D(n)$-m-tuples. Precisely, they proved that for $n \neq 0$, there does not exist a polynomial $D(n)$-11-tuple. Dujella and Fuchs [4] also proved that there does not exist a polynomial $D(-1)$-quadruple, and the same authors [5] have proven a strong Diophantine quintuple conjecture for polynomials with integer coefficients. In this paper we prove the analogous result in the case $n = 4$.

Definition 2. *A polynomial $D(4)$-quadruple $\{a, b, c, d\}$ is called regular if*

$$(a + b - c - d)^2 = (ab + 4)(cd + 4),$$

or equivalently if $d = d_+$ or $d = d_-$, where

$$d_\pm = a + b + c + \frac{1}{2}(abc \pm rst)$$

and $r, s, t \in \mathbb{Z}[x]$ are defined by

$$ab + 4 = r^2,\ ac + 4 = s^2,\ bc + 4 = t^2.$$

In this paper d_+ and d_- will always have this meaning. If $\{a,b,c\}$ is a polynomial $D(4)$-triple, it is easy to check that $\{a,b,c,d_\pm\}$ is a polynomial $D(4)$-quadruple. Indeed, since

$$ad_\pm + 4 = \left(\frac{1}{2}(at \pm rs)\right)^2, \; bd_\pm + 4 = \left(\frac{1}{2}(bs \pm rt)\right)^2, \; cd_\pm + 4 = \left(\frac{1}{2}(cr \pm st)\right)^2,$$

it suffices to check that $d_\pm \in \mathbb{Z}[x]$. But from the fact that $2\mathbb{Z}[x]$ is a prime ideal in $\mathbb{Z}[x]$ and from $(abc+rst)(abc-rst) \in 2\mathbb{Z}[x]$, we conclude that $(abc+rst)$ or $(abc-rst)$ is in $2\mathbb{Z}[x]$. Since these are obviously equivalent, we see that $d_\pm \in \mathbb{Z}[x]$. The same argument can be used in several places to ensure that the terms of binary recurring sequences or both sides of congruences have integer coefficients.

We will write our main result in the form of a theorem.

Theorem 1. *All polynomial $D(4)$-quadruples are regular.*

In the proof of the theorem we use the strategy from the paper of Dujella and Fuchs [5] together with the results the first author has proved in [10, 11]. Because some of the proofs are exactly the same as in [5], we will not give all details. We first transform the problem into solving the system of simultaneous polynomial Pellian equations, which then reduces to finding the intersection of binary recurring sequences of polynomials. Assuming that we have an irregular $D(4)$-quadruple $\{a,b,c,d\}$ with a minimal d, we will obtain a contradiction.

2. System of Pellian equations

Let $\mathbb{Z}^+[x]$ denote the set of all polynomials with integer coefficients with positive leading coefficients. For $a,b \in \mathbb{Z}^+[x]$, $a<b$ means that $b-a \in \mathbb{Z}^+[x]$. It can be checked that usual fundamental properties of the inequality hold for this order. For $a \in \mathbb{Z}^+[x]$, we define $|a| = a$ if $a \geq 0$, and $|a| = -a$ if $a < 0$. It is clear that all leading coefficients of the polynomials in a $D(4)$-m-tuple have the same sign. So without loss of generality we may assume that they are all positive, i.e. that all polynomials are in $\mathbb{Z}^+[x]$.

If $\{a,b,c,d\}$ is a polynomial $D(4)$-quadruple such that $a<b<c<d$, then d is not a constant by definition. Assume now that a and b are constant polynomials. Considering leading coefficients of $ad+4$ and $bd+4$ we can conclude that ab is a square, which contradicts the assumption that $ab+4$ is also a perfect square. This proves that in a polynomial $D(4)$-quadruple we can have at most one constant polynomial.

Assume now that $\{a,b,c,d\}$, such that $0<a<b<c<d$, is an irregular $D(4)$-quadruple with minimal d among all such quadruples. Under this assumption we will end up with a contradiction, which will finish the proof of our theorem.

Let us fix some notation at the beginning. Let $r,s,t \in \mathbb{Z}^+[x]$ be defined by

$$ab+4 = r^2, \; ac+4 = s^2, \; bc+4 = t^2.$$

Moreover, let

$$ad+4 = x^2, \; bd+4 = y^2, \; cd+4 = z^2,$$

with $x, y, z \in \mathbb{Z}^+[x]$. Eliminating d we get the following system of simultaneous Pellian equations

$$az^2 - cx^2 = 4(a-c), \tag{1}$$

$$bz^2 - cy^2 = 4(b-c). \tag{2}$$

Let

$$\deg a = A,\ \deg b = B,\ \deg c = C.$$

The letters A, B, C will always have this meaning. We can now describe the sets of solutions of equations (1) and (2) in the following lemma.

Lemma 1. *There exist $z_0, x_0, z_1, y_1 \in \mathbb{Z}[x]$, with the following properties:*

(i) *(z_0, x_0) and (z_1, y_1) are solutions of (1) and (2), respectively.*

(ii) *The following inequalities are satisfied:*

$$\deg x_0 \leq \frac{A+C}{4} < \deg s,\ \deg z_0 \leq \frac{3C-A}{4} < C, \tag{3}$$

$$\deg y_1 \leq \frac{B+C}{4} < \deg t,\ \deg z_1 \leq \frac{3C-B}{4} < C, \tag{4}$$

and

$$x_0, |z_0|, y_1, |z_1| > 0.$$

(iii) *If (z, x) and (z, y) are polynomial solutions of (1) and (2), with $x, y, z \in \mathbb{Z}^+[x]$, then there exist integers $m, n \geq 0$ such that*

$$z\sqrt{a} + x\sqrt{c} = (z_0\sqrt{a} + x_0\sqrt{c})\left(\frac{s+\sqrt{ac}}{2}\right)^m, \tag{5}$$

$$z\sqrt{b} + y\sqrt{c} = (z_1\sqrt{b} + y_1\sqrt{c})\left(\frac{t+\sqrt{bc}}{2}\right)^n, \tag{6}$$

where this means that coefficients of $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{c}$ respectively on both sides are equal.

Proof. The statement can be proven in exactly the same way as [7, Lemma 4] or directly as in [4, 8]. □

From (5) we get $z = v_m$ for some integer $m \geq 0$, where

$$v_0 = z_0,\ v_1 = \frac{1}{2}\left(sz_0 + cx_0\right),\ v_{m+2} = sv_{m+1} - v_m, \tag{7}$$

for some solution (z_0, x_0) of (1) satisfying (3). From (6) we conclude that $z = w_n$ for some integer $n \geq 0$, where

$$w_0 = z_1,\ w_1 = \frac{1}{2}\left(tz_1 + cy_1\right),\ v_{n+2} = tv_{n+1} - v_n, \tag{8}$$

for some solution (z_1, y_1) of (2) with (4).

We have now reduced our problem to solving equations of the form $v_m = w_n$. By induction we get information on the degrees of these sequences.

Lemma 2. *Let (v_m) and (w_n) be the sequences from the above. Then*

$$\deg v_m = \deg v_1 + (m-1)\frac{A+C}{2},$$

$$\deg w_n = \deg w_1 + (n-1)\frac{B+C}{2}.$$

3. Gap principle and congruence relations

We start with the same construction as in [5].

Lemma 3. *Let us define u, v, w such that*

$$ad_{\pm}+4 = u^2, bd_{\pm}+4 = v^2, cd_{\pm}+4 = w^2,$$

where

$$u = \frac{1}{2}\left(at \pm rs\right), v = \frac{1}{2}\left(bs \pm rt\right), w = \frac{1}{2}\left(cr \pm st\right).$$

Then,

$$c = a+b+d_{\pm} + \frac{1}{2}(abd_{\pm} \mp ruv),$$

and

$$d_+ \cdot d_- = (c-a-b-2r)(c-a-b+2r).$$

Observation that if $d_- \neq 0$, then $d_- \geq 1$, implies the following lemma.

Lemma 4. *If $\{a, b, c\}$ is a polynomial $D(4)$-triple such that $a < b < c$, then $c = a+b+2r$ or $c \geq \frac{1}{2}abd_- + 1$ with $d_- \neq 0$.*

From this lemma we get useful gap principle: $C \geq A+B$ or $c = a+b+2r$.

For the sequences (v_m) and (w_n) we have

$$v_{2m} \equiv v_0 \pmod{c}, v_{2m+1} \equiv v_1 \pmod{c},$$

$$w_{2n} \equiv w_0 \pmod{c}, w_{2n+1} \equiv w_1 \pmod{c}.$$

The following lemma can be proven in the same way as [5, Lemma 5] and [10, Lemma 4].

Lemma 5.

(i) If the equation $v_{2m} = w_{2n}$ has a solution, then $z_0 = z_1$.

(ii) If the equation $v_{2m+1} = w_{2n}$ has a solution, then $z_0 \cdot z_1 < 0$ and $|z_1| = \frac{1}{2}(cx_0 - s|z_0|)$.

(iii) If the equation $v_{2m} = w_{2n+1}$ has a solution, then $z_0 \cdot z_1 < 0$ and $|z_0| = \frac{1}{2}(cy_1 - t|z_1|)$.

(iv) If the equation $v_{2m+1} = w_{2n+1}$ has a solution, then $z_0 \cdot z_1 > 0$ and $cx_0 - s|z_0| = cy_1 - t|z_1|$.

Considering our sequences modulo c^2 we have the following lemma.

Lemma 6.

$$v_{2m} \equiv z_0 + \frac{1}{2}c(az_0m^2 + sx_0m) \pmod{c^2},$$

$$2v_{2m+1} \equiv sz_0 + c(\frac{1}{2}asz_0m(m+1) + x_0(2m+1)) \pmod{c^2},$$

$$w_{2n} \equiv z_1 + \frac{1}{2}c(bz_1n^2 + ty_1n) \pmod{c^2},$$

$$2w_{2n+1} \equiv tz_1 + c(\frac{1}{2}btz_1n(n+1) + y_1(2n+1)) \pmod{c^2}.$$

Proof. We will give details only for v_{2m}. First we consider the case $c \in 2\mathbb{Z}[x]$. Notice $v_m^2 \equiv z_0^2 \pmod{c}$, for any m. Because we are interested only in the sequences that will give us the extension to a quadruple we must have $z_0^2 \equiv 4 \pmod{c}$. But this implies $z_0^2 \in 2\mathbb{Z}[x]$, i.e. $z_0 \in 2\mathbb{Z}[x]$. Also $s \in 2\mathbb{Z}[x]$. When z_0, $s \in 2\mathbb{Z}[x]$, we can easily prove congruence relation by induction. In this case we do not have any problem with polynomials that are not in $\mathbb{Z}[x]$.

On the other hand, if $c \notin 2\mathbb{Z}[x]$, we can prove by induction

$$2v_{2m} \equiv 2z_0 + c(az_0m^2 + sx_0m) \pmod{c^2}.$$

Moreover, it can be shown that $az_0m^2 + sx_0m \in 2\mathbb{Z}[x]$. If m is even, it is obviously true, and if m is odd, it follows from

$$(az_0 + sx_0)(az_0 - sx_0) = 4a(a-c) - 4x_0^2 \in 2\mathbb{Z}[x].$$

At the end, because $c \notin 2\mathbb{Z}[x]$, we can divide both sides of congruence by 2 and get the statement of the lemma. $\square$

4. Determination of initial terms

From the estimates of initial terms and congruence conditions modulo c we see that if the equation $v_m = w_n$ has a solution, then there exists a solution with $m = 0$ or $m = 1$. However, since such a small solution induces a polynomial $D(4)$-quadruple $\{a, b, c, d_0\}$, with $d_0 < c$, it follows from the minimality of d that $d_0 = 0$ or $d_0 = d_-$, which gives us very precise determination of initial terms.

Lemma 7.

(i) *If the equation $v_{2m} = w_{2n}$ has a solution, then $z_0 = z_1$. Moreover, $|z_0| = 2$, or $|z_0| = \frac{1}{2}(cr - st)$.*

(ii) *If the equation $v_{2m+1} = w_{2n}$ has a solution, then $|z_0| = t$, $|z_1| = \frac{1}{2}(cr - st)$, $z_0z_1 < 0$.*

(iii) *If the equation $v_{2m} = w_{2n+1}$ has a solution, then $|z_1| = s$, $|z_0| = \frac{1}{2}(cr - st)$, $z_0z_1 < 0$.*

(iv) *If the equation $v_{2m+1} = w_{2n+1}$ has a solution, then $|z_0| = t$, $|z_1| = s$, $z_0z_1 > 0$.*

Proof. Since the proof is similar to [5, Lemma 7] and [10, Lemma 9], we will give only the proof of (i). From Lemma 5 we get $z_0 = z_1$. Define $d_0 = \frac{z_0^2-4}{c}$. Then $d_0 \in \mathbb{Z}^+[x]$ and

$$cd_0 + 4 = z_0^2,\ ad_0 + 4 = x_0^2,\ bd_0 + 4 = y_1^2.$$

If $d_0 = 0$, we conclude $|z_0| = 2$. If $d_0 \neq 0$, then $\{a, b, c, d_0\}$ is a polynomial $D(4)$-quadruple. From $d_0 < c$ and minimality of d we get $d_0 = d_-$ and $|z_0| = \frac{1}{2}(cr - st)$. □

Now we need some technical lemmas. For the first two we will omit the proofs, since they are exactly the same as the ones of Lemmas 8 and 9 in [5].

Lemma 8. *Assume that $B < C$. Then,*

$$\deg(cr - st) = C - \frac{A+B}{2}.$$

Moreover, if $z_0 = \frac{1}{2}(cr - st)$, then

$$\deg(cx_0 - sz_0) = \frac{B+C}{2},$$

and if $z_1 = \frac{1}{2}(cr - st)$, then

$$\deg(cy_1 - tz_1) = \frac{A+C}{2}.$$

Lemma 9. *Let $e = \frac{1}{2}(rst - cr^2) + c$. Then*

(i) $\deg e = C - A - B < C$, *if* $C > A + 2B$,

(ii) $\deg e \leq B = C - A - B < C$, *if* $C = A + 2B$,

(iii) $\deg e = B$, *if* $C < A + 2B$.

Lemma 10. *Let $\{a, b, c\}$ be a polynomial $D(4)$-triple such that $a < b < c$ and $B < C = A + 2B$. Then*

$$\{a, b, d_-, c\} = \{a, b, a + b \pm 2r, r(r \pm a)(b \pm r)\}.$$

Moreover, in this case $e = \mp 2r$.

Proof. From

$$d_+ \cdot d_- = (c - a - b - 2r)(c - a - b + 2r),$$

we conclude $C = \deg abd_-$, which implies $\deg d_- = B$. Trivially $d_- \neq 0, a, b$. Since $A \leq B = \deg d_-$ we can apply construction from Lemma 3 to the triple $\{a, b, d_-\}$. So we have $c = e_+$ and

$$e_+ \cdot e_- = (d_- - a - b - 2r)(d_- - a - b + 2r).$$

From the exactly the same argument as the proof of Lemma 10 in [5], we see that $e_- = 0$ and $d_- = a + b \pm 2r$. Now,

$$c = a + b + d_- + \frac{1}{2}(abd_- + ruv),$$

where $u^2 = ad_- + 4, v^2 = bd_- + 4$. After some computation we get $u = r \pm a$, $v = b \pm r$ and $c = ruv = r(r \pm a)(b \pm r)$. This furthermore implies $s = r^2 \pm ar - 2$ and $t = br \pm (r^2 - 2)$. If we define $e = \frac{1}{2}(rst - cr^2) + c$, direct computation shows that $e = \mp 2r$. □

5. Proof of the Theorem

We finish the proof of our theorem considering intersections of the recurring sequences obtained with the initial values described in Lemma 5. Because the proof uses the same strategy as in [5], we will not give all details.

Case **1.1)** $v_{2m} = w_{2n}$, $z_0 = z_1 = \pm 2$.

From (1), (2) and Lemma 1 we conclude $x_0 = y_1 = 2$. Therefore Lemma 2 gives us

$$\deg v_{2m} = C + (2m - 1)\frac{A + C}{2},$$
$$\deg w_{2n} = C + (2n - 1)\frac{B + C}{2},$$

if $B < C$ or $z_0 = 2$ and

$$\deg v_{2m} = C + (2m - 1)\frac{A + C}{2},$$
$$\deg w_{2n} = \frac{A + B}{2} + (2n - 1)\frac{B + C}{2},$$

if $B = C$ and $z_0 = -2$. Now from Lemma 6 we get

$$\pm am^2 + sm \equiv \pm bn^2 + tn \pmod{c}. \tag{9}$$

We can assume that $m, n \neq 0$, because $m = n = 0$ corresponds to $d = 0$, which contradicts the assumption. Consider three cases seperately:

$$B < C,\ A < B = C,\ A = B = C.$$

Then, in exactly the same arguments as in the proof of the theorem in [5], we can arrive at a contradiction in each case (note that the case $A < B = C$ leads to $m = n = 1$, which contradicts $d \neq d_\pm$). This finishes the proof in Case 1.1.

In all other cases we may assume $B < C$. Otherwise $B = C$ implies $c = a+b+2r$ and $\frac{1}{2}(cr - st) = 2$. The case $z_0 = z_1 = \pm 2$ we have just solved, and in all other cases we get contradiction. If $z_0 = \pm t$, we get $\deg t = \deg(b + r) = C > \deg z_0$, a contradiction. If $z_1 = \pm s$, from $z_1^2 = ac+4$, and Lemma 1 which gives us $\deg z_1 \leq \frac{C}{2}$, we conclude $A = 0$. We also have $y_1 = r$, and relation $(t - 2)y_1^2 \leq b(c - b)$. Since $\deg((t-2)r^2) = 2B$ and $\deg b(c-b) = \deg(b(a+2r)) = \frac{3B}{2}$, we get a contradiction.

Case **1.2)** $v_{2m} = w_{2n}$, $z_0 = z_1 = \pm\frac{1}{2}(cr - st)$.

Let us first assume $A = B$. Then from Lemma 2 we get

$$\deg v_{2m} = \frac{3C - B}{2} + (2m-1)\frac{A+C}{2}, \ \deg w_{2n} = \frac{3C - B}{2} + (2n-1)\frac{A+C}{2},$$

or

$$\deg v_{2m} = \frac{B + C}{2} + (2m-1)\frac{A+C}{2}, \ \deg w_{2n} = \frac{B+C}{2} + (2n-1)\frac{A+C}{2},$$

where the first case is for $z_0 = z_1 > 0$ and the second for $z_0 = z_1 < 0$. Comparing degrees in $\deg v_{2m} = \deg w_{2n}$, we get $m = n$. Since $x_0 = \frac{1}{2}(rs - at)$, $y_1 = \frac{1}{2}(rt - bs)$, Lemma 6 implies

$$\mp astm(m \pm 1) + 4rm \equiv \mp bstn(n \pm 1) + 4rn \pmod{c}.$$

Multiplying this by st we have

$$\mp 16(am(m \pm 1) - bn(n \pm 1)) \equiv 4rst(n - m) \pmod{c},$$

which implies $m(m \pm 1)(a - b) = 0$. Since $a \neq b$, we obtain $m = 0$ or $m = 1$, which contradicts $d \neq 0, d_{\pm}$.

Thus we may assume $A < B < C$. From Lemmas 1 and 8, we get $C \leq 2A + B$. From relation

$$d_+ \cdot d_- = (c - a - b - 2r)(c - a - b + 2r)$$

we conclude $C = \deg abd_-$, which yields $\deg d_- \leq A$. Now we have $e = \frac{1}{2}(rst - cr^2) + c$, which after some computation gives us $d_- = a + b - e$. It follows from $\deg d_- \leq A < B = \deg e$, that the leading coefficients of b and e are equal. Also our congruence becomes the equation

$$\pm 16(am(m \pm 1) - bn(n \pm 1)) = 8e(n - m),$$

and by comparing the leading coefficients we get

$$\pm 2n(n \pm 1) = n - m.$$

The rest of the proof in this case is exactly the same as in [5].

Case **2)** $v_{2m+1} = w_{2n}$, $z_0 = \pm t$, $z_1 = \pm\frac{1}{2}(st - cr)$.

Case **3)** $v_{2m} = w_{2n+1}$, $z_0 = \pm\frac{1}{2}(cr - st)$, $z_1 = \mp s$.

The proof in these two cases is exactly the same as in [5], with slightly different coefficients in congruences, using lemmas 8, 9 and 10.

Case **4)** $v_{2m+1} = w_{2n+1}$, $z_0 = \pm t$, $z_1 = \pm s$.

In this case from Lemma 1 and equations (1) and (2) we get $x_0 = y_1 = r$. Then Lemma 6 implies

$$\pm astm(m+1) + 2r(2m+1) \equiv \pm bstn(n+1) + 2r(2n+1) \pmod{c}. \tag{10}$$

In this case by computing degrees we get

$$\deg v_{2m+1} = C \pm \frac{A+B}{2} + (A+C)m,$$

$$\deg w_{2n+1} = C \pm \frac{A+B}{2} + (B+C)n.$$

It implies $(A+C)m = (B+C)n$. If $A = B$, we conclude $m = n$ and we have congruence

$$m(m+1)(a-b) \equiv 0 \pmod{c},$$

which contradicts $d \neq 0$. Hence $A < B$. If we compare the degree of $z_0 = \pm t$ with the estimate from Lemma 1, we get $C \geq A + 2B$. Moreover, multipying (10) by st yields

$$\pm 16(am(m+1) - bn(n+1)) \equiv 4rst(n-m) \pmod{c}. \tag{11}$$

Let $e = \frac{1}{2}(rst - cr^2) + c$. Then $4rst \equiv 8e \pmod{c}$. Since by Lemma 9 we know $\deg e < C$, (11) implies

$$\pm 16(am(m+1) - bn(n+1)) = 8e(n-m). \tag{12}$$

If $C > A + 2B$, Lemma 9 yields $\deg e = C - A - B$, and by comparing degrees in (12) we get $C = A + 2B$, a contradiction. If $C = A + 2B$, by Lemma 10 we conclude $e = \mp 2r$ and $\deg e = \frac{A+B}{2} < B$, which contradicts (12). This finishes the proof of this case and of Theorem 1.

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