An extension of Knopp’s core theorem for complex bounded sequences

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Abstract. In this paper, using the idea used by Choudhary, we extend previously known results on the core for complex bounded sequences.

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1. Introduction

Let $m, c$ be linear spaces of complex bounded and convergent sequences $x = \{x_n\}$, respectively, normed by $\|x\| = \sup |x_n|$. We define functionals $l$ and $L$ on linear space of a real bounded sequence by

$$l(x) = \liminf x_n; L(x) = \limsup x_n.$$ 

Let $A = (a_{nk})$ be an infinite matrix and write

$$(Ax)_n := \sum_k a_{nk} x_k$$

if the series converges for each $n \in \mathbb{N}$. By $Ax$ we denote the sequence $\{(Ax)_n\}$. If $\lim Ax = \lim x$ for each $x \in c$, we say that $A$ is regular [2], [9] and write $A \in (c, c; p)$. Silverman Toeplitz theorem gives the necessary and sufficient conditions for regularity of the matrix $A$ [2], [9].

Matrix $A = (a_{nk})$ is called normal if it is a lower semi-triangular matrix with non-zero diagonal entries [2].

For brevity we shall denote the Knopp core of $x$ by $K - core \{x\}$; recall [2], [4] that

$$K - core \{x\} := \bigcap_{n=1}^{\infty} C_n(x)$$

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where $C_n(x)$ is the least closed convex hull of $\{x_k\}_{k \geq n}$. If $x$ is a real bounded sequence, then $K - \text{core} \{x\}$ will be a closed interval $[\bar{f}(x), L(x)]$.

The famous Knopp’s core theorem (see [2], [3], [5], [6], [8], [11]) determines a class of regular matrices for which $L(Ax) \leq L(x)$ for all real bounded sequences $x$; that is $K - \text{core} \{Ax\} \subseteq K - \text{core} \{x\}$.

Let $\mathbb{C}$ denote the set of complex numbers. In Shcherbakoff [10] it is shown that for every bounded $x$,

$$K - \text{core} \{x\} := \bigcap_{z \in \mathbb{C}} B_x(z),$$

where

$$B_x(z) := \left\{ w \in \mathbb{C} : |w - z| \leq \limsup_k |x_k - z| \right\}.$$  

Shcherbakoff [10] generalized the notion of the core of a bounded complex sequence by introducing the idea of the generalized $\alpha$-core of a bounded complex sequence $x$ as

$$K^{(\alpha)} - \text{core} \{x\} := \bigcap_{z \in \mathbb{C}} B^{(\alpha)}_x(z),$$

where

$$B^{(\alpha)}_x(z) := \left\{ w \in \mathbb{C} : |w - z| \leq \alpha \limsup_k |x_k - z|, \alpha \geq 1 \right\}.$$  

when $\alpha = 1$, $K^{(\alpha)} - \text{core} \{x\}$ reduces the usual Knopp core.

In [7] Natarajan has proved the following theorem.

**Theorem A.** When $K = \mathbb{R}$ or $\mathbb{C}$, an infinite matrix $A = (a_{nk})$, $a_{nk} \in K$, $n, k = 0, 1, 2, \ldots$ is such that

$$K - \text{core} \{Ax\} \subseteq K^{(\alpha)} - \text{core} \{x\}, \alpha \geq 1$$

for any bounded sequence $x$ if and only if $A$ is regular and satisfies

$$\limsup_{n \to \infty} \left( \sum_{k=0}^{\infty} |a_{nk}| \right) \leq \alpha.$$

This result for the case $\alpha = 1$ yields a simple proof of Knopp’s core theorem.

In this paper, using the idea used by Choudhary [1], we generalize inclusion (1).

### 2. Main results

Before giving the main result we first state a result due to Choudhary [1] that we need for our purposes.

**Lemma 1.** Let $n$ be fixed. In order that, whenever $Bx$ is bounded, $(Ax)_n$ should be defined, it is necessary and sufficient that

(i) $c_{nk} = \sum_{v=k}^{\infty} a_{nv} b_{vk}^{-1}$ exist for all $k$;
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(ii) \( \sum_{k=0}^{\infty} |c_{nk}| < \infty \) (for all \( n \));

(iii) \( \sum_{k=0}^{\infty} \left| \sum_{v=j+1}^{\infty} a_{nv} b_{vk}^{-1} \right| \to 0 \) (\( j \to \infty \))

should hold for the \( n \) considered. If these conditions are satisfied, then for bounded \( Bx \),

\[
(Ax)_n = (Cy)_n
\]  

(2)

where \( y := Bx \).

Whenever \( B \) is normal, \( B \) has a reciprocal. Denote its reciprocal by \( B^{-1} = (b_{nk}^{-1}) \). Note that if \( B \) is a normal matrix, then the space \( m_B := \{ x : Bx \in m \} \) is isometrically isomorphic to \( m \). Hence given a sequence \( y \in m_B \), there exists a unique sequence \( x \in m_B \) so that \( y := Bx \).

Now we are ready to state our first result:

**Theorem 1.** Let \( B = (b_{nk}) \) be a normal matrix and \( A \) any matrix. In order that, whenever \( Bx \) is bounded, \( Ax \) should exist and be bounded and that

\[
K^{(\alpha)} - \text{core} \{ Ax \} \subseteq K^{(\alpha)} - \text{core} \{ Bx \}, \ \alpha \geq 1
\]  

(3)

it is necessary and sufficient that

(i) \( C = AB^{-1} \) exists;

(ii) \( C \) is regular;

(iii) \( \limsup_{n \to \infty} \left( \sum_{k=0}^{\infty} |c_{nk}| \right) \leq \alpha \)

(iv) for any fixed \( n \)

\[
\sum_{k=0}^{j} \left| \sum_{v=j+1}^{\infty} a_{nv} b_{vk}^{-1} \right| \to 0 \quad (j \to \infty)
\]

**Proof.** Assume that (3) holds. Write \( y := Bx \). Let \( (Ax)_n \) be exist for each \( n \) whenever \( y \) is bounded. Then by Lemma 1, (i) and (iv) of Theorem hold. Moreover, for every bounded \( y \) we have (2). Hence, by (3) we get

\[
K^{(\alpha)} - \text{core} \{ Cy \} \subseteq K^{(\alpha)} - \text{core} \{ y \}, \ \alpha \geq 1
\]

for every bounded \( y \). Now it follows from Theorem A that (ii) and (iii) hold.

**Sufficiency.** Observe that conditions (i) and (iv) imply the conditions of Lemma 1. So (2) holds and \( Cy \) is bounded whenever \( y \in m \). Now from Theorem A, (ii) and (iii) imply that

\[
K^{(\alpha)} - \text{core} \{ Cy \} \subseteq K^{(\alpha)} - \text{core} \{ y \}, \ \alpha \geq 1
\]

provided \( y \) is bounded. Writing \( y = Bx \) we immediately get (3), whence the result.

Recall that the matrix \( A \) is called row-finite if every row contains only a finite number of non-zero elements. In this case (iii) of Theorem 1 is zero for sufficiently large \( j \); hence (iii) is evidently satisfied. So, Theorem 1 reduces to the following
Theorem 2. Let $B = (b_{nk})$ be a normal matrix. Then for a row-finite matrix $A$,

$$K - \text{core } \{Ax\} \subseteq K^{(\alpha)} - \text{core } \{x\}, \alpha \geq 1, \text{ (for all } x \in m_B)$$

if and only if (i) and (iii) hold.

If we interchange the roles of matrices $A$ and $B$ in Theorem 1, we immediately get the following

Theorem 3. Let $B = (b_{nk})$ and $A = (a_{nk})$ be normal matrices. Then for all $x \in m_B \cap m_A$ we have that

$$K - \text{core } \{Ax\} = K^{(\alpha)} - \text{core } \{Bx\}$$

if and only if

a) $C = AB^{-1}$ and $D = BA^{-1}$ exist;

b) $C$ and $D$ are regular;

c) $\limsup_{n \to \infty} \left( \sum_{k=0}^{\infty} |c_{nk}| \right) \leq \alpha$

$$\lim_{n \to \infty} \left( \sum_{k=0}^{\infty} |d_{nk}| \right) \leq \alpha$$

d) for any fixed $n$

$$\sum_{k=0}^{j} \sum_{v=j+1}^{\infty} a_{nv} b_{vk}^{-1} \to 0 (j \to \infty),$$

and

$$\sum_{k=0}^{j} \sum_{v=j+1}^{\infty} b_{nv} a_{vk}^{-1} \to 0 (j \to \infty).$$

References


