On the distance spectra of some graphs

Gopalapillai Induḷal∗ and Ivan Gutman†

Abstract. The $D$-eigenvalues of a connected graph $G$ are the eigenvalues of its distance matrix $D$, and form the $D$-spectrum of $G$. The $D$-energy $E_D(G)$ of the graph $G$ is the sum of the absolute values of its $D$-eigenvalues. Two (connected) graphs are said to be $D$-equienergetic if they have equal $D$-energies. The $D$-spectra of some graphs and their $D$-energies are calculated. A pair of $D$-equienergetic bipartite graphs on $24t$, $t \geq 3$, vertices is constructed.

Key words: distance eigenvalue (of a graph), distance spectrum (of a graph), distance energy (of a graph), distance–equienergetic graphs

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1. Introduction

Let $G$ be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$. The distance matrix $D = D(G)$ of $G$ is defined so that its $(i,j)$-entry is equal to $d_G(v_i, v_j)$, the distance (= length of the shortest path [2]) between the vertices $v_i$ and $v_j$ of $G$. The eigenvalues of the $D(G)$ are said to be the $D$-eigenvalues of $G$ and form the $D$-spectrum of $G$, denoted by $\text{spec}_D(G)$.

The ordinary graph spectrum is formed by the eigenvalues of the adjacency matrix [4]. In what follows we denote the ordinary eigenvalues of the graph $G$ by $\lambda_i$, $i = 1, 2, \ldots, p$, and the respective spectrum by $\text{spec}(G)$.

Since the distance matrix is symmetric, all its eigenvalues $\mu_i$, $i = 1, 2, \ldots, p$, are real and can be labelled so that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$. If $\mu_1 > \mu_2 > \cdots > \mu_g$ are the distinct $D$-eigenvalues, then the $D$-spectrum can be written as

$$\text{spec}_D(G) = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_g \\ m_1 & m_2 & \cdots & m_g \end{pmatrix}$$

where $m_j$ indicates the algebraic multiplicity of the eigenvalue $\mu_j$. Of course, $m_1 + m_2 + \cdots + m_g = p$.

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Two graphs $G$ and $H$ for which $\text{spec}_D(G) = \text{spec}_D(H)$ are said to be $D$-
cospectral. Otherwise, they are non-$D$-cospectral.

The $D$-energy, $E_D(G)$, of $G$ is defined as

$$E_D(G) = \sum_{i=1}^{p} |\mu_i|.$$  

(1)

Two graphs with equal $D$-energy are said to be $D$-
equienergetic. $D$-cospectral graphs are evidently $D$-
equienergetic. Therefore, in what follows we focus our attention to $D$-
equienergetic non-$D$-cospectral graphs.

The concept of $D$-energy, Eq. (1), was recently introduced [11]. This definition was motivated by the much older [7] and nowadays extensively studied [8, 9, 10, 13, 14, 15, 16] graph energy, defined in a manner fully analogous to Eq. (1), but in terms of the ordinary graph eigenvalues (eigenvalues of the adjacency matrix, see [4]).

In this paper we first derive a Hoffman–type relation for the distance matrix of distance regular graphs. By means of it, the distance spectra of some graphs and their energies are obtained. Also pairs of $D$-equienergetic bipartite graphs on $24t$, $t \geq 3$, vertices are constructed. All graphs considered in this paper are simple and we follow [4] for spectral graph theoretic terminology.

The considerations in the subsequent sections are based on the applications of the following lemmas:

**Lemma 1** [see [4]]. Let $G$ be a graph with adjacency matrix $A$ and $\text{spec}(G) = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$. Then $\det A = \prod_{i=1}^{p} \lambda_i$. In addition, for any polynomial $P(x)$, $P(\lambda)$ is an eigenvalue of $P(A)$ and hence $\det P(A) = \prod_{i=1}^{p} P(\lambda_i)$.

**Lemma 2** [see [5]]. Let

$$A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$$

be a $2 \times 2$ block symmetric matrix. Then the eigenvalues of $A$ are those of $A_0 + A_1$ together with those of $A_0 - A_1$.

**Lemma 3** [see [4]]. Let $M$, $N$, $P$, and $Q$ be matrices, and let $M$ be invertible. Let

$$S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}.$$  

Then $\det S = \det M \det(Q - PM^{-1}N)$. Besides, if $M$ and $P$ commute, then $\det S = \det(MQ - PN)$.

**Lemma 4** [see [4]]. Let $G$ be a connected $r$-regular graph, $r \geq 3$, with ordinary spectrum $\text{spec}(G) = \{r, \lambda_2, \ldots, \lambda_p\}$. Then

$$\text{spec}(L(G)) = \begin{pmatrix} 2r - 2 & \lambda_2 + r - 2 & \cdots & \lambda_p + r - 2 & -2 \\ 1 & 1 & \cdots & 1 & p(r - 2)/2 \end{pmatrix}.$$
Lemma 5 [see [4]]. For every $t \geq 3$, there exists a pair of non-cospectral cubic graphs on $2t$ vertices.

Lemma 6 [see [6]]. The distance spectrum of the cycle $C_n$ is given by

<table>
<thead>
<tr>
<th>$n$</th>
<th>greatest eigenvalue</th>
<th>$j$ even</th>
<th>$j$ odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>even</td>
<td>$n^2 / 4$</td>
<td>0</td>
<td>$-\csc^2 \left( \frac{\pi j}{2n} \right)$</td>
</tr>
<tr>
<td>odd</td>
<td>$n^2 - 1 / 4$</td>
<td>$-\sec^2 \left( \frac{\pi j}{2n} \right) / 4$</td>
<td>$-\csc^2 \left( \frac{\pi j}{2n} \right) / 4$</td>
</tr>
</tbody>
</table>

Definition 1 [see [12]]. Let $G$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$. Take another copy of $G$ with the vertices labelled by $\{u_1, u_2, \ldots, u_p\}$, where $u_i$ corresponds to $v_i$ for each $i$. Make $u_i$ adjacent to all the vertices in $N(v_i)$ in $G$, for each $i$. The resulting graph, denoted by $D_2G$, is called the double graph of $G$.

Definition 2 [see [4]]. Let $G$ be a graph. Attach a pendant vertex to each vertex of $G$. The resulting graph, denoted by $G \circ K_1$, is called the corona of $G$ with $K_1$.

We first prove the following auxiliary theorem.

Theorem 1. Let $M$ be a real symmetric irreducible square matrix of order $p$ in which each row sum is equal to a constant $k$. Then there exists a polynomial $Q(x)$ such that $Q(M) = J$, where $J$ is the all-one square matrix whose order is same as that of $M$.

Proof. Since $M$ is a real symmetric irreducible matrix in which each row sums to $k$, by the Frobenius theorem [4], $k$ is a simple and greatest eigenvalue of $M$. The matrix $M$ is diagonalizable because it is real and symmetric. Therefore there exists an orthonormal basis of characteristic vectors of $M$, associated with the eigenvalues of $M$.

Let $\lambda_1 = k, \lambda_2, \ldots, \lambda_g$ be the distinct eigenvalues of $M$. Let $\mathcal{S}(\lambda_i)$ be the eigenspace spanned by the orthonormal set of characteristic vectors $\{x^1_i, x^2_i, \ldots, x^p_i\}$ associated with $\lambda_i$, $i = 1, 2, \ldots, g$. Then $M$ has a spectral decomposition

$$M = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_g T_g$$

where $T_i$ is the projection of $M$ onto $\mathcal{S}(\lambda_i)$, treating $M$ as a linear operator. Then $T_i^2 = T_i$, $T_i T_j = 0$, $i \neq j$ and

$$T_i = x^1_i (x^1_i)^T + x^2_i (x^2_i)^T + \cdots + x^p_i (x^p_i)^T.$$

Now, corresponding to the greatest eigenvalue $k$ of $M$, there exists a unique
In this case, the distance matrix of a connected distance regular graph is irreducible, symmetric and each row sums to a constant.

Let \( D = -\lambda T_1 + \lambda_2 T_2 + \cdots + \lambda_\nu T_\nu \) for any polynomial \( f(x) \). As \( M \) is diagonalizable, the minimal polynomial of \( M \) is \( (x - k)(x - \lambda_2) \cdots (x - \lambda_\nu) \).

Let \( S(x) = (x - \lambda_2) \cdots (x - \lambda_\nu) \). Then \( S(\lambda_i) = 0 \), \( \lambda_i \neq k \). Thus \( S(M) = S(k) T_1 s(k) (1/p) J \). Choose \( Q(x) = p S(x) / S(k) \). This \( Q(x) \) satisfies the requirement of the theorem.

**Theorem 2.** Let \( D \) be the distance matrix of a connected distance regular graph \( G \). Then \( D \) is irreducible and there exists a polynomial \( P(x) \) such that \( P(D) = J \).

In this case
\[
P(x) = p \times \frac{(x - \lambda_2)(x - \lambda_3) \cdots (x - \lambda_\nu)}{(k - \lambda_2)(k - \lambda_3) \cdots (k - \lambda_\nu)}
\]

where \( k \) is the unique sum of each row which is also the greatest simple eigenvalue of \( D \), whereas \( \lambda_2, \lambda_3, \ldots, \lambda_\nu \) are the other distinct eigenvalues of \( D \).

**Proof.** The theorem follows from Theorem 1 due to the observation that the distance matrix of a connected distance regular graph is irreducible, symmetric and each row sums to a constant.

The rest of this paper is organized as follows. In the next section we obtain the distance spectra of \( D_2(G) \), \( G \times K_2 \), \( G[K_2] \), the lexicographic product of \( G \) with \( K_2 \), and \( G \circ K_1 \). Using this, the distance energies of \( D_2(C_{2n}) \), \( C_n \times K_2 \), \( C_{2n}[K_2] \), and \( C_n \circ K_1 \) are calculated. In the third section the \( D \)-spectrum of the extended double cover graphs of regular graphs of diameter 2 is discussed and a pair of \( D \)-equienergetic bipartite graphs on \( 24t \), \( t \geq 3 \) vertices is constructed.

For operations on graphs that are not defined in this paper see [4].

## 2. Distance spectra of some graphs

In this section we obtain the distance spectra of the double graph of \( C_n \), the Cartesian product of \( C_n \) with \( K_2 \) and the corona of \( C_n \) with \( K_1 \).
2.1. The double graph of $G$

**Theorem 3.** Let $G$ be a graph with distance spectrum $\text{spec}_D(G) = \{\mu_1, \mu_2, \ldots, \mu_p\}$. Then

$$\text{spec}_D(D_2G) = \begin{pmatrix} 2(\mu_i + 1) & -2 \\ 1 & p \end{pmatrix}, \; i = 1, 2, \ldots, p.$$ 

**Proof.** By definition of $D_2(G)$ we have:

$$d_{D_2G}(v_i, v_j) = d_G(v_i, v_j)$$
$$d_{D_2G}(v_i, u_i) = 2$$
$$d_{D_2G}(v_i, u_j) = d_G(v_i, v_j)$$
$$d_{D_2G}(v_j, u_i) = d_G(v_j, v_i).$$

Hence a suitable ordering of vertices yields the distance matrix of $D_2G$ of the form

$$\begin{bmatrix} D & D + 2I \\ D + 2I & D \end{bmatrix}$$

and the theorem follows from Lemma 2.

**Theorem 4.** $E_D(D_2C_{2n}) = 4n(n + 1)$.

**Proof.** By Lemma 6 and Theorem 3 we have

$$\text{spec}_D(D_2C_{2n}) = \begin{pmatrix} 2(n^2 + 1) & 2 & -2\cot^2(\pi j/2n) & -2 \\ 1 & n - 1 & 1 & 2n \end{pmatrix}, \; j = 1, 3, 5, \ldots, 2n - 1.$$ 

Thus $E_D(D_2C_{2n}) = 2 \times [2(n^2 + 1) + 2(n - 1)]4n(n + 1)$.

2.2. The Cartesian product $G \times K_2$

**Theorem 5.** Let $G$ be a distance regular graph with distance regularity $k$, distance matrix $D$, and $D$-spectrum $\{\mu_1 = k, \mu_2, \ldots, \mu_p\}$. Then

$$\text{spec}_D(G \times K_2) = \begin{pmatrix} 2k + p & -p & 2\mu_i & 0 \\ 1 & 1 & 1 & p - 1 \end{pmatrix}, \; i = 2, 3, \ldots, p.$$ 

**Proof.** The theorem follows from the fact that the distance matrix of $G \times K_2$ has the form

$$\begin{bmatrix} D & D + J \\ D + J & D \end{bmatrix}$$

and from Theorem 1 and Lemma 2.

**Corollary 1.** $E_D(G \times K_2) = 2(E_D(G) + p)$.
2.3. The corona of $G$ and $K_1$

**Theorem 6.** Let $G$ be a connected distance regular graph with distance regularity $k$, distance matrix $D$, and $\text{spec}_D(G) = \{\mu_1 = k, \mu_2, \ldots, \mu_p\}$. Then $\text{spec}_D(G \circ K_1)$ consists of the numbers

$$p + k - 1 + \sqrt{(p + k)^2 + (p - 1)^2}, \quad p + k - 1 - \sqrt{(p + k)^2 + (p - 1)^2}$$

$$\mu_i - 1 + \sqrt{\mu_i^2 + 1}, \quad \mu_i - 1 - \sqrt{\mu_i^2 + 1}, \quad i = 2, 3, \ldots, p.$$

**Proof.** From the definition of $G \circ K_1$, it follows that the distance matrix $H$ of $G \circ K_1$ is of the form

$$\begin{bmatrix} D & D + J \\ D + J & D + 2(J - I) \end{bmatrix}.$$  

Now the characteristic equation of $H$ is

$$|\lambda I - H| = 0 \Rightarrow \begin{vmatrix} \lambda I - D & -(D + J) \\ -(D + J) & \lambda I - D - 2(J - I) \end{vmatrix} = 0$$

$$\Rightarrow (\lambda I - D)(\lambda I - D - 2(J - I)) - (D + J)^2 = 0 \text{ by Lemma 3}$$

Now $D$ being the distance matrix of a distance regular graph, it satisfies the requirement in Theorem 2. Then the $D$-spectrum of $G \circ K_1$ follows from Theorem 2 and Lemma 1.

**Corollary 2.**

$$E_D(C_{2n} \circ K_1) = 2\left[(n - 1)^2 + \sqrt{(n - 1)^4 + 6n^2}\right]$$

$$E_D(C_{2n+1} \circ K_1) = 2\left[n^2 + 3n + \sqrt{(n^2 + 3n)^2 + 6n^2 + 6n + 1}\right].$$

2.4. The lexicographic product of $G$ with $K_2$

**Theorem 7.** Let $G$ be a connected graph with distance spectrum $\text{spec}_D(G) = \{\mu_1 = k, \mu_2, \ldots, \mu_p\}$. Then

$$\text{spec}_D(G[K_2]) = \begin{pmatrix} 2\mu_i + 1 & -1 \\ 1 & p \end{pmatrix}, \quad i = 1, 2, \ldots, p.$$

**Proof.** From the definition of the lexicographic product of $G$ with $K_2$, its distance matrix can be written as

$$\begin{bmatrix} D & D + I \\ D + I & D \end{bmatrix}$$

and the theorem follows from Lemma 2. □
Corollary 3. \( E_D(C_{2n}[K_2]) = 2n(2n + 1) \).

Proof. From Lemma 6 and Theorem 7 we have

\[
\text{spec}_D(C_{2n}[K_2]) = \begin{pmatrix} 2n^2 + 1 & 1 & -1 & 1 - 2\csc^2(\pi j/2n) \\ n - 1 & 1 & 2n & 1 \end{pmatrix}, \quad j = 1, 3, 5, \ldots .
\]

Since \( 1 - 2\csc^2\theta = -(\cot^2\theta + \csc^2\theta) \), the only positive eigenvalues are \( 4n^2 + 1 \) and 1 with multiplicities 1 and \( n - 1 \), respectively. Thus \( E_D(C_{2n}[K_2]) = 2n(2n + 1) \).

3. The extended double cover graph of regular graphs of diameter 2

In [1] N. Alon introduced the concept of extended double cover graph of a graph as follows.

Let \( G \) be a graph on the vertex set \( \{v_1, v_2, \ldots, v_p\} \). Define a bipartite graph \( H \) with \( V(H) = \{v_1, v_2, \ldots, v_p, u_1, u_2, \ldots, u_p\} \) in which \( v_i \) is adjacent to \( u_i \) for each \( i = 1, 2, \ldots, p \) and \( v_i \) is adjacent to \( u_j \) if \( v_i \) is adjacent to \( v_j \) in \( G \). The graph \( H \) is known as the extended double cover graph (EDC-graph) of \( G \). The ordinary spectrum of \( H \) has been determined in [3].

In this section we obtain the distance spectrum of the EDC-graph of a regular graph of diameter 2 and use it to construct regular \( D \)-equienergetic bipartite graphs on \( 24t \) vertices, for \( t \geq 3 \).

Theorem 8. Let \( G \) be an \( r \)-regular graph of diameter 2 on \( p \) vertices with (ordinary) spectrum \( \{r, \lambda_2, \ldots, \lambda_p\} \). Then the \( D \)-spectrum of the EDC-graph of \( G \) consists of the numbers \( 5p - 2r - 4, \quad 2r - p, \quad -2(\lambda_i + 2), \quad i = 2, 3, \ldots, p \), and \( 2\lambda_i, \quad i = 2, 3, \ldots, p \).

Proof. Let \( A \) and \( \overline{A} \) be, respectively, the adjacency matrices of \( G \) and \( \overline{G} \). Then by the definition of the EDC-graph, its distance matrix can be written as

\[
\begin{pmatrix}
2(J - I) & A + 3\overline{A} + I \\
A + 3\overline{A} + I & 2(J - I)
\end{pmatrix}
\]

and the theorem follows from Lemmas 1 and 3 and also from the observation that \( \overline{A} = J - I - A \).

Corollary 4.

\[
E_D(EDC(C_p \nabla C_p)) = \begin{cases} 
40, & p = 3 \\
4[E(C_p) + 5p - 10], & p \geq 4
\end{cases}
\]

where \( C_p \nabla C_p \) is the join [4] of \( C_p \) with itself.

Proof. The join of \( C_p \) with itself is a regular graph diameter 2 with the ordinary spectrum

\[
\begin{pmatrix} 
p + 2 & 2 - p & \lambda_i \\
1 & 1 & 2
\end{pmatrix}, \quad i = 2, 3, \ldots, p
\]
where \( \{2, \lambda_2, \ldots, \lambda_p\} \) is the ordinary spectrum of \( C_p \). Then by the above theorem, the distance spectrum of \( EDC(C_p \nabla C_p) \) is
\[
\begin{pmatrix}
8p - 8 & 4 & -2(\lambda_i + 2) & 2p - 8 & 4 - 2p & 2\lambda_i \\
1 & 1 & 2 & 1 & 1 & 2
\end{pmatrix}, \quad i = 2, 3, \ldots, p
\]
and hence the corollary follows as \( E(C_3) = 4 \).

\[\square\]

### 3.1. On a pair of \( D \)-equienergetic bipartite graphs

**Theorem 9.** There exists a pair of regular non-\( D \)-cospectral \( D \)-equienergetic bipartite graphs on \( 24t \) vertices, for each \( t \geq 3 \).

**Proof.** Let \( G \) be a cubic graph on \( 2t \) vertices, \( t \geq 3 \). Consider \( L^2(G) \), its second iterated line graph. Then by Lemma 4 and Theorem 8, we calculate that for \( F = L^2(G) \nabla L^2(G) \), the \( D \)-spectrum of \( EDC(F) \) is
\[
\begin{pmatrix}
16(3t - 1) & 12 & 0 & 2(\lambda_i + 3) & 12t - 16 & -4 & -12(t - 1) & -2(\lambda_i + 5) \\
1 & 1 & 8t & 2 & 1 & 8t & 1 & 2
\end{pmatrix}, \quad i = 2, 3, \ldots, 2t.
\]
Thus
\[
ED(E_D(F)) = 2 \times \left[ 12(t - 1) + 32t + 4 \sum_{i=2}^{2t} (\lambda_i + 5) \right]
\]
\[
= 2 \times [12t - 12 + 32t + 4(-3 + 5(2t - 1))] = 8(21t - 11).
\]
Now let \( G_1 \) and \( G_2 \) be the two non-cospectral cubic graphs on \( 2t \) vertices as given by Lemma 5. Further, let \( H_1 \) and \( H_2 \) be the \( EDC \)-graphs of \( L^2(G_1) \nabla L^2(G_1) \) and \( L^2(G_2) \nabla L^2(G_2) \), respectively. Then \( H_1 \) and \( H_2 \) are bipartite and \( E_D(H_1) = E_D(H_2) = 8(21t - 11) \), proving the theorem.

\[\square\]

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**References**


