Optimal Control and Filtering of Weakly Coupled Linear Discrete-Time Stochastic Systems by the Eigenvector Approach

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1 INTRODUCTION

The work in this paper is influenced by the work done in the theory of weakly coupled systems. The theory of weakly coupled control systems has attracted a lot of attention in the control literature [1, 2, 3, 4]. In [3] a transformation was introduced for decomposition of the weakly coupled algebraic Riccati equation, which is based on the closed-loop decomposition technique. The algebraic equations comprising the transformation have the form of general nonsymmetric nonsquare Riccati equations. These equations can be efficiently solved by iterative methods (fixed point iterations, Newton method) for a small value of coupling between subsystems [2]. For a larger value of coupling between subsystems, iterative methods could diverge and the desired transformation could not be found. In [5], the transformation was used in order to decompose corresponding algebraic Riccati equations of the optimal regulator and Kalman filter of weakly coupled linear discrete stochastic systems. The eigenvector approach to the solution of optimal control of continuous-time singularly perturbed and weakly coupled systems was introduced in [10], [11]. This work extends previous ideas to the problem of optimal control and filtering of weakly coupled linear discrete-time stochastic systems.

2 DECOMPOSITION OF THE LINEAR-QUADRATIC CONTROL PROBLEMS

Consider a linear time-invariant discrete system

\[ x(k+1) = Ax(k) + Bu(k) \] (1)

with the quadratic performance criterion

\[ J = \frac{1}{2} \sum_{k=0}^{\infty} [x(k)^T Q x(k) + u(k)^T R u(k)] \] (2)

The weakly coupled structure of (1) and (2) implies the following partitions

\[
\begin{align*}
x(k) &= \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \\
u(k) &= \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}, \\
A &= \begin{bmatrix} A_1 & \varepsilon A_2 \\ \varepsilon A_3 & A_4 \end{bmatrix}, \\
B &= \begin{bmatrix} B_1 & \varepsilon B_2 \\ \varepsilon B_3 & B_4 \end{bmatrix}, \\
Q &= \begin{bmatrix} Q_1 & \varepsilon Q_2 \\ \varepsilon Q_2^T & Q_3 \end{bmatrix}, \\
R &= \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}, \\
S &= BR^{-1}B^T = \begin{bmatrix} S_1 & \varepsilon S_2 \\ \varepsilon S_2^T & S_3 \end{bmatrix}
\end{align*}
\] (3)
where \( x_1, x_2 \) are vectors of subsystem state variables of appropriate dimensions \((n_1, n_2)\), \( u_1, u_2 \) are vectors of control inputs \((m_1, m_2)\), and \( \varepsilon \) is a small coupling parameter. \( A, B \) are system constant matrices, \( Q \) and \( R \) are constant weighting matrices. In addition, it is assumed that \( A_1 \) and \( A_4 \) are nonsingular.

The well known solution to the above optimal control problem is given by

\[
u(k) = -R^{-1}B^T \lambda(k+1) = -(R + B^TP_rB)^{-1}B^TP_rAx(k)
\]

(4)

\[
u(k) = -Fx(k)
\]

where \( \lambda(k) \) is a costate variable and \( P_r \) is the positive-semidefinite stabilizing solution of the discrete Riccati equation given by

\[
P_r = Q + A^TP_rA - A^TP_rB(R + B^TP_rB)^{-1}B^TP_rA
\]

(5)

Partitioning the state vector \( x \), the corresponding costate vector \( \lambda \) and interchanging second and third rows, the Hamiltonian form can be written as [3]

\[
\begin{bmatrix}
\eta(k+1) \\
\xi(k+1)
\end{bmatrix} =
\begin{bmatrix}
I & -\varepsilon H \\
0 & I
\end{bmatrix}
\begin{bmatrix}
T_{1r} & \varepsilon T_{2r} \\
\varepsilon T_{3r} & T_{4r}
\end{bmatrix}
\begin{bmatrix}
I \\
0
\end{bmatrix}
\begin{bmatrix}
I & \varepsilon H \\
\varepsilon L & I
\end{bmatrix}
\begin{bmatrix}
\eta(k) \\
\xi(k)
\end{bmatrix}
\]

(10)

or

\[
\begin{bmatrix}
\eta(k+1) \\
\xi(k+1)
\end{bmatrix} =
\begin{bmatrix}
I - \varepsilon^2 H_rL_r \\
\varepsilon L_r & I
\end{bmatrix}
\begin{bmatrix}
T_{1r} & \varepsilon T_{2r} \\
\varepsilon T_{3r} & T_{4r}
\end{bmatrix}
\begin{bmatrix}
I \\
0
\end{bmatrix}
\begin{bmatrix}
I & \varepsilon H_r \\
-\varepsilon L_r & I - \varepsilon^2 L_rH_r
\end{bmatrix}
\begin{bmatrix}
\eta(k) \\
\xi(k)
\end{bmatrix}
\]

(11)

The solution to this equation exists under the standard stabilizability-detectability assumption imposed on the triple \((A, B, Q)\).

The Hamiltonian form of the optimal control problem is given by [9]

\[
\begin{bmatrix}
x(k+1) \\
\lambda(k+1)
\end{bmatrix} = H
\begin{bmatrix}
x(k) \\
\lambda(k)
\end{bmatrix}
\]

(6)

where

\[
H =
\begin{bmatrix}
A + BR^{-1}B^TA^T & -BR^{-1}B^TA^T \\
-A^TQ & A^T
\end{bmatrix}
\]

(7)

The Hamiltonian form represents the closed-loop solution to the optimal control problem, where \( \lambda(k) = P_r x(k) \).

The transformation leads to two completely decoupled subsystems

\[
\begin{align*}
\eta(k+1) &= (T_{1r} - \varepsilon^2 T_{2r}L_r) \eta(k) \\
\xi(k+1) &= (T_{4r} + \varepsilon^2 L_rT_{2r}) \xi(k)
\end{align*}
\]

(13)

\[
\begin{bmatrix}
\eta_1(k) \\
\eta_2(k)
\end{bmatrix}, \quad
\begin{bmatrix}
\xi_1(k) \\
\xi_2(k)
\end{bmatrix}
\]

(14)
and $L_r$ and $H_r$ satisfying

$$L_rT_{1r} - T_{4r}L_r + T_{3r} - \epsilon^2 L_rT_{2r}L_r = 0$$

$$H_r\left(T_{4r} + \epsilon^2 L_rT_{2r}\right) - \left(T_{1r} - \epsilon^2 T_{2r}L_r\right)H_r - T_{2r} = 0$$

The rearrangement of variables in (8) is done by the means of a similarity transformation $E$ of the form

$$\begin{bmatrix} x_1(k) \\ \lambda_1(k) \\ x_2(k) \\ \lambda_2(k) \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} \begin{bmatrix} x_1(k) \\ \lambda_1(k) \\ x_2(k) \\ \lambda_2(k) \end{bmatrix} = E \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix}$$

The relationship between original and new coordinates is given by

$$\begin{bmatrix} \eta_1(k) \\ \xi_1(k) \\ \eta_2(k) \\ \xi_2(k) \end{bmatrix} = E^T T_r E \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} = \Pi_r \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix}$$

$$= \begin{bmatrix} \Pi_{1r} & \Pi_{2r} \\ \Pi_{3r} & \Pi_{4r} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix}$$

Since $\lambda = P_r x$, where $P_r$ satisfies the discrete algebraic Riccati equations (5), it follows

$$\begin{bmatrix} \eta_1(k+1) \\ \xi_1(k+1) \\ \eta_2(k+1) \\ \xi_2(k+1) \end{bmatrix} = \begin{bmatrix} \Pi_{1r} + \Pi_{2r}P_r \\ \Pi_{3r} + \Pi_{4r}P_r \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

The decoupled subsystems (13) also represent the closed-loop solution of the optimal control problem in the new coordinates. Based on this fact the equations (13) can be written as

$$\begin{bmatrix} \eta_1(k+1) \\ \eta_2(k+1) \\ \xi_1(k+1) \\ \xi_2(k+1) \end{bmatrix} = \begin{bmatrix} a_{1r} & a_{2r} \\ a_{3r} & a_{4r} \\ b_{1r} & b_{2r} \\ b_{3r} & b_{4r} \end{bmatrix} \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \\ \xi_1(k) \\ \xi_2(k) \end{bmatrix}$$

or

$$\begin{bmatrix} \eta_2(k) \\ \xi_2(k) \end{bmatrix} = \begin{bmatrix} P_{ra} & 0 \\ 0 & P_{rb} \end{bmatrix} \begin{bmatrix} \eta_1(k) \\ \xi_1(k) \end{bmatrix}$$

and $P_a$ and $P_b$ satisfy nonsymmetric Riccati equations of the form

$$P_{ra}a_1 - a_4 P_{ra} - a_3 + P_{ra}a_2 P_{ra} = 0$$

$$P_{rb}b_1 - b_4 P_{rb} - b_3 + P_{rb}b_2 P_{rb} = 0$$

leading to

$$\eta_1(k+1) = (a_1 + a_2 P_{ra}) \eta_1(k)$$

$$\xi_1(k+1) = (b_1 + b_2 P_{rb}) \xi_1(k)$$

It follows from (18) and (20)

$$\begin{bmatrix} P_{ra} & 0 \\ 0 & P_{rb} \end{bmatrix} = (\Pi_3 + \Pi_4 P_r) (\Pi_1 + \Pi_2 P_r)^{-1}$$

This equation can be solved for $P_r$ giving

$$P_r = \begin{bmatrix} P_{ra} & 0 \\ 0 & P_{rb} \end{bmatrix} (\Pi_3 - P_{ra} P_{rb}) (\Pi_1)$$

which gives the solution of the global discrete Riccati equation (5) in terms of reduced order continuous time nonsymmetric Riccati equations (21) and decoupling transformation matrix (12). In order to realize the above presented decomposition procedure, it is necessary to solve continuous-time nonsquare and nonsymmetric Riccati Equations (15) and (21).
3 DECOMPOSITION OF THE OPTIMAL FILTERING PROBLEM

Let the linear discrete-time invariant stochastic system be given by

\[
\begin{bmatrix}
  x_1(k+1) \\
  x_2(k+1)
\end{bmatrix} = \begin{bmatrix}
  A_1 & \varepsilon A_2 \\
  \varepsilon A_3 & A_4
\end{bmatrix} \begin{bmatrix}
  x_1(k) \\
  x_2(k)
\end{bmatrix} + \begin{bmatrix}
  G_1 \\
  \varepsilon G_2
\end{bmatrix} w_k + \begin{bmatrix}
  G_3 \\
  \varepsilon G_4
\end{bmatrix} v_k
\]

with corresponding measurements

\[
\begin{bmatrix}
  y_1(k) \\
  y_2(k)
\end{bmatrix} = \begin{bmatrix}
  C_1 & \varepsilon C_2 \\
  \varepsilon C_3 & C_4
\end{bmatrix} \begin{bmatrix}
  x_1(k) \\
  x_2(k)
\end{bmatrix} + \begin{bmatrix}
  v_1(k) \\
  v_2(k)
\end{bmatrix}
\]

where \( x_i \) are state vectors, \( w_i \) and \( v_i \) are independent zero-mean white Gaussian processes with intensities \( W \) and \( V \), and \( y_i \) are system measurements. \( A_i, G_i, C_i \) are constant system matrices \((i = 1, 2, 3, 4)\). The well known optimal Kalman filter is given by

\[
\hat{x}(k+1) = A\hat{x}(k) + K(y(k) - C\hat{x}(k))
\]

or in the closed-loop form as

\[
\hat{x}(k+1) = (A - KC)\hat{x}(k) + Ky(k)
\]

where

\[
A = \begin{bmatrix}
  A_1 & \varepsilon A_2 \\
  \varepsilon A_3 & A_4
\end{bmatrix},
\quad
C = \begin{bmatrix}
  C_1 & \varepsilon C_2 \\
  \varepsilon C_3 & C_4
\end{bmatrix},
\quad
K = \begin{bmatrix}
  K_1 & \varepsilon K_2 \\
  \varepsilon K_3 & K_4
\end{bmatrix}
\]

The Kalman gain is given by

\[
K = APfC^T(V + CPfC^T)^{-1},
\]

where \( P_f \) is the positive-semidefinite stabilizing solution of the discrete-time algebraic Riccati equation given by

\[
P_f = AP_f A^T - AP_f C^T(V + CP_f C^T)^{-1} CP_f A^T + GWG^T
\]

with

\[
G = \begin{bmatrix}
  G_1 & \varepsilon \epsilon G_2 \\
  \epsilon G_3 & G_4
\end{bmatrix},
\quad
W = \begin{bmatrix}
  W_1 & 0 \\
  0 & W_2
\end{bmatrix}
\]

Using the decomposition procedure given in the previous section and the duality property between the optimal regulator and optimal filter, will result in the decomposition of the global filter to the completely decoupled reduced order subsystem filters both driven by system measurements.

By duality between the optimal filter and regulator, the filter Riccati equation (31) can be solved by using the same decomposition method presented in the previous section with

\[
A \rightarrow A^T, Q \rightarrow GWG^T, B \rightarrow C^T,
\quad
BR^{-1}B^T \rightarrow C^T V^{-1} C
\]

which leads to the Hamiltonian state-costate filter closed-loop form

\[
\begin{bmatrix}
  x_1(k+1) \\
  \lambda_1(k+1) \\
  x_2(k+1) \\
  \lambda_2(k+1)
\end{bmatrix} =
\begin{bmatrix}
  \bar{A}_1f & \bar{S}_1f & \varepsilon \bar{A}_2f & \varepsilon \bar{S}_2f \\
  \bar{Q}_1f & \bar{A}_1f & \varepsilon \bar{Q}_2f & \varepsilon \bar{A}_2f \\
  \varepsilon \bar{Q}_3f & \varepsilon \bar{A}_2f & \bar{Q}_4f & \bar{A}_2f
\end{bmatrix} \begin{bmatrix}
  x_1(k) \\
  \lambda_1(k) \\
  x_2(k) \\
  \lambda_2(k)
\end{bmatrix}
\]

Partitioning the state vector \( x \), the corresponding costate vector \( \lambda \) and interchanging second and third rows, the Hamiltonian form can be written as

\[
\begin{bmatrix}
  U(k+1) \\
  V(k+1)
\end{bmatrix} =
\begin{bmatrix}
  T_{1f} & \epsilon T_{2f} & U(k) \\
  \epsilon T_{3f} & T_{4f} & V(k)
\end{bmatrix}
\]

As it was shown in the previous section, this system can be diagonalized by the means of the similarity transformation given by

\[
\begin{bmatrix}
  U(k) \\
  V(k)
\end{bmatrix} =
\begin{bmatrix}
  I - \epsilon^2 H_f L_f & -\epsilon H_f \\
  \epsilon L_f & I
\end{bmatrix} \begin{bmatrix}
  \eta(k) \\
  \xi(k)
\end{bmatrix} = T_f \begin{bmatrix}
  \eta(k) \\
  \xi(k)
\end{bmatrix}
\]

and \( L_f \) and \( H_f \) satisfy
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\[ LfT_f - T_4fL_f + T_3f - \varepsilon^2 L_fT_2fL_f = 0 \]

\[ Hf \left( T_4f + \varepsilon^2 L_fT_2f \right) - 
\left( T_1f - \varepsilon^2 T_2fL_f \right)H_f - T_2f = 0 \]

The transformation leads to two decoupled sub systems

\[ \begin{bmatrix} \eta_1(k+1) \\ \eta_2(k+1) \end{bmatrix} = \begin{bmatrix} a_1f & a_2f \\ a_3f & a_4f \end{bmatrix} \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix} \]

\[ \begin{bmatrix} \xi_1(k+1) \\ \xi_2(k+1) \end{bmatrix} = \begin{bmatrix} b_1f & b_2f \\ b_3f & b_4f \end{bmatrix} \begin{bmatrix} \xi_1(k) \\ \xi_2(k) \end{bmatrix} \]

where

\[ \eta_2(k) = Pfa\eta_1(k), \xi_2(k) = Pb\xi_1(k) \]

or

\[ \begin{bmatrix} \eta_2(k) \\ \xi_2(k) \end{bmatrix} = \begin{bmatrix} Pfa & 0 \\ 0 & Pb \end{bmatrix} \begin{bmatrix} \eta_1(k) \\ \xi_1(k) \end{bmatrix} \]

and \( Pfa \) and \( Pb \) satisfy nonsymmetric Riccati equations of the form

\[ Pfaa_1f - a_4fPfa - a_3f + Pfaa_2fPfa = 0 \]

\[ Pbfb_1f - b_4fPfb - b_3f + Pbfb_2fPfb = 0 \]

leading to

\[ \eta_1(k+1) = (a_1f + a_2fPfa)\eta_1(k) \]

\[ \xi_1(k+1) = (b_1f + b_2fPfb)\xi_1(k) \]

The overall transformation between the new and original coordinates is given by

\[ \begin{bmatrix} \eta_1 \\ \xi_1 \\ \eta_2 \\ \xi_2 \end{bmatrix} = E^T T_f E \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} = \Pi_f \]

Since \( \lambda = P_f x \), where \( P_f \) satisfies the discrete algebraic Riccati equations (31), it follows

\[ \begin{bmatrix} \eta_1(k+1) \\ \xi_1(k+1) \end{bmatrix} = (\Pi_1f + \Pi_2fP_f) \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \Omega \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \]

\[ \begin{bmatrix} \eta_2(k+1) \\ \xi_2(k+1) \end{bmatrix} = (\Pi_3fP_f + \Pi_4f) \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \]

It follows from (43) and (39)

\[ \begin{bmatrix} Pfa & 0 \\ 0 & Pb \end{bmatrix} = (\Pi_3f + \Pi_4fP_f)(\Pi_1f + \Pi_2fP_f)^{-1} \]

This equation can be solved for \( P_f \) giving

\[ P_f = \begin{bmatrix} Pfa & 0 \\ 0 & Pb \end{bmatrix} \begin{bmatrix} \Pi_2f - \Pi_4f \\ \Pi_3f - \Pi_1f \end{bmatrix} \]

which gives the solution of the filter global discrete Riccati equation (31).

Applying the transformation (43) to the Kalman filter equation (28) leads to

\[ \begin{bmatrix} \hat{\eta}_1(k+1) \\ \hat{\xi}_1(k+1) \end{bmatrix} = \Omega^{-T}(A - KC)\Omega^T \begin{bmatrix} \eta_1(k) \\ \xi_1(k) \end{bmatrix} + \Omega^{-T}K_y(k) \]

or

\[ \begin{bmatrix} \hat{\eta}_1(k+1) \\ \hat{\xi}_1(k+1) \end{bmatrix} = (a_1f + a_2fPfa)^T \eta_1(k) + K_1y(k) \]

\[ \begin{bmatrix} \hat{\xi}_1(k+1) \\ \hat{\xi}_1(k+1) \end{bmatrix} = (b_1f + b_2fPfb)^T \xi_1(k) + K_2y(k) \]
which completely decomposes the global Kalman filter into two reduced order subfilters, that can be implemented independently. Again, as it was the case in the previous section, in order to realize the above presented decomposition procedure it is necessary to solve continues-time nonsquare and non-symmetric Riccati equations (37) and (40).

4 LQG CONTROL PROBLEM

The well known linear quadratic Gaussian control problem is defined as follows. Given the linear discrete-time stochastic system

\[
\begin{bmatrix}
    x_1(k+1) \\
    x_2(k+1)
\end{bmatrix} = \begin{bmatrix}
    A_1 & \epsilon A_2 \\
    \epsilon A_3 & A_4
\end{bmatrix} \begin{bmatrix}
    x_1(k) \\
    x_2(k)
\end{bmatrix} + \begin{bmatrix}
    B_1 & \epsilon B_2 \\
    \epsilon B_3 & B_4
\end{bmatrix} \begin{bmatrix}
    u_1(k) \\
    u_2(k)
\end{bmatrix} + \begin{bmatrix}
    G_1 & \epsilon G_2 \\
    \epsilon G_3 & G_4
\end{bmatrix} \begin{bmatrix}
    w_1(k) \\
    w_2(k)
\end{bmatrix} \begin{bmatrix}
    y_1(k) \\
    y_2(k)
\end{bmatrix} = \begin{bmatrix}
    C_1 & \epsilon C_2 \\
    \epsilon C_3 & C_4
\end{bmatrix} \begin{bmatrix}
    x_1(k) \\
    x_2(k)
\end{bmatrix} + \begin{bmatrix}
    v_1(k) \\
    v_2(k)
\end{bmatrix}
\]  

(48)

with performance criterion

\[
J = \frac{1}{2} \mathbb{E} \left\{ \sum_{k=0}^{\infty} x(k)^T Q x(k) + u(k)^T R u(k) \right\}
\]  

(49)

Find the control law which minimizes the criterion. The optimal control law is given by [8]

\[
u(k) = -F \hat{x}(k)
\]  

(50)

where \( F \) is found according to the section II with the optimal filter

\[
\hat{x}(k+1) = (A - KC)\hat{x}(k) + Ky(k) + Bu(k)
\]  

(51)

which is decomposed into reduced order filters according to the section III as

\[
\hat{\eta}(k+1) = (a_1 + a_2 \varphi_f) \hat{\eta}(k) + K_1 y(k) + \varphi_i u(k)
\]

(52)

\[
\hat{\xi}_1(k+1) = (b_1 + a_2 \varphi_f) \hat{\eta}(k) + K_2 y(k) + \varphi_i u(k)
\]

(53)

5 THE EIGENVECTOR SOLUTION TO NONSYMMETRIC ALGEBRAIC RICCATI EQUATION

The eigenvector method for solving the algebraic symmetric and square, nonsymmetric and non-square Riccati equations has received considerable attention in the literature [8, 9]. Without loss of generality, let us consider the algebraic square and nonsymmetric Riccati equation (ARE) given by

\[
AX + XB + C + XDX = 0
\]  

(54)

where matrices \( A, B, C, D \) are of appropriate dimensions \((n \times n)\) and \( X \) is the sought solution of dimension \((n \times n)\).

Let the matrix \( R \) be associated with the ARE

\[
R = \begin{bmatrix}
B & D \\
-C & -A
\end{bmatrix}
\]  

(55)

The matrix \( R \) can be diagonalized by the matrix \( M \) consisting of eigenvectors of the matrix \( R \) as follows. Calculate all \( 2n \) eigenvalues of \( R \), \( \lambda_i = a_i + jb_i \) and all corresponding eigenvectors \( v_i = x_i + jy_i \). Arrange in the \((2n \times 2n)\) matrix \( M \) all real eigenvectors \( (x_i) \) and for each complex-conjugate pair use consecutively the real and imaginary parts of one eigenvector only \((x_i, y_i)\). There are many ways to form matrix \( M \).

Then, it follows that

\[
M^{-1}RM = \Lambda, RM = MA = \begin{bmatrix}
M_1 & M_2 \\
0 & A_2
\end{bmatrix}
\]  

(56)

where \( M_1 \) contains the first \( n \) columns and \( M_2 \) contains the remaining \( n \) columns of \( M \). \( A_1 \) and \( A_2 \) are diagonal or block diagonal matrices.

The equation (56) may be rewritten as

\[
RM_1 = M_1 A_1, RM_2 = M_2 A_2
\]  

(57)

By partitioning \( M_1 \) as

\[
M_1 = \begin{bmatrix}
M_{11} \\
M_{21}
\end{bmatrix}
\]  

(58)
we get from (57)
\[ BM_{11} + DM_{21} = M_{11}A, \]
\[ -CM_{11} - AM_{21} = M_{21}A \] (59)

Rearranging the last two equations and using the substitution
\[ X = M_{21}M_{11}^{-1} \] (60)
leads to
\[ AX + XB + C + XDX = 0 \] (61)
which proves that \( X \) is a solution to (54). Since the matrix \( M \) can be formed in many ways it follows that all solutions to (54) have the form
\[ X_k = M_{k21}M_{k22}^{-1} \] (62)

Let the spectrum of \( R \) be \( S = \{\lambda_1, \ldots, \lambda_{2n}\} \) or \( S = S_1US_2 \), where \( S_1 = \{\lambda_1, \ldots, \lambda_n\} \) and \( S_2 = \{\lambda_{n+1}, \ldots, \lambda_{2n}\} \). If corresponding eigenvalues of eigenvectors used to form \( M_1 \) are \( S_1 = \{\lambda_1, \ldots, \lambda_n\} \) and to form \( M_2 \) are \( S_2 = \{\lambda_{n+1}, \ldots, \lambda_{2n}\} \), then eigenvalues of \( (B + DX) \) are \( S_1 \) and eigenvalues of \( -(A + DX) \) are \( S_2 \) [9]. This is easily justified by transforming the matrix \( R \) as follows
\[
\begin{bmatrix}
I & 0 \\
-X & I
\end{bmatrix}
\begin{bmatrix}
B & D \\
-C & -A
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
X & 0
\end{bmatrix}
= 
\begin{bmatrix}
B + DX & D \\
0 & -(A + XD)
\end{bmatrix}
\] (63)

Further, the matrix \( R \) can be put in the block diagonal form by using another transformation matrix
\[
\begin{bmatrix}
I & -Y \\
0 & I
\end{bmatrix}
\begin{bmatrix}
B + DX & D \\
0 & -(A + XD)
\end{bmatrix}
\begin{bmatrix}
I & Y \\
0 & I
\end{bmatrix}
= 
\begin{bmatrix}
B + DX & 0 \\
0 & -(A + XD)
\end{bmatrix}
\] (64)
where \( Y \) satisfies the Sylvester equation
\[ (B + DX)Y + Y(A + XD) + D = 0 \] (65)

6 EXAMPLE

Consider the system with problem matrices given by (\( \varepsilon = 1 \))

\[
Q = 0.1I_4, R = I_2, W = I_2, V = I_2
\]
The obtained solutions for LQG problem according to the presented methodology (note that iterative methods in this case do not converge) are summarized as follows

\[
P_r = \begin{bmatrix}
391.0855 & -50.8930 & 96.3664 & 49.4185 \\
-50.8930 & 8.3975 & -14.4404 & -7.3477 \\
96.3664 & -14.4404 & 26.0110 & 13.2175 \\
\end{bmatrix}
\]

\[
P_f = \begin{bmatrix}
1.7500 & -0.9361 & 0.7142 & -0.4013 \\
-0.9361 & 0.6377 & -0.4928 & 0.2313 \\
0.7142 & -0.4928 & 0.4316 & -0.2111 \\
-0.4013 & 0.2313 & -0.2111 & 0.1718
\end{bmatrix}
\]

\[
\hat{y}(k+1) = \begin{bmatrix}
-0.1339 & -1.0531 \\
-0.6604 & 0.1458
\end{bmatrix} \hat{y}(k) + \\
\begin{bmatrix}
0.2742 & 0.3128 \\
-0.4997 & -0.1248
\end{bmatrix} y(k) + \begin{bmatrix}
0.0799 & -0.0984 \\
0.1798 & 0.0495
\end{bmatrix} u(k)
\]

\[
\hat{\xi}(k+1) = \begin{bmatrix}
0.1849 & -0.1127 \\
0.0289 & 0.2919
\end{bmatrix} \hat{\xi}(k) + \\
\begin{bmatrix}
0.0066 & -0.0114 \\
-0.0053 & 0.0185
\end{bmatrix} y(k) + \begin{bmatrix}
0.1479 & -0.0104 \\
-0.1987 & 0.1899
\end{bmatrix} u(k)
\]

\[
u(k) = \begin{bmatrix}
-5.7198 & 5.5900 \\
6.2312 & -6.6806
\end{bmatrix} \hat{y}(k) + \\
\begin{bmatrix}
-11.0231 & -5.7617 \\
12.5405 & 6.5756
\end{bmatrix} \hat{\xi}(k)
\]

7 CONCLUSION

In this paper the algebraic Riccati equation decomposition and eigenvector method have been
used in order to solve the optimal control and filtering of the discrete linear weakly coupled stochastic system. This approach can be used in case of higher level of coupling between the subsystems. Beside providing reduction and parallelism in on-line computation of control and filtering tasks, it gives new insights into the optimal control and filtering of weakly coupled systems.

REFERENCES


Primjena svojstvenih vektora pri optimalnom upravljanju i filtriranju slabo spregnutih linearnih stohastičkih sustava. U članku je opisan postupak rješavanja regulatorskih i filtrirajućih Riccatijevih jednadžbi koje se dobiju prilikom definiranja ravnovežnog rješenja problema optimalnog upravljanja i filtriranja slabo spregnutih linearnih diskretnih stohastičkih sustava. Postupak je zasnovan na primjeni svojstvenih vektora u rješavanju podproblema nižeg reda. Takav postupak pokazuje bolje značajke u odnosu na iterativne postupke (iteracije u fiksnoj aritmetici, Newtonovi postupci) u rješavanju podproblema nižeg reda u slučaju kada su podsustavijače spregnuti.

Ključne riječi: optimalno upravljanje i filtriranje, slabo spregnuti sustav, blokovsko dijagonaliziranje, raspravljanje

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