A single-amplifier, active-RC filters design procedure for common filter types, such as Butterworth and Chebyshev, using tables with normalized filter component values, is presented. The considered filters consist of RC ladder network in operational amplifier's positive feedback path. Tables with normalized component values having equal capacitors and equal resistors were presented by some authors [1, 2]. In this paper, we presented new tables for designing filters with optimized sensitivity to passive circuit components. A considerable improvement in sensitivity is achieved using the design technique called »impedance tapering«. The presented filters are up to the 6th-order. The sensitivity problem generally limits the use of higher than 6th-order single-amplifier filters. Low-pass and high-pass filters design is presented.

**Key words:** active-RC filters, minimal-RC filters, low power, low sensitivity, normalized components

1 INTRODUCTION

Passive filters are usually designed using handbooks with filter tables [3–5]. They are typically applied for the design of passive ladder-RLC filters. Besides filter transfer function parameters for many common filter types, handbooks usually contain tables with normalized filter component values, as well [3, 4].

Active filters are commonly designed using closed-form design equations and/or filter-design programs [6, 7]. In a majority of applications they are realized as cascade structures of second-order (»biquads«) and/or third-order (»bitriplets«) sections filters, and explicit formulas for calculation of filter components, as well as, step by step design procedures are available.

However, the high-order, single-amplifier, non-cascade filter designs have very often complicated design procedures and since there exists no explicit formulas, numerical calculations have to be performed. Those filters are referred to as canonical or minimal-RC because they have minimum number of components. In this paper we present filter tables with components for designing single-amplifier, active-RC filters up to the 6th-order, for Butterworth and Chebyshev approximation types. Based on the preliminary results from previous paper [8], extended (final) results including practical design examples as guidance for engineers are presented. The need for design tables with normalized filter component values in the form of a handbook is approved. Compared to the filter design tables for some common active-RC filters given in [1, 2], the tables in this paper are produced using an optimization procedure for the low sensitivity to passive component tolerances.

In active-RC filter design, the most important parameters for evaluation of a filter’s section quality are: simple realizability, repeatability, a possibility of straightforward procedure of parameter calculation, small number of components, low power consumption, low noise performance and the most often low filter’s magnitude sensitivities to passive component tolerances and/or active gain variations. The filters, designed in this paper, possess minimum passive sensitivities for a given topology. The sensitivity is reduced using the design technique, called »impedance tapering« [9]. All calculations are performed numerically, including the calculation of filters components, as well as, optimizations for filters low sensitivity. Schoeffler’s sensitivity measure is used as a basis for comparison of sensitivity to component tolerances of the various filters. Monte Carlo runs are performed as a double check to the Butterworth and Chebyshev filter examples.
2 SINGLE-AMPLIFIER, MINIMAL-RC, ACTIVE-RC FILTER

Consider an $n$th-order, allpole, single-amplifier, low-pass filter circuit with positive feedback presented in Fig. 1. The filter belongs to the type of class-4 or Sallen and Key [2, 10]. The voltage transfer function $T(s) = \frac{N(s)}{D(s)}$ of the filter in Fig. 1 is given by

$$T(s) = \frac{N(s)}{D(s)} = \frac{Ka_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_is^i + \ldots + a_1s + a_0}$$

(1)

where the pass-band gain $K$ is given by

$$K = \beta = 1 + \frac{R_F}{R_G}$$

(2)

The coefficients $a_i$ ($i = 0, \ldots, n-1$) as functions of components $R_k$ and $C_k$ ($k = 1, \ldots, n$) are presented in Appendix I for the canonical filters up to the 6th-order. There are many different ways how those coefficients can be calculated. In this paper, in Appendix I, we present one recursive way which is easily programmed in any program that can, besides numeric, perform symbolic calculations (e.g. Mathematica, Matlab or MathCad).

Knowing coefficients $a_i$ in the transfer function's (1) denominator polynomial, we can develop a set of nonlinear equations by equating each of the coefficients in the polynomial to the coefficient values of the appropriate Butterworth or Chebyshev polynomials. Butterworth and Chebyshev polynomials are readily obtained from tables [3, 6] or can be calculated using closed-form equations. We can, finally, numerically solve a set of nonlinear equations and calculate passive (normalized) components $R_k$, $C_k$ ($k = 1, \ldots, n$) and $\beta$ to build the filter circuit in Fig. 1.

The results of such computer calculation for equal capacitors and equal resistors cases are presented in [1] (and repeated in [2] p. 252). Tables presented there, are calculated for some typical amplifier gains ($\beta = 2.0$ and $2.2$) and are not optimized for sensitivity.

2.1 Optimization of Sensitivity

In this paper we calculate component values of the filters that are optimized for minimum sensitivity to component tolerances. Component values are obtained numerically and shown in Tables 1–2. The low sensitivity performance is achieved by impedance taperings design method (more precisely by capacitive tapering), first introduced in [9], and applied to the LP filter structure in Fig. 1. At the same time a value of the so-called «design frequency» $\omega_0$ is optimized iteratively to converge to the value which provides minimum sensitivity filter. Therefore, in Tables 1–2 we obtain various optimal values for $R_1$ and the corresponding values of the gain $\beta$. (Suppose we have chosen $C_1 = 1$, thus for optimal $\omega_0$, an optimal value of $R_1 = (\omega_0 C_1)^{-1}$ readily follows.) Optimal values for $R_1$ are calculated using the procedure shown in Fig. 2, which is implemented with program Mathematica [11].

It should be stressed that the capacitive-tapering factor $\rho_C$ for higher-order filters is smaller, than that for filters of lower-order $n$. For example, if we try to find solution of the 6th-order Butterworth filter with larger $\rho_C$ (for example $\rho_C \geq 3$), it is not possible, because the Newton's method does not converge. Furthermore, larger capacitive tapering factor $\rho_C$ is not permitted since the last capacitor $C_n$ (e.g. $C_0 = C_1 / \rho_C^5$) becomes too small and comparable to the parasitic capacitance of the circuit. Luckily, in high-order filters, even the small tapering factor satisfies enough our needs in degree of desensitization (e.g. $\rho_C = 2.0$ is good enough for the filter order $n = 6$).

In the sequel, we briefly explain the block diagram shown in Fig. 2. The procedure in Fig. 2 uses Newton’s iterative method to calculate the components of the high-order (i.e. 4th-, 5th- and 6th-order) allpole low-pass filters. In addition, the filters are capacitively tapered and numerically optimized for low sensitivity. Optimization of 2nd- and 3rd-order filters follows the same steps in the block diagram shown in Fig. 2, but theirs procedures are simpler in that the calculation of filter components can be performed analytically.

The input data are the values of Butterworth or Chebyshev coefficients $a_i$ ($i = 0, \ldots, n-1$), chosen values of $\rho_C$, and $C_1$. Because of capacitive tapering, the procedure calculates $C_k = C_1 / \rho_C^{k-1}$ ($k = 2, \ldots$,
In the first cycle the initial value of the resistor $R_1$ has to be defined. The resistor $R_1$ is the design parameter to be adjusted. We choose $C_1=1$, then by varying the value of $R_1$ we indeed vary the value of the design frequency $\omega_0=(R_1C_1)^{-1}$. With the value of $R_1$ we solve the system of non-linear equations for the vector $R_2,\ldots,R_n$ and $\beta$. To achieve a solution, we start with vector of random values for $R_2,\ldots,R_n$ and $\beta_0$. Random initial resistors’ values $R_k^0$ are within the interval $<0.2, 20>$ and gain $\beta$ has a value in the range $<1.0, 5.0>$. If the proper starting vector is chosen, the Newton’s method will, with prescribed accuracy, converge in several steps to the solution, i.e. to the vector $R_2,\ldots,R_n$ and $\beta$. If the method fails to converge, we must try another random starting vector. If the convergence is achieved but we have solution with negative resistor values or gain $\beta$ less than unity, then we, again, choose another random starting vector. We perform random starting vectors for maximum 1000 times. This process of finding solution is known as random search. If we choose starting vectors for Newton’s iterative solving method by applying some rule, which tries to find all possible solutions, we have exhaustive search (brute force algorithm). In solving some problems, if there exist several local minimums, exhaustive search tries to find global minimum, but only for a particular type of problems.

If we do not find all real and positive component values $R_2,\ldots,R_n$ and gain $\beta \geq 1$, we proceed with another value of resistor $R_1$. Finally, when we find realizable elements we proceed to the sensitivity optimization. To accomplish it we calculate multi-parametrical statistical measure $M$, which is defined in [12], with

$$M = \omega_2 S_2(\omega) d\omega$$

where $S_2(\omega)$ is Schoeffler’s sensitivity function defined by

$$S_2(\omega) d\omega = \sum_{i=1}^{N} \left(S_{F_i}^F \right)^2$$

Function $S_{F_i}^F$ in (4) represents relative sensitivity of the function $F$ (it is the transfer function magnitude, i.e. $F=|T(j\omega)|$) to the parameter $x_i$ (there are $N=2n+2$ passive filter components $x_i$ in an $n$th-order filter). $S_2(\omega)$ in (3) is a function of frequency, while $M$ is a number instead of a function, and can be used as a goal of optimizing process. $M$ represents area under the function $S_2(\omega)$, with borders of integration from $\omega_1$ to $\omega_2$. Disadvantage is the dependence of number $M$ on the selected borders of integration $\omega_1$ and $\omega_2$. Therefore, during whole optimization process, we choose the same pair of $\omega_1$ and $\omega_2$. The whole above procedure is repeated with new value for $R_1$, until minimum value of $M$ is found.

For the 2nd-order filters we do not perform numerical optimization. Instead we apply capacitive tapering factors $\rho_C$ in Tables 1–2 that must obey the following constraint

$$\rho_C \leq \rho_{\text{maks}} = \frac{q}{p} \left( \frac{R_2}{R_1} \right)^2$$

The value $\rho_{\text{maks}}$ in (5) represents an upper bound of resistive tapering factor $\rho_C$. For equal resistors
optimization should be carried out.

minimum sensitivity design; instead the numerical
than-third order filters no such rules exist for the
minimum sensitivity derived in the 3rd-order filter we have the values of
of 2nd-order filters (having unity gain, i.e.
that for the 3rd-order low-pass circuits with
clusions for the 3rd-order low-pass circuits with
standard deviation
is related to the Schoeffler's sensitivities) of the
values for Butterworth filters given in Table 1, as
elements, is calculated for the optimized element
variation of the logarithmic gain
is always preferable to a Chebyshev filter with
Chebyshev filter and a low-ripple Chebyshev filter
a Butterworth filter is always preferable to a
filter approximations lead to the same conclusions,
but they are not presented here.
filter approximations lead to the same conclusions,
are repeated in Fig. 4, sorted by different designs.
(5) given by
\[
\rho_C \leq \rho_{\text{max}} \bigg|_{R_1=R_2} = 4q_p^2
\]  
(6)
If we use maximum desensitization with \(\rho_C=4q_p^2\)
in (6) we obtain in Tables 1–2 »low-Q« realization of 2nd-order filters (having unity gain, i.e. \(\beta=1\)) [6]. Furthermore, it is apparent from tables that for the 3rd-order filter we have the values of \(R_2\) and \(R_3\) close to each other. This corresponds to the conclusions for the 3rd-order low-pass circuits with minimum sensitivity derived in [9]. For higher-than-third order filters no such rules exist for the minimum sensitivity design; instead the numerical optimization should be carried out.

### 2.2 Sensitivity Analysis

In this section we compare sensitivities of newly designed capacitively-tapered filters to the filters having equal capacitors and equal resistors.

A sensitivity analysis was performed to the logarithmic gain function \(a(\omega)=20\log|T(\omega)|/\text{dB}\), assuming the relative changes of the resistors and capacitors to be uncorrelated random variables, with a zero-mean Gaussian distribution and 1% standard deviation. The standard deviation \(\sigma_a(\omega)/\text{dB}\) (which is related to the Schoeffler’s sensitivities) of the variation of the logarithmic gain \(\Delta a=8.68588\) \(\Delta |T(\omega)|/|T(\omega)|/\text{dB}\), with respect to the passive elements, is calculated for the optimized element values for Butterworth filters given in Table 1, as well as for the equal capacitors and equal resistors filters designed using tables in [1, 2]. For all those

filters the standard deviations \(\sigma_a(\omega)\) are presented in Fig. 3. Observing Fig. 3 we conclude that the capacitively impedance tapered filters have minimum sensitivities to component tolerances of the circuits for all filter orders. Very close results are achieved with filter circuits, having equal resistors. One possible explanation is in the slightly tapered capacitor values, which can be seen in Table 1. The worst sensitivity performances have filters with equal capacitors. Although it is usually not practical to mass produce discrete component active-RC filters having unequal capacitors, in the case of IC design various (tapered) capacitor values are acceptable. The sensitivity curves in Fig. 3 are repeated in Fig. 4, sorted by different designs.

Same investigations performed for Chebyshev filter approximations lead to the same conclusions, but they are not presented here.

It is well known that Butterworth filter compared to a Chebyshev filter of equal order has lower pole Qs. Since the sensitivities are related to the magnitudes of filter Q factors, from the sensitivity point, a Butterworth filter is always preferable to a Chebyshev filter and a low-ripple Chebyshev filter is always preferable to a Chebyshev filter with higher ripple. Preferable filters with regard to low sensitivity have lowest possible pole Qs. Furthermore, filters with high order \(n\) have both larger pole Qs and larger number of components than low-order filters and therefore much higher sensitivities. The reason for high sensitivities to component variations of high-order canonical filters is obvious from filter coefficients, presented as functions of components, in Appendix I. Qualitatively this can be explained by the fact that in a canonic

| Table 1 Normalized components of min. sensitivity LP filters using capacitive tapering: Butterworth approximation |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(n\) | \(\rho_C\) | \(C_1\) | \(C_2\) | \(C_3\) | \(C_4\) | \(C_5\) | \(C_6\) | \(R_1\) | \(R_2\) | \(R_3\) | \(R_4\) | \(R_5\) | \(R_6\) | \(\beta\) |
| 2 | 2 | 1 | 0.5 | 0.1111 | 0.0370 | 0.0256 | 0.03125 | 1.41421 | 1.41421 | 1.41421 | 1.41421 | 1.41421 | 1.41421 | 1.0 |
| 3 | 3 | 1 | 0.3333 | 0.1111 | 0.0370 | 0.0256 | 0.03125 | 1.09 | 6.01255 | 4.11983 | 8.13008 | 8.13008 | 8.13008 | 1.14231 |
| 4 | 3 | 1 | 0.3333 | 0.1111 | 0.0370 | 0.0256 | 0.03125 | 0.7 | 7.07694 | 18.1004 | 8.13008 | 8.13008 | 8.13008 | 1.34647 |
| 5 | 2.5 | 1 | 0.4 | 0.16 | 0.064 | 0.0256 | 0.03125 | 1.29 | 2.26474 | 8.21287 | 28.67896 | 32.8969 | 32.8969 | 1.5333 |
| 6 | 2 | 1 | 0.5 | 0.25 | 0.125 | 0.0625 | 0.03125 | 0.675 | 8.63542 | 5.19291 | 8.97413 | 20.3341 | 20.3341 | 1.74164 |

| Table 2 Normalized components of min. sensitivity LP filters using capacitive tapering: 0.5 dB Chebyshev approximation |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(n\) | \(\rho_C\) | \(C_1\) | \(C_2\) | \(C_3\) | \(C_4\) | \(C_5\) | \(C_6\) | \(R_1\) | \(R_2\) | \(R_3\) | \(R_4\) | \(R_5\) | \(R_6\) | \(\beta\) |
| 2 | 2.9841 | 1 | 0.3351 | 0.1111 | 0.0370 | 0.0256 | 0.03125 | 1.40289 | 1.40289 | 1.40289 | 1.40289 | 1.40289 | 1.40289 | 1.0 |
| 3 | 3 | 1 | 0.3333 | 0.1111 | 0.0370 | 0.0256 | 0.03125 | 1.71 | 6.5827 | 3.35148 | 7.6595 | 7.6595 | 7.6595 | 1.31082 |
| 4 | 3 | 1 | 0.3333 | 0.1111 | 0.0370 | 0.0256 | 0.03125 | 1.31 | 9.61917 | 19.9327 | 7.6595 | 7.6595 | 7.6595 | 1.4989 |
| 5 | 2.5 | 1 | 0.4 | 0.16 | 0.064 | 0.0256 | 0.03125 | 1.96 | 4.86466 | 11.4959 | 29.8471 | 8.06382 | 8.06382 | 1.66703 |
| 6 | 2 | 1 | 0.5 | 0.25 | 0.125 | 0.0625 | 0.03125 | 1.8 | 14.2659 | 9.77697 | 11.7128 | 23.8853 | 23.8853 | 1.84611 |
realization all the components interact with each other, and a change in any component will be magnified through its repeated occurrence in all coefficients. The larger order \( n \) of the filter, the more complicated and larger are the filter coefficients and therefore the larger sensitivities. In cascade realization, in contrast, the change in a component will affect only a localized segment, i.e. only one biquad in a cascade. Therefore, the cascade design has lower sensitivities. But it has more amplifiers (consumes more power) and passive components (resistors) than single-amplifier filters.

As stated above, in the design of active-RC filters, very important quality of the filter is its low power consumption. Thus, for low-pass filters of reasonably low order (for example \( n \leq 6 \)), the use of single-amplifier filters, as in Fig. 1 is advantageous over the cascaded 2\(^{nd}\)- and 3\(^{rd}\)-order Sallen and Key sections [10], although the latter have smaller sensitivities to component tolerances of the circuit. Another advantage of canonical, single-amplifier filters, which are presented here, is that they have only \( n \) capacitors and \( n \) resistors as well, and their sensitivities are improved applying »impedance tapering« technique. Finally, advantages of single-amplifier filters lie also in the fact that there is no need for signal level optimization procedure which must be performed in multiple amplifier configurations.

![Fig. 3 Schoeffler's sensitivity of normalized Butterworth low-pass filter circuits (up to 6\(^{th}\)-order), with components given in Table 1 and in tables presented in [1] and [2] p. 252](image1)

![Fig. 4 Schoeffler's sensitivity of normalized Butterworth LP filter circuits in Fig. 3, sorted by type of impedance tapering](image2)
3 TRANSFORMATIONS

3.1 LP-LP Transformation

Low-pass to low-pass (LP-LP) frequency transformation (i.e. denormalization) defined by

\[ s \rightarrow \frac{s}{\omega_0} \]  

(7)

can be performed by new elements calculations. From normalized element values in Tables 1–2 we calculate denormalized element values by applying

\[ R_{LP} = R_0 \cdot R_{nLP}, \quad C_{LP} = \frac{C_{nLP}}{\omega_0 R_0} \]  

(8)

where index \( n \) denotes normalized components values in tables. It is an advantage of normalized component values in that they can be used as a template to directly construct the filter with any cut-off frequency. We do not consider inductances because we deal with active-RC filters. Denormalization constants \( \omega_0 \) is in \([\text{rad/s}]\) and \( R_0 \) in \(\Omega\).

3.2 LP-HP Transformation

Low-pass to high-pass (LP-HP) frequency transformation defined by

\[ s \rightarrow \frac{\omega_0}{s} \]  

(9)

can be performed by new elements calculations. From normalized LP filter components in Tables 1–2, HP filter components can be readily calculated using \((RC-CR)\) transformation

\[ R_{HP} = \frac{R_0}{C_{nLP}}, \quad C_{HP} = \left( R_{nLP} \omega_0 R_0 \right)^{-1} \]  

(10)

Note, that while applying LP-HP transformation (9) [i.e. while calculating HP filter components using (10)]; capacitive tapering by a factor \( \rho_C \) at LP filters, transforms itself into resistive tapering factor at HP filters. The value of gain \( \beta \) remains the same as in the LP prototypes.

4 DESIGN OF LOWPASS FILTERS

Suppose we build mixed-signal-processing circuitry on the chip for the hearing aids. It consists of continuous-time to discrete-time converter (C/D) which prepares the signal for the digital signal processing circuitry. Before C/D converter we have to prepare voice signal using an integrated low-pass filter to suppress high frequency components and eliminate aliasing. On the other hand the low-pass filter has to be as simple as possible (therefore we realize it using an active-RC filter), must have low power consumption (it has a single amplifier), and must be selective (it is a high order filter).

Because of relatively high filter order it must have acceptably small sensitivity to component tolerance to be realizable. For those reasons we decided to use the \( n^{\text{th}} \)-order allpole low-pass filter circuit presented in Fig. 1. Because, this filter is rather complicated to calculate (especially for the orders higher than 3rd, where there are no closed form equations) the Tables 1–2 with component values shows to be of enormous help. In the design process we will first decide which type of filter (Butterworth or Chebyshev) best suites our needs.
The filter order $n$ for the Butterworth filter can be readily calculated

\[ n \geq \frac{1}{2} \log \left( \frac{A_p}{10^{10} - 1} \right) \log \left( \frac{f_s}{f_p} \right) \]  

where $A_p = 2 \pi f_p$. The cut-off frequency $\omega_0$ is given by

\[ \omega_0 = \frac{\omega_p}{2^{\frac{1}{n}} A_p 10^{10} - 1} \]  

where the radian frequency $\omega_p = 2 \pi f_p$. Note the similarity between (16) and the Butterworth formula (12). The Table 1 provides a set of necessary component values for the designing required Butterworth filter.

Chebyshev transfer function magnitude has the following form

\[ A_{\text{Loss}}(\omega) = 20 \log \left( 1 + \left( \frac{\omega}{\omega_0} \right)^{2n} \right) \]  

where $0 < \varepsilon \leq 1$ describes the pass-band ripple and $T_n(\omega)$ an $n^{th}$ order Chebyshev polynomial. The constant $\varepsilon$ readily follows from minimum pass-band attenuation (i.e. pass-band ripple) $A_p$ given in dB

\[ \varepsilon = \sqrt{e^{10^{-1}} - 1} \]  

The filter order $n$ for the Chebyshev filter can be calculated using

\[ n \geq \frac{1}{2} \log \frac{A_p}{10^{10} - 1} \log \frac{f_s}{f_p} \]  

where $\omega_0 = \omega_p$.

If the cut-off frequency in the Chebyshev filters design is required to be the frequency at which the magnitude reaches $-3\text{dB}$ value (not the pass-band ripple value $A_p$) then we have to calculate the value of $\omega_0$ in (14) another way using

\[ \omega_0 = \frac{\omega_p}{\omega_{-3\text{dB}}} \]  

where

\[ \omega_{-3\text{dB}} = \text{Cosh} \left( \frac{1}{n} \right) \text{Arch} \left( \frac{1}{\varepsilon} \right) \]  

The $\varepsilon$ value is given by (15). In this case we also use normalized component values of the LP low-sensitivity Chebyshev filter in Table 2. This case will not be considered in this paper, instead we will design Chebyshev filters having cut-off frequency at the pass-band ripple value $A_p$ /dB.

4.1 Example

Using equations (11) to (19) we can readily calculate the filter order $n$ and the cut-off frequency $\omega_0$ for the design of the Butterworth or Chebyshev filters. Indeed those calculations are the only that we need. The filter design then proceeds using tables and is straightforward.

As descriptive examples we present the designs of filters that satisfy our specifications in Fig. 5: i) Butterworth, ii) Chebyshev with pass-band ripple $R_p=0.2$ dB and iii) Chebyshev with $R_p=0.5$ dB.
Using (12) and (16) we calculate required filter order $n$. Thus we have for the three possible realizations the following filter orders and cut-off frequencies: i) Butterworth $n = 5$, $\omega_0 = 155084$ rad/s; ii) Chebyshev $n = 4$, $\omega_0 = 125664$ rad/s; and iii) Chebyshev $n = 3$, $\omega_0 = 125664$ rad/s. Note that the order $n$ of the Chebyshev filters is smaller than the order of the Butterworth filter. Also note that the Chebyshev filter with higher ripple requires lower filter order. Consequently, in what follows we realize the Butterworth filter and the Chebyshev filter with 0.5 dB ripple.

Using normalized predefined component values from above design tables, we proceed with the denormalization procedure. To calculate filter components we need to choose denormalizing resistor $R_0$ value and the denormalization frequency $\omega_0$. Generally speaking $\omega_0$ and $R_0$ are free constants. But, in our case, the frequency $\omega_0$ is already known and follow from filter specifications (e.g. it is $\omega_0 = 155084$ rad/s for the Butterworth filter example). The resistor value $R_0$ could be chosen to obtain, for example, desired total capacitance value $C_{TOT}$. The latter constraint arises because we intend to realize the filter circuit on the chip, thus we are limited with maximum allowable capacitance still realizable on the chip.

Suppose that we want to obtain a total capacitance $C_{TOT} = 300$ pF, we have to calculate denormalization resistance $R_0$. For the filters with tapered capacitors by a factor $\rho_C$ of the order $n$ the total capacitance equals

$$C_{TOT} = \sum_{i=1}^{n} C_i = C_1 \cdot \sum_{i=0}^{n-1} \rho_C^{-i} = C_1 \cdot \frac{\rho_C^n - 1}{\rho_C - 1} \quad (20)$$

For the 5th-order Butterworth filter from Table 1 and using (20) we obtain normalized value $C_{TOT} = C_1 = 1.6496$ and therefore maximum value for capacitor $C_1$ should be $C_1 = 300$ pF/1.6496 = 181,862 pF. Starting from this we calculate minimum denormalization resistance $R_0$

$$R_{0\min} = \frac{1}{\omega_0 C_1} = \frac{1}{155084 \cdot 181.86 \text{ pF}} = 35.4561 \text{ k}\Omega \quad (21)$$

We can choose for example reference resistor $R_0 = 36 \text{ k}\Omega$ thus we have $R_0 \geq R_{0\min}$. We calculate reference capacitor value $C_0 = 1/(\omega_0 R_0) = 179.114$ pF and according to (8) the circuit capacitor is then $C = C_0 \cdot C_{nLP}$ and the circuit resistor is $R = R_0 \cdot R_{nLP}$, where $C_{nLP}$ and $R_{nLP}$ represent normalized values from Tables 1–2. The fifth capacitor $C_5$ has a value of $C_5 = C_0 \cdot C_{5nLP} = 4.5853$ pF. This is very small value and it is dangerously near to the order of the parasitic capacitance in the circuit. We choose $R_G = 10 \text{ k}\Omega$ and calculate

$$R_F = R_G (\beta - 1) = 5.333 \text{ k}\Omega \quad (22)$$

Figure 6a shows the normalized network with the reference values $\omega_0 = 155084$ rad/s and $R_0 = 36 \text{ k}\Omega$ (all normalized values follow directly from the 5th row in Table 1). Figure 6b shows the denormalized network with resistor values in kΩ and capacitors in pF. A simple first-order check for the correctness of these results is to verify that $\omega_0 = (R_1 R_2 R_3 R_4 R_5) C_1 C_2 C_3 C_4 C_5)^{-1}$.

Now we can continue with the next example of the 3rd-order Cheby-0.5 dB filter. Using the same design steps as above we choose $R_0 = 39 \text{ k}\Omega$. This provides that the total capacitance is less than 300 pF. The reference capacitor value is $C_0 = 1/(\omega_0 R_0) = 204.044$ pF.

We choose $R_G = 10 \text{ k}\Omega$ and calculate $R_F = 3.1082 \text{ k}\Omega$. Note that the third capacitor $C_3$ has a value of $C_3 = 22.67$ pF, which is, in this case, quite larger than the parasitic capacitance in the integrated circuit.
Figure 7a shows the normalized network with the reference values $\omega_0 = 125664$ rad/s and $R_0 = 39$ kΩ, whereas Figure 7b shows the denormalized network.

In what follows for both denormalized filters shown in Fig. 6b and Fig. 7b, magnitudes are shown in Fig. 8 to demonstrate their functionality and the satisfaction of filter specifications. Notice that magnitudes in Fig. 8 have unity pass-band gain ($K = 1$, i.e. 0 dB). The realization of the desired pass-band gain value $K$ will be presented in the section 4.2, below. As a double check of filter sensitivities, Monte Carlo runs (using PSPICE simulation) are performed and shown in Fig. 9.

Note that Chebyshev filter although has 0.5 dB ripple in the pass-band and larger pole Q factors, shows lower sensitivity than the other realization, a higher order Butterworth filter. This is because lower filter order has substantially lower sensitivity, especially when filters are realized using single opamp circuit as in Fig. 1. Therefore our choice will be the Chebyshev filter realization.

Recall that the filters presented in design tables above are capacitively tapered and optimized for low sensitivity.

### 4.2 The Gain Factor $K$

The DC forward gain $K$ of the filter transfer function will generally not coincide with the am-
plifier gain $\beta$, required to obtain filter transfer function parameters. The gain factor $K$ may be specified by the filter designer, but the amplifier gain is determined by the values for $\beta$ given in design tables. Recall that all values in above tables including $\beta$ are obtained using optimization procedure for low sensitivity filter. Thus, the value of $\beta$ cannot be freely chosen; it depends on the design equations for the filter, whereas the overall DC filter gain $K$ may very likely be required to have a different value. Fortunately, there are various schemes for the decoupling of $K$ and $\beta$ [9], one of which will be presented in what follows. In terms of our filter, this implies that

$$\alpha = \frac{K}{\beta}$$  \hspace{1cm} (23)

If the desired value of $\alpha$ is less than unity, i.e. $\beta > K$ (and because $\beta \geq 1$), then a resistive voltage divider can be inserted at the input of the network, as shown in Fig. 10. The gain decoupling was applied to the 3rd-order Chebyshev filter with 0.5 dB shown in Fig. 7b. In this case input resistor $R_1$ was substituted by voltage divider (by factor $\alpha$) consisting of resistors $R'_1$ and $R''_1$. We have

$$\alpha = \frac{R'_1}{R'_1 + R''_1} \quad \text{and} \quad R_1 = \frac{R'_1 R''_1}{R'_1 + R''_1}$$  \hspace{1cm} (24)

i.e. $R'_1 = R_1/\alpha$ and $R''_1 = R_1/(1-\alpha)$. Since $\alpha$ is, in this case, less than unity, $R''_1$ is always positive. In our example to realize $K=1$ we have $\alpha = K/\beta = 0.7629$, and $R'_1 = R_1/\alpha = 87.419 \, \text{k}\Omega$; $R''_1 = 281.251 \, \text{k}\Omega$.

5 DESIGN OF HIGHPASS FILTERS

In this section we present the design of high-pass (HP) filters starting from given specifications and using the same tables as in the low-pass filter (LP) case presented above. The filters are designed in the same straightforward way and final filter circuits are optimal in the sense that they have minimum sensitivity to passive components variations of the circuit, and low power consumption.

Consider an example of HP filter which satisfies the specifications shown in Fig. 11. The specification requires the maximum pass-band attenuation of $A_p = 0.5 \, \text{dB}$ for the frequencies above $f_p = 32 \, \text{kHz}$ and the minimum stop-band attenuation of $A_s = 10 \, \text{dB}$ for the frequencies below $f_s = 20 \, \text{kHz}$. The filter should have a unity gain in the pass band ($K=1$).

When we include specifications values into (12) and (16) we readily calculate required filter order $n$. Recall that the HP specifications can be transformed to the normalized LP specifications with the normalized cut-off frequency $F_s = f_p/f_s$. In both LP and HP filter examples we obtain the same orders $n$ because both specifications have the same requirements, i.e. we have the same frequency $F_s$.

In what follows we are going to design a HP filter of the 3rd-order with Chebyshev approximation 0.5 dB ripple, which is dual to the filter circuit in Fig. 7b.

To design HP filter we should calculate capacitors values as reciprocals of resistors values in Tables 1—2 and vice versa, resistors values are reciprocals of capacitors values ($RC$-$CR$ transformation), which is accomplished by (10). In (10) the denormalization of components values is performed, as well. The denormalization is performed to the cut-off frequency $f_p = 32 \, \text{kHz}$ or $\omega_0 = 2\pi f_p = 201061.9 \, \text{rad/s}$. To properly select the denor-
malization resistance $R_0$ we must be able to realize capacitors on the chip.

Therefore we calculate (with $C_{TOT}=300$ pF)

$$R_{0\min} = \frac{1}{\omega_0 C_0} = \frac{\sum C_{in}}{\omega_0 C_{TOT}} = \frac{1.03508}{201062 \cdot 300} = 17160 \Omega \quad (25)$$

We choose reference resistor value $R_0=18$ kΩ and this provides reference capacitor value $C_0=1/(\omega_0 R_0)=276.311$ pF. The circuit capacitor is then $C=C_0 \cdot C_{nHP}$ and the circuit resistor is $R=R_0 \cdot R_{nHP}$, where index $n$ denotes normalized values. Thus we have the following values for components in HP filter

$$R_{1HP} = \frac{R_0}{C_{nLP}} = R_0 \cdot R_{nLP} = 18 \text{ kΩ}$$
$$R_{2HP} = 54 \text{ kΩ}; \quad R_{3HP} = 162 \text{ kΩ};$$
$$C_{1HP} = \frac{1}{R_{nLP} \omega_0 R_0} = C_0 \cdot C_{nLP} = 161.6 \text{ pF}; \quad (26)$$
$$C_{2HP} = 41.98 \text{ pF}; \quad C_{3HP} = 82.44 \text{ pF}$$

Note that in (26) $R_{nHP}=1/C_{nLP}$ and $C_{nHP}=1/R_{nLP}$. Fig. 12a shows the normalized network with the $\omega_0=201061.9$ rad/s and $R_0=18$ kΩ, whereas the de-normalized network is shown in Fig. 12b. It is obvious from Fig. 12a that the circuit is resistively tapered by the factor 3 (as in the LP case, where the capacitors are scaled by the factor $\rho_C=3$). The resistors $R_G$ and $R_F$ in the amplifier feedback realize the same gain $\beta$ as in the LP case.

To realize the gain $K=1$ we substitute an input capacitor $C_1$ by voltage divider (by factor $\alpha$) consisting of capacitors $C'_1$ and $C''_1$. We have

$$\alpha = \frac{C'_1}{C'_1 + C''_1} \quad \text{and} \quad C_1 = C'_1 + C''_1 \quad (27)$$

i.e. $C'_1=C_1/(1-\alpha)$ and $C''_1=C_1/\alpha$. In our example we have $\alpha=K/\beta=0.7629$, thus the input capacitor $C_1$
is split into $C'_1=681.45 \text{ pF}$ and $C''_1=211.81 \text{ pF}$ ($C'_1$ is connected to the input signal generator). Corresponding magnitudes and MC runs are shown in Fig. 13 and Fig. 14, respectively. The obtained HP filter circuit in Fig. 12b satisfies the specifications in Fig. 11.

Comparing MC runs in Fig. 14 (for the HP 0.5 dB Chebyshev filter) to those in Fig. 9b for its dual counterpart LP filter (Fig. 7b), both satisfying the same LP prototype specifications, we can conclude that both of these filters have identical sensitivities. This is because they are designed starting from the same optimized component values in the second row (for $n=3$) in Table 2.

6 CONCLUSIONS

A procedure for the design of allpole low-sensitivity, low-power, active-RC filters using tables with predefined normalized filter component values for some common filter types (Butterworth and Chebyshev 0.5 dB) is presented. The filter uses only one operational amplifier, and a minimum number of passive components. The amplifier itself ensures realization of conjugate-complex filter poles, and low output impedance. For reasons related to the filter topology, the application of the capacitive impedance tapering has improved the sensitivity of the low-pass filters’ magnitude to component tolerances [9]. The proposed design is universal and straightforward by using design tables; thus there is no need for numeric calculations. It can be extended to the design of single-amplifier, low-sensitivity high-pass filters, as well. Because the high-pass filters are dual to the low-pass filters, resistive tapering is applied to reduce the sensitivity of the high-pass filter.

Furthermore, the reduction in power and component count achieved with the single-amplifier LP filters is obtained at a price: a cascade of impedance-tapered second-order and/or third-order sections has lower sensitivity than impedance-tapered single-amplifier filter. Thus the decision on which way to go is typically one of tradeoffs: low power and component count versus low sensitivity. In our example, of hearing aid circuit realization, we preferred former solution having low power and low component count and acceptably low sensitivity.

APPENDIX I: COEFFICIENTS OF $T(s)$

Consider the $n^{th}$-order allpole low-pass filter circuit with positive feedback presented in Fig. A.1.

Note the descending notation of $R_n$, $C_n$ to $R_1$, $C_1$ from the driving source to the amplifier input. This reverse order notation is convenient to develop recursive formulas for determining transfer function coefficients $d_j$ to $d_n$ in eq. (A.2) of the $n^{th}$-order polynomial $D'_n(s)$ as functions of resistors $R_j$, capacitors $C_j$ and gain $\beta$.

Recursive formulas follow from characteristics of the continuants. The continuants are used to solve ladder-networks, and they can be calculated recursively. The filter in Fig. A.1 has a ladder network in the amplifier’s positive feedback loop with gain $\beta$, where gain $\beta=1+R_C/R_G$ represents the gain in the class-4 filter circuit. The transfer function of the $n^{th}$-order filter as presented in the Fig. A.1 has the form given by

$$T(s) = \frac{\beta}{D_n'(s)} \quad (A.1)$$

As shown in [1] and [2] p. 252, the $n^{th}$-order denominator polynomial in transfer function (A.1), i.e.

$$D'_n(s) = \sum_{j=1}^{n} d_j s^{j-1} + 1 \quad (A.2)$$

can be calculated from polynomials of $n-1$ and $n-2$ order:

$$D'_{n-1}(s) = \sum_{j=1}^{n-1} c_j s^{j-1} + 1 \quad (A.3)$$

and

$$D'_{n-2}(s) = \sum_{j=1}^{n-2} b_j s^{j-1} + 1 \quad (A.4)$$

using

$$d_j = (c_j - b_j) \frac{R_n}{R_{n-1}} + c_j + c_{j-1} R_n C_n - \delta_{1j} \left[ \frac{(-1)^n + 1}{2} \right] R_n C_n \beta \quad (A.5)$$

where $\delta_{1j}=0$, for $j \neq 1$; and $\delta_{1j}=1$, for $j=1$ where $1 \leq j \leq n$. Note that $b_0=c_0=d_0=1$. Note also that, for the start of the recursive process polynomials $D'_0=1$ and $D'_1=R_1 C_1 s+1$.

To perform the symbolic calculations (to calculate coefficients in terms of components) we use symbolic calculation program Mathematica. At the end of recursive process we change to the ascending notation, i.e. we substitute $R_n \rightarrow R_1$, $C_n \rightarrow C_1$,
R_{n-1} \rightarrow R_2, C_{n-1} \rightarrow C_2, \ldots, R_1 \rightarrow R_n, C_1 \rightarrow C_n$, resulting in the filter circuit with notation shown in Fig. 1. Consequently, we perform multiplication of nominator and denominator by the same factor

$$N(s) = \frac{N'(s)}{d_n} = \beta \cdot a_0, \quad D(s) = \frac{D'(s)}{d_n} \quad (A.6a)$$

$$a_j = \frac{d_j}{d_n}, \quad 0 \leq j \leq n \quad (A.6b)$$

We obtain the form of the transfer function of the $n^{th}$-order filter given by (note the unity coefficient with the highest, i.e. $n^{th}$ power of $s$)

$$T(s) = \frac{N(s)}{D(s)} = \frac{\beta a_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \quad (A.7)$$

The transfer function (A.7) has the same form as the transfer function (1). In what follows we present the transfer function’s coefficients up to reasonable ($6^{th}$) order obtained by the recursive procedure shown above. Those coefficients correspond to the filter circuit with ascending notation shown in Fig. 1, and are used in this paper for all numerical calculations carried out by Mathematica. It can be seen that increasing the filter order $n$, coefficients become even more complicated, and therefore the canonical high-order filter’s sensitivity rapidly increases.

2nd Order

$$a_1 = (C_1 R_1 + C_2 R_1 + C_3 R_2 + C_1 R_1 \beta) / (C_1 C_2 R_1 R_2)$$

$$a_0 = 1 / (C_1 C_2 R_1 R_2)$$

3rd Order

$$a_2 = (C_1 C_2 R_1 R_2 + C_1 C_3 R_1 R_2 + C_1 C_3 R_1 R_3 + C_2 C_3 R_2 R_3 - C_1 C_2 R_1 R_2 \beta) / (C_1 C_2 C_3 R_1 R_2 R_3)$$

$$a_1 = (C_1 R_1 + C_2 R_1 + C_3 R_1 + C_2 R_2 + C_3 R_2 + C_1 R_1 \beta - C_2 R_2 \beta) / (C_1 C_2 C_3 R_1 R_2 R_3)$$

$$a_0 = 1 / (C_1 C_2 C_3 R_1 R_2 R_3)$$

4th Order

$$a_3 = (C_2 C_3 C_4 (R_1 + R_2) R_3 R_4 + C_1 R_1 (C_3 C_4 (R_2 + R_3) R_4 + C_2 R_2 (C_4 (R_3 + R_4) + C_3 R_3 (R_3 - R_3 \beta))) / (C_1 C_2 C_3 C_4 R_1 R_2 R_3 R_4)$$

$$a_2 = (C_1 C_2 C_3 R_1 R_2 + C_4 (C_3 (R_1 + R_2) R_3 + C_2 (R_1 + R_2) R_3 + C_1 R_1 (R_2 + R_3)) (-1 + \beta) / (C_1 C_2 C_3 R_1 R_2 R_3 R_4)$$

$$a_1 = (C_1 R_1 + C_2 R_1 + C_2 R_2 + C_4 (R_1 + R_2 + R_3 + R_4) - C_3 (R_1 + R_2 + R_3) (-1 + \beta) - C_1 R_1 \beta) / (C_1 C_2 C_3 R_1 R_2 R_3 R_4)$$

$$a_0 = 1 / (C_1 C_2 C_3 C_4 R_1 R_2 R_3 R_4)$$

5th Order

$$a_4 = (C_2 C_3 C_4 C_5 (R_1 + R_2) R_3 R_4 R_5 + C_1 R_1 (C_3 C_4 C_5 (R_2 + R_3) R_4 R_5 + C_2 C_3 R_2 (C_4 C_5 C_6 (R_3 + R_4) + C_3 R_3 (R_4 - R_4 \beta)))) / (C_1 C_2 C_3 C_4 C_5 R_1 R_2 R_3 R_4 R_5)$$

$$a_3 = (C_1 C_2 C_3 R_1 R_2 R_3 + C_5 (C_4 (C_3 (R_1 + R_2 + R_3) R_4 + C_2 R_2 (R_1 + R_2) R_3 + C_1 R_1 (R_2 + R_3)) R_5 + C_3 (C_2 (R_1 + R_2) R_3 + C_1 R_1 (R_2 + R_3)) (R_4 + R_5) + C_1 C_2 R_1 R_2 (R_3 + R_4 + R_5) - C_4 (C_2 C_3 (R_1 + R_2) R_3 R_4 + C_1 C_2 R_1 R_2 (R_3 + R_4))) (-1 + \beta) / (C_1 C_2 C_3 C_4 C_5 R_1 R_2 R_3 R_4 R_5)$$

$$a_2 = (C_1 C_2 R_1 R_2 + C_1 C_3 R_1 R_2 + C_3 R_1 R_3 + C_2 C_3 R_2 R_3 + C_1 R_1 R_3 + C_2 C_3 R_2 R_3 + C_1 R_2 R_3) / (C_1 C_2 C_3 C_4 C_5 R_1 R_2 R_3 R_4 R_5)$$

$$a_1 = (C_1 R_1 + C_2 R_1 + C_3 R_1 + C_2 R_2 + C_2 R_2 + C_3 R_2 + C_1 R_1 \beta - C_2 R_2 \beta) / (C_1 C_2 C_3 C_4 C_5 R_1 R_2 R_3 R_4 R_5)$$

$$a_0 = 1 / (C_1 C_2 C_3 C_4 C_5 R_1 R_2 R_3 R_4 R_5)$$

6th Order

$$a_5 = (C_2 C_3 C_4 C_5 C_6 (R_1 + R_2) R_3 R_4 R_5 R_6 + C_1 R_1 (C_3 C_4 C_5 C_6 (R_2 + R_3) R_4 R_5 R_6 + C_2 C_3 C_4 C_5 C_6 (R_3 + R_4) R_5 R_6 + C_3 R_3 (C_5 C_6 (R_4 + R_5) R_6 + C_4 R_4 (C_6 (R_5 + R_6) + C_5 (R_5 - R_5 \beta)))) / (C_1 C_2 C_3 C_4 C_5 C_6 R_1 R_2 R_3 R_4 R_5 R_6)$$

$$a_4 = (C_2 C_3 C_4 (R_1 + R_2) R_3 R_4 + C_1 R_1 (C_3 C_4 (R_2 + R_3) R_4 + C_2 R_2 (C_4 (R_3 + R_4) + C_3 R_3 (R_3 - R_3 \beta))) / (C_1 C_2 C_3 C_4 R_1 R_2 R_3 R_4)$$

$$a_3 = (C_1 C_2 C_3 R_1 R_2 R_3 + C_5 (C_4 (C_3 (R_1 + R_2 + R_3) R_4 + C_2 R_2 (R_1 + R_2) R_3 + C_1 R_1 (R_2 + R_3)) R_5 + C_3 (C_2 (R_1 + R_2) R_3 + C_1 R_1 (R_2 + R_3)) (R_4 + R_5) + C_1 C_2 R_1 R_2 (R_3 + R_4 + R_5) - C_4 (C_2 C_3 (R_1 + R_2) R_3 R_4 + C_1 C_2 R_1 R_2 (R_3 + R_4))) (-1 + \beta) / (C_1 C_2 C_3 C_4 C_5 R_1 R_2 R_3 R_4 R_5)$$

$$a_2 = (C_1 C_2 R_1 R_2 + C_1 C_3 R_1 R_2 + C_3 R_1 R_3 + C_2 C_3 R_2 R_3 + C_1 R_1 R_3 + C_2 C_3 R_2 R_3 + C_1 R_2 R_3) / (C_1 C_2 C_3 C_4 C_5 R_1 R_2 R_3 R_4 R_5)$$

$$a_1 = (C_1 R_1 + C_2 R_1 + C_3 R_1 + C_2 R_2 + C_2 R_2 + C_3 R_2 + C_1 R_1 \beta - C_2 R_2 \beta) / (C_1 C_2 C_3 C_4 C_5 R_1 R_2 R_3 R_4 R_5)$$

$$a_0 = 1 / (C_1 C_2 C_3 C_4 C_5 R_1 R_2 R_3 R_4 R_5)$$
a₄= \frac{1}{(C₁ C₂ C₃ C₄ C₅ C₆ R₁ R₂ R₃ R₄ R₅ R₆)} \left[ C₅ (C₁ C₂ C₃ C₄ C₅ C₆ R₁ R₂ R₃ R₄ R₅ R₆) \right]

a₅= \frac{1}{(C₁ C₂ C₃ C₄ C₅ C₆ R₁ R₂ R₃ R₄ R₅ R₆)} \left[ C₅ (C₁ C₂ C₃ C₄ C₅ C₆ R₁ R₂ R₃ R₄ R₅ R₆) \right]

REFERENCES


**Ključne riječi:** aktivni $RC$ filtri, minimalni $RC$ filtri, mala potrošnja, niska osjetljivost, normalizirane vrijednosti komponenata

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