Common fixed point theorems for occasionally weakly compatible mappings satisfying a generalized contractive condition

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Abstract. *We obtain several fixed point theorems for a class of operators called occasionally weakly compatible maps defined on a symmetric space satisfying a generalized contractive condition. These results establish some of the most general fixed point theorems for four maps. Our theorem generalizes Theorem 1 of [A. Aliouche, A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type, J. Math. Anal. Appl. 322(2006),796-802] and Theorem 1 of [X. Zhang, Common fixed point theorems for some new generalized contractive type mappings, J. Math. Anal. Appl. 333(2007), 780-786] and those contained therein.*

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Prior to 1968 all work involving fixed points used the Banach contraction principle. In 1968 Kannan [10] proved a fixed point theorem for a map satisfying a contractive condition that did not require continuity at each point. This paper was a genesis for a multitude of fixed point papers over the next two decades. (see e.g. [12] for a listing and comparison of many of these definitions). Sessa [15] coined the notion of weakly commuting. Then Jungck generalized this idea, first to compatible mappings [7] and then to weakly compatible mappings [8]. There are examples that show that each of these generalizations of commutativity is a proper extension of the previous definition. We shall list here only the definition of weakly compatible. Also during this time a number of authors established fixed point theorems for pairs of maps (see for example [4], [11] and references therein). Thagafi and Shahzad [2] gave a definition which is proper generalization of nontrivial weakly compatible maps which have coincidence points. The second author [9]

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proved two fixed point theorems involving more general contractive conditions (see also [16]). Recently, Zhang [17] obtained common fixed point theorems for some new generalized contractive type mappings. The aim of this paper is to obtain some fixed points theorem involving occasionally weakly compatible maps in the setting of a symmetric space satisfying a generalized contractive condition. Our results complement, extend and unify several well known results.

Two maps S and T are said to be weakly compatible if they commute at coincidence points.

Definition 1. *Let X be a set, f, g selfmaps of X. A point x in X is called a coincidence point of f and g iff $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g.*

The following concept [2] is a proper generalization of nontrivial weakly compatible maps which have a coincidence point.

Definition 2. *Two selfmaps f and g of a set X are occasionally weakly compatible (owc) iff there is a point x in X which is a coincidence point of f and g at which f and g commute.*

We shall also need the following lemma from [9].

Lemma 1. *Let X be a set, f, g owc selfmaps of X. If f and g have a unique point of coincidence, $w := fx = gx$, then w is a unique common fixed point of f and g.*

Our theorems are proved in symmetric spaces, which are more general than metric spaces.

Definition 3. *Let X be a set. A symmetric on X is a mapping $r : X \times X \to [0, \infty)$ such that*

$$r(x, y) = 0 \quad \textit{iff} \quad x = y, \quad \textit{and} \quad r(x, y) = r(y, x) \quad \textit{for} \quad x, y \in X.$$

Let $A \in (0, \infty]$, $R_A^+ = [0, A)$. Let $F : R_A^+ \to R$ satisfy

(i) $F(0) = 0$ and $F(t) > 0$ for each $t \in (0, A)$ and

(ii) F is nondecreasing on R_A^+.

Define, $\mathcal{F}[0, A) = \{F : R_A^+ \longrightarrow R : F \text{ satisfies } (i) - (ii)\}$.
Let $A \in (0, \infty]$. Let $\psi : R_A^+ \to R$ satisfies

(i) $\psi(t) < t$ for each $t \in (0, A)$ and

(ii) ψ is nondecreasing.

Define, $\Psi[0, A) = \{\psi : R_A^+ \longrightarrow R : F \text{ satisfies } (i) - (ii) \text{ above}\}$.
For some examples of mappings $F : R_A^+ \longrightarrow R : F$ which satisfies $(i) - (ii)$, we refer to [17].

Theorem 1. *Let X be a set with a symmetric r. Let $D = \sup\{r(x, y) : x, y \in X\}$. Suppose that f, g, S, T are selfmaps of X and that the pairs $\{f, S\}$ and $\{g, T\}$ are each owc. If for each $x, y \in X$ for which $fx \neq gy$ we have*

$$F(r(fx, gy)) < \psi(F(M(x, y))), \tag{1}$$

for each $x, y \in X$, $F \in \mathcal{F}[0, A)$ *and* $\psi \in \Psi[0, F(A-0))$, *where* $A = D$ *if* $D = \infty$ *and* $A > D$ *if* $D < \infty$, *and*

$$M(x,y) := \max\{r(Sx,Ty), r(Sx,fx), r(Ty,gy), r(Sx,gy), r(Ty,fx)\},$$

then there is a unique point $w \in X$ *such that* $fw = gw = w$ *and a unique point* $z \in X$ *such that* $gz = Tz = z$. *Moreover,* $z = w$, *so that there is a unique common fixed point of* f, g, S, *and* T.

Proof. Since the pairs $\{f, S\}$ and $\{g, T\}$ are each owc, there exist points $x, y \in X$ such that $fx = Sx$ and $gy = Ty$. We claim that $fx = gy$. For, otherwise, consider,

$$M(x,y) := \max\{r(Sx,Ty), r(Sx,fx), r(Ty,gy), r(Sx,gy), r(Ty,fx)\} = r(fx,gy).$$

Then (1) implies

$$F(r(fx,gy)) < \psi(F(M(x,y))) = \psi(F(r(fx,gy))) < F(r(fx,gy))$$

a contradiction. Therefore, $fx = gy$; i.e., $fx = Sx = gy = Ty$. Moreover, if there is another point z such that $fz = Sz$, then, using (1) it follows that $fz = Sz = gy = Ty$, or $fx = fz$, and $w = fx = Sx$ is the unique point of coincidence of f and S. By Lemma 1, w is the only common fixed point of f and S. By symmetry there is a unique point $z \in X$ such that $z = gz = Tz$.

Suppose that $w \neq z$. Using (1),

$$F(r(w,z)) = F(r(fw,gz)) < \psi(F(M(w,z))) < F(r(w,z)))$$

a contradiction. Therefore $w = z$ and w is a common fixed point. By the preceding argument it is clear that w is unique. □

Corollary 1. *Let* X *be a set with a symmetric* r. *Suppose that* $f, g,$ S, T *are selfmaps of* X *such that* $\{f, S\}$ *and* $\{g, T\}$ *are owc. If*

$$F(r(fx,gy)) \leq \psi(F(m(x,y))), \tag{2}$$

for each $x, y \in X$, $F \in \mathcal{F}[0, A)$ *and* $\psi \in \Psi[0, F(A-0))$, *where* $A = D$ *if* $D = \infty$ *and* $A > D$ *if* $D < \infty$, *and*

$$m(x,y) := \max\big\{r(Sx,Ty), r(Sx,fx), r(Ty,gy), [r(Sx,gy) + r(Ty,fx)]/2\big\},$$

and $0 \leq h < 1$, *then* f, g, S, T *have a unique common fixed point.*

Proof. Since (2) is a special case of (1), the result follows immediately from Theorem 1. □

In proving fixed point theorems for four maps, step one is by far the most difficult part of the proof. In this paper we have imposed the condition owc, which automatically gives the result of step one. Other authors have circumvented this difficulty by hypothesizing a property, known as property (E, A), which implies owc.

Two maps f, S are said to satisfy property (E, A) if there exists a sequence $\{x_n\}$ such that $\lim_n Sx_n = \lim_n fx_n = t$ for some $t \in X$. Some papers in which this property has appeared are [1] and [3]. Two maps S and T are said to be pointwise

R-commuting if, for each $x \in X$ there exists an $R > 0$ such that $d(STx, TSx) \leq Rd(Sx, Tx)$. The definition of R$-$pointwise commuting is equivalent to S and T commuting at coincidence points; i.e. S and T are weakly compatible.

Corollary 2. *Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that T and S satisfy property (E, A), $T(X) \subset S(X)$, and*

$$F(d(Sx, Ty)) \leq F(M(x, y))), \tag{3}$$

for each $x, y \in X$, $F \in \mathcal{F}[0, A)$, where $A = D$ if $D = \infty$ and $A > D$ if $D < \infty$, and

$$M(x, y) = \max\left\{d(Sx, Sy), [d(Tx, Sx) + d(Ty, Sy)]/2, [d(Ty, Sx) + d(Tx, Sy)]/2\right\}.$$

If SX or TX is a complete subspace of X, then T and S have a unique common fixed point.

Proof. Condition (3) is a special case of condition (1). Property (E, A) implies that S and T have owc. The conclusion now follows from Theorem 1. □

Corollary 3. *Let (X, d) be a symmetric space with symmetric r and f, S selfmaps of X such that f and S are owc, and*

$$F(r(fx, fy)) \leq \psi(F(M(x, y))), \tag{4}$$

for each $x, y \in X$, $F \in \mathcal{F}[0, A)$ and $\psi \in \Psi[0, F(A - 0))$, where $A = D$ if $D = \infty$ and $A > D$ if $D < \infty$, and

$$\begin{aligned} M(x, y) = {} & ar(Sx, Sy) + b\max\{r(fx, Sx), r(fy, Sy)\} \\ & + c\max\{r(Sx, Sy), r(Sx, fx), r(Sy, fy)\} \end{aligned} \tag{5}$$

for all $x, y \in X$, where $a, b, c > 0, a + b + c = 1$. Then f and S have a unique common fixed point.

Proof. From (5)

$$\begin{aligned} M(x, y) &= ar(Sx, Sy) + b\max\{r(fx, Sx), r(fy, Sy)\} \\ &\quad + c\max\{r(Sx, Sy), r(Sx, fx), r(Sy, fy)\} \\ &\leq (a + b + c)\max\{r(Sx, Sy), r(fx, Sx), r(fy, Sy)\} \\ &\leq \max\{r(Sx, Sy), r(fx, Sx), r(fy, Sy), r(fx, Sy), r(fy, Sx)\}. \end{aligned}$$

Thus Condition (4) is a special case of condition (1) when $f = g$ and $S = T$. Therefore the result now follows from Theorem 1. □

Theorem 2. *Let X be a symmetric space with symmetric r, f, g, S, and T selfmaps of X and*

$$F(r(fx, gy))^p) \leq \psi(F(M_p(x, y))), \tag{6}$$

for each $x, y \in X$, $F \in \mathcal{F}[0, A)$ and $\psi \in \Psi[0, F(A - 0))$, where $A = D$ if $D = \infty$ and $A > D$ if $D < \infty$, and

$$\begin{aligned} M_p(x, y) = {} & a(r(fx, Ty))^p + (1 - a)\max\Big\{(r(fx, Sx))^p, (r(gy, Ty))^p, \\ & (r(fx, Sx))^{p/2}(r(fx, Ty))^{p/2}, (r(Ty, fx))^{p/2}(r(Sx, gy))^{p/2}\Big\}, \end{aligned} \tag{7}$$

for all $x, y \in X$, where $0 < a \leq 1$, and $p \geq 1$. If $\{f, S\}$ and $\{g, T\}$ are owc, then f, g, S, and T have a unique common fixed point.

Proof. By hypothesis there exist points x and y such that $fx = Sx$ and $gy = Ty$. Suppose that $fx \neq gy$. Then, from (7)

$$M_p(x, y) = a(r(fx, gy))^p + (1-a)\max\{0, 0, 0, (r(fx, gy))^p\} = (r(fx, gy))^p$$

$$\begin{aligned} F(r(fx, gy))^p) &\leq \psi(F(M_p(x, y))) \\ &= \psi(F((r(fx, gy))^p)) < F((r(fx, gy))^p) \end{aligned}$$

a contradiction. Therefore $r(fx, gy) = 0$, which implies that $fx = gy$. Suppose that there exists another point z such that $fz = Sz$. Then, using (6) one obtains $fz = Sz = gy = Ty = fx = Sx$ and hence $w = fx = fz$ is the unique point of coincidence of f and S. By symmetry there exists a unique point $v \in X$ such that $v = gz = Tv$. It then follows that $w = v$, w is a common fixed point of f, g, S, and T, and w is unique. □

Define $\mathcal{G} = \{\dot{g} : \mathbb{R}^5 \to \mathbb{R}^5\}$ such that

(g_1) $\dot{g}$ is nondecreasing in the 4th and the 5th variable,

(g_2) If $u, v, \in \mathbb{R}^+$ are such that $u \leq \dot{g}(v, v, u, u+v, 0)$, or $u \leq \dot{g}(v, u, v, u+v, 0)$ or $v \leq \dot{g}(u, u, v, u+v, 0)$, or $u \leq \dot{g}(v, u, v, u, u+v)$, then $u \leq hv$, where $0 < h < 1$ is a constant,

(g_3) If $u \in \mathbb{R}^+$ is such that $u \leq \dot{g}(u, 0, 0, u, u)$ or $u \leq \dot{g}(0, u, 0, u, u)$ or $u \leq \dot{g}(0, 0, u, u, u)$, then $u = 0$.

Theorem 3. *Let X be a set, r a symmetric on X. Let f, g, S, T be selfmaps of X satisfying*

$$\begin{aligned} F(r(fx, gy)) \leq \dot{g}(&F(r(Sx, Ty)), F(r(fx, Sx)), F(r(gy, Ty)), \\ &F(r(fx, Ty)), F(r(gy, Sx))), \end{aligned} \tag{8}$$

for all $x, y \in X$, $F \in \digamma[0, A)$, where $A = D$ if $D = \infty$ and $A > D$ if $D < \infty$, and $\dot{g}$ satisfies (g_3). If $\{f, S\}$ and $\{g, T\}$ are owc, then f, g, S, T have a unique common fixed point.

Proof. By hypothesis there exist points $x, y \in X$ such that $fx = Sx$ and $gy = Ty$. Suppose that $fx \neq gy$. Then, from (8)

$$F(r(fx, gy)) \leq \dot{g}(F(r(fx, gy)), 0, 0, F(r(fx, gy)), F(r(gy, fx))),$$

which from (g_3) implies that $F(r(fx, gy)) = 0$ and hence $r(fx, gy) = 0$. Hence $fx = gy$. As in the previous theorems it can then be shown that fx is unique and that $u = fx$ is a common fixed point of the four mappings. Condition (8) implies uniqueness. □

A control function Φ is defined by $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies $\Phi(t) = 0$ iff $t = 0$.

Theorem 4. *Let $\{f, S\}$ and $\{g, T\}$ be owc pairs of selfmaps of a space X, with symmetric r and*

$$F(\Phi(r(fx, gy))) < \psi(F(M_\Phi(x, y))), \tag{9}$$

for each $x, y \in X$, $F \in \mathcal{F}[0, A)$ *and* $\psi \in \Psi[0, F(A-0))$, *where* $A = D$ *if* $D = \infty$ *and* $A > D$ *if* $D < \infty$, *and*

$$M_\Phi(x,y) := \max\Big\{\Phi(r(Sx,Ty)), \Phi(r(Sx,fx)), \Phi((r(gy,Ty)),$$
$$[\Phi(r(fx,Ty)), +\Phi(r(Sx,gy))]/2\Big\}. \tag{10}$$

Then f, g, S, *and* T *have a unique common fixed point.*

Proof. By hypothesis there exist points $x, y \in X$ for which $fx = Sx$ and $gy = Ty$. Suppose that $fx \neq gy$. Then, from (10)

$$M_\Phi(x,y) := \max\{\Phi(r(fx,gy)), \Phi(0), \Phi(0), \Phi(r(fx,gy))\}.$$

Thus

$$0 < F(\Phi(r(fx,gy))) < \psi(F(M_\Phi(x,y)))$$
$$= \psi(F(\Phi(r(fx,gy))) < F(\Phi(r(fx,gy)),$$

a contradiction. Therefore

$$\Phi((r(fx,gy)) = 0,$$

which implies that $(r(fx,gy) = 0$, implying that $fx = gy$. It then follows that f, g, S, and T have a common fixed point. Condition (9) gives uniqueness. □

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