Trigonometric Proof of Steiner-Lehmus Theorem in Hyperbolic Geometry

Trigonometric Proof of Steiner-Lehmus Theorem in Hyperbolic Geometry

ABSTRACT
In this study, we give a trigonometric proof of the Steiner-Lehmus Theorem in hyperbolic geometry.

Key words: hyperbolic geometry, hyperbolic triangle
MSC 2000: 51K05, 51K99

1 Introduction
Elementary hyperbolic geometry was born in 1903 when Hilbert provided, using the end-calculus to introduce coordinates, a first-order axiomatization for it by adding to the axioms for plane absolute geometry a hyperbolic parallel axiom stating that “Through any point \( P \) not lying on a line \( l \) there are two rays \( r_1 \) and \( r_2 \), not belonging to the same line, which do not intersect \( l \), and such that every ray through \( P \) contained in the angle formed by \( r_1 \) and \( r_2 \) does intersect \( l' \)” [2]. The hyperbolic geometry is a non-euclidean geometry. Here in this study, we give hyperbolic version of Steiner-Lehmus theorem. The well-known Steiner-Lehmus theorem states that if the internal angle bisectors of two angles of a triangle are equal, then the triangle is isosceles [1].

Lemma 1 (Sines Theorem) In the hyperbolic triangle \( ABC \) let \( \alpha, \beta, \gamma \) denote at \( A, B, C \) and \( a, b, c \) denote the hyperbolic lengths of the sides opposite \( A, B, C \), respectively, then

\[
\frac{\sin \alpha}{\sinh a} = \frac{\sin \beta}{\sinh b} = \frac{\sin \gamma}{\sinh c} \tag{1}\]

[3, p.125].

2 Main results
Theorem 2 If the internal angle bisectors of two angles of a triangle are equal, then the triangle isn’t isosceles.

Proof. Let \( BB' \) and \( CC' \) be the respective internal angle bisectors of angles \( B \) and \( C \) in triangle \( ABC \), and let \( a, b \) and \( c \) denote the sidelengths in the standard order. As shown in Figure 1, we set

\[
B = 2\beta, \ C = 2\gamma, \ u = AB', \ U = B'C, \ v = AC', \ V = C'B. \]

Figure 1

If we use the sines theorem in the triangles \( ABC, BB'C, BB'A, BCC', ACC' \) respectively (See Lemma 1), then

\[
\frac{\sin A}{\sinh a} = \frac{\sin 2\gamma}{\sinh c} = \frac{\sin 2\beta}{\sinh b} \tag{2}\]
If we put the values of $\sinh U = \sin A \sinh V = \sin A \sinh\gamma$ in the equations (7) and (8), then

$$\frac{\sinh b - \sinh c}{\sinh u} = \frac{\sinh a}{\sinh b} - \frac{\sinh a}{\sinh b} < 0$$

Because of $C > B$, $V > U$, $\nu > u$. Hence, $\sinh b < \sinh c$ (and $c > b$). Consequently, the case $C > B$ is satisfied while $BB' = CC'$. The triangle $ABC$ can’t isosceles.

References


Nilgün Sönmez

e-mail: nceylan@aku.edu.tr

Afyon Kocatepe University
Faculty of Science and Literatures
Department of Mathematics
ANS Campus, 03200 - Afyonkarahisar, Turkey