FINITE 2-GROUPS $G$ WITH $\Omega_2^2(G)$ METACYCLIC

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Abstract. In this paper we classify finite non-metacyclic 2-groups $G$ such that $\Omega_2^2(G)$ (the subgroup generated by all elements of order 4) is metacyclic. However, if $G$ is a finite 2-group such that $\Omega_2(G)$ (the subgroup generated by all elements of order $\leq 4$) is metacyclic, then $G$ is metacyclic.

1. Introduction

A famous result of N. Blackburn (see Proposition 1.4) states that if $G$ is a finite 2-group such that the subgroup $\Omega_2(G)$ (the subgroup generated by all elements of order $\leq 4$) is metacyclic, then $G$ is metacyclic. What can we say in the case, where $G$ is a finite 2-group and we know that only the subgroup $\Omega_2^2(G)$ (the subgroup generated by all elements of order 4) is metacyclic? The purpose of this paper is to classify finite non-metacyclic 2-groups $G$ such that $\Omega_2^2(G)$ is metacyclic. We have seen in Janko [3] that such a subgroup $\Omega_2^2(G)$ has the strong influence on the structure of the whole group $G$ so that the structure of the 2-group $G$ is almost uniquely determined, when $\Omega_2^2(G)$ is known.

All groups considered here are finite and our notation is standard. In particular,

$$M_{2^n} = \langle a, t \mid a^{2^{n-1}} = t^2 = 1, a^t = a^{1+2^{n-2}}, n \geq 4 \rangle,$$

and 2-groups of maximal class are dihedral groups $D_{2^n}$ (of order $2^n$, $n \geq 3$), generalized quaternion groups $Q_{2^n}$ (of order $2^n$, $n \geq 3$), and semi-dihedral groups $SD_{2^n}$ (of order $2^n$, $n \geq 4$).

For convenience, we state here some known results which are used in this paper.

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Proposition 1.1 ([5, Proposition 1.4]). Let $G$ be a 2-group of order $\geq 2^4$ satisfying $|\Omega_2(G)| \leq 2^3$. If $Z(G)$ is noncyclic, then $G$ is abelian of type $(2, 2^n)$, $n \geq 3$.

Proposition 1.2 ([5, Theorem 2.1]). Let $G$ be a metacyclic 2-group which is neither cyclic nor of maximal class. Then $G$ has exactly three involutions.

Proposition 1.3 ([2, Theorem 4.1]). Let $G$ be a 2-group of order $> 2^4$ all of whose elements of order 4 generate the subgroup $H = \Omega_2^2(G)$ of order $2^4$. Assume in addition that $G$ has exactly 6 cyclic subgroups of order 4 and $|\Omega_2(G)| > 2^4$. Then we have the following possibilities:

(a) $H \cong Q_8 \times C_2$ and $G \cong SD_{24} \times C_2$.
(b) $H \cong \langle a, b \mid a^2 = b^4 = 1, a^b = a^{-1} \rangle$ and $G = \langle b, t \mid b^4 = t^2 = 1, b^t = ab, a^t = a^{-1} \rangle$.

Here $|G| = 2^5$, $H = \langle a, b \rangle$, $\Phi(G) = \langle a, b^2 \rangle \cong C_4 \times C_2$, $\Omega_2(G) = G$, and $Z(G) = \langle a^2, b^2 \rangle \cong E_4$.
(c) $H \cong C_4 \times C_4$ and $G$ has a metacyclic maximal subgroup $M$ such that $\Omega_2(M) = H$, $G = M \langle t \rangle$, where $t$ is an involution with $C_M(t) = \Omega_1(M) \cong E_4$ and $t$ inverts each element of $C_M(H)$ so that $C_M(H)$ is abelian. (The last statement actually follows from the proof of Theorem 4.1 in [2].)

Proposition 1.4 (N. Blackburn, [2, Proposition 1.8]). If $G$ is a 2-group such that $\Omega_2(G)$ is metacyclic, then $G$ is metacyclic, too.

Proposition 1.5 ([4, Theorem 5.1]). Let $G$ be a 2-group containing exactly one abelian subgroup of type $(4, 2)$. Then one of the following holds:

(a) $|\Omega_2(G)| = 8$.
(b) $G \cong C_2 \times D_{2^{n+1}}$, $n \geq 2$.
(c) $G = \langle b, t \mid b^{2^{n+1}} = t^2 = 1, b^t = b^{-1}2^{n-1}u, u^2 = [u, t] = 1, b^u = b^{1+2^n}, n \geq 2 \rangle$. Here $|G| = 2^{n+3}$, $Z(G) = \langle b^{2^n} \rangle$ is of order 2, $\Phi(G) = \langle b^2, u \rangle$, $E = \langle b^{2^n}, u, t \rangle \cong E_8$ is self-centralizing in $G$, $\Omega_2(G) = \langle u \rangle \times \langle b^2, t \rangle \cong C_2 \times D_{2^{n+1}}$, $G' = \langle b^{2^n}, u \rangle \cong E_4$ in case $n = 2$, and $G' = \langle b^2u \rangle \cong C_{2^n}$ for $n \geq 3$.

Proposition 1.6 ([4, Proposition 1.12]). Let $G$ be a $p$-group with a nonabelian subgroup $P$ of order $p^3$. If $C_p(P) \leq P$, then $G$ is of maximal class.

Proposition 1.7 (Janko, [1, Proposition 1.10]). Let $\tau$ be an involutory automorphism acting on an abelian 2-group $B$ so that $C_B(\tau) = W_0$ is contained in $\Omega_1(B)$. Then $\tau$ acts invertingly on $\Omega_1(B)$ and on $B/W_0$.

Proposition 1.8 ([2, Introduction]). Suppose that a 2-group $G$ is neither cyclic nor of maximal class. Then the number $c_n(G)$ ($n > 1, n$ fixed) of cyclic
subgroups of order $2^n$ is even. (This result is also due to G. A. Miller and appears in section 51 of the 1915 book on “Finite groups” by Miller-Blichfeldt-Dickson.)

We prove here the following new result.

**Theorem 1.9.** Let $G$ be a non-metacyclic 2-group of exponent $> 2$ such that $H = \Omega_2(G)$ is metacyclic. Then one of the following holds:

(i) $H \cong C_4 \times C_2$ is the unique abelian subgroup of $G$ of type $(4, 2)$ and $G$ is isomorphic to one of the groups given in (b) and (c) of Proposition 1.5.

(ii) $H \cong C_4 \times C_4$ and $G$ is isomorphic to one of the groups given in (c) of Proposition 1.3.

(iii) $G = \langle t, c \mid t^2 = c^{2^{n+1}} = 1, n \geq 2, tc = b, b^4 = [b^2, c] = 1 \rangle$, where $|G| = 2^{n+2}$, $n \geq 2$, $H = \Omega_2^h(G) = \langle c^2, b \rangle$ with $(c^2)^b = c^{-2}$ and $H$ is a splitting metacyclic maximal subgroup, $\langle b^2 \rangle \times \langle c \rangle$ is the unique abelian maximal subgroup (of type $(2, 2^{n+1})$), $Z(G) = \langle b^2, c^{2^n} \rangle \cong E_4$, $G' = \langle c^2b^2 \rangle \cong C_{2^n}$, and $\langle t, b^2, c^{2^n} \rangle \cong E_8$ (so that $G$ is non-metacyclic).

2. **Proof of Theorem 1.9**

Let $G$ be a non-metacyclic 2-group of exponent $> 2$ such that the subgroup $H = \Omega_2^h(G)$ is metacyclic. If $H = \Omega_2(G)$, then a result of N. Blackburn (Proposition 1.4) implies that $G$ is metacyclic, a contradiction. Hence $\Omega_2(G) > H$ and so there exist involutions in $G - H$.

Suppose that $H$ is cyclic. Then $H \cong C_4$ and so $G$ has exactly one cyclic subgroup of order 4. But then Proposition 1.8 implies that $G$ is metacyclic, a contradiction. Hence $H$ is noncyclic.

Assume that $H$ is abelian (of rank 2). Since $H = \Omega_2^h(G)$, we have either $H \cong C_4 \times C_2$ or $H \cong C_4 \times C_4$.

Suppose $H \cong C_4 \times C_2$. In that case $H$ is the unique abelian subgroup of type $(4, 2)$ in $G$. Suppose this is not the case. Then there is an involution $i \in G - H$ which centralizes an element $v \in H$ of order 4, where $\langle i \rangle \times \langle v \rangle \cong C_2 \times C_4$. But then $o(iv) = 4$ and $iv \in G - H$, a contradiction. (We need this uniqueness proof so that we are able to use Proposition 1.5.) By Proposition 1.5, $G$ is isomorphic to a group given in parts (b) and (c) of that proposition.

Suppose $H \cong C_4 \times C_4$. In that case $G$ has exactly 6 cyclic subgroups of order 4 and $\Omega_2(G) > H$ and so $G$ is isomorphic to a group given in the part (c) of Proposition 1.3.

From now on we assume that $H$ is nonabelian. Suppose in addition that $H$ has a cyclic subgroup of index 2. Since $\Omega_2^h(H) = H$, we get $H \cong Q_{2^n}$, $n \geq 3$. Let $H_0 \cong Q_8$ be a quaternion subgroup of $H$ so that $C_H(H_0) = Z(H_0) = Z(H) \cong C_2$. If $C_G(H_0) \leq H_0$, then $G$ is of maximal class (Proposition 1.6)
and so $G$ is metacyclic, a contradiction. Hence $D = C_G(H_0) \not\leq H_0$ so that $D \cap H = Z(H_0) \supseteq D > Z(H_0)$, and $D$ must be elementary abelian. Let $d \in D - Z(H_0)$ and $s \in H_0$ with $o(s) = 4$. Then $o(ds) = 4$ and $ds \not\in H$, a contradiction.

Our subgroup $H = \Omega_2^3(G)$ is metacyclic nonabelian and $H$ has no cyclic subgroup of index 2 and so, by Proposition 1.2, $H$ has exactly three involutions and $\Omega_1(H) \cong E_4$. Let $Z = \langle a \rangle$ be a cyclic normal subgroup of $H$ such that $H/Z$ is cyclic and we have $|H/Z| \geq 4$. Let $K/Z$ be the subgroup of index 2 in $H/Z$. Since $\Omega_2^3(H) = H$, there is an element $b$ of order 4 in $H - K$. This implies $|H/Z| = 4$, $H = \langle a \rangle \langle b \rangle$ with $\langle a \rangle \cap \langle b \rangle = \{1\}$ and so $H$ is splitting over $Z$. We set $o(a) = 2^n$ with $n \geq 2$ since $H$ is nonabelian. Since $K = \langle a \rangle \langle b^2 \rangle$ contains exactly three involutions, $K$ is either abelian of type $(2, 2^n)$, $n \geq 2$ or $K \cong M_{2n+1}$, $n \geq 3$. In the last case, $\langle b \rangle \cong C_4$ acts faithfully on $\langle a \rangle$ and so in that case $n \geq 4$.

First assume $K \cong M_{2n+1}$, $n \geq 4$, where $\langle b \rangle$ acts faithfully on $Z = \langle a \rangle$. We have $a^b = av$ or $a^b = a^{-1}v$, where $v$ is an element of order 4 in $\langle a \rangle$. Set $v^2 = z$, where $z \in Z(H)$.

Suppose $a^b = av$ so that $H' = \langle v \rangle$ and $(a^4)^b = (av)^4 = a^4$. Since $\langle v \rangle \leq \langle a^4 \rangle$, we have $H' \leq Z(H)$. If $x, y \in H$ with $o(x) \leq 8$ and $o(y) \leq 8$, then $(xy)^8 = x^8y^8[y, x]^{28} = 1$ and so $\Omega_3(H) < H$ because $o(a) \geq 2^4$. This is a contradiction since we must have $\Omega_2^3(H) = H$ but $\Omega_2^3(H) \leq \Omega_3(H)$.

Assume $a^b = a^{-1}v$ so that $(a^2)^b = (a^{-1}v)^2 = a^{-2}z$ and $(a^4)^b = (a^{-1}v)^4 = a^{-4}$. Therefore $b$ inverts $\langle v \rangle$ and so $v^b = v^{-1}$. Also,

$$a^{b^2} = (a^{-1}v)^b = (a^{-1}v)^{-1}v^{-1} = av^{-2} = az,$$

and so $o(ba^2) = 4$. This implies $\Omega_2^3(H) \geq \langle b, a^2 \rangle$, where $L = \langle b, a^2 \rangle$ is a maximal subgroup of $H$. We claim that the set $H - L$ contains no elements of order 4 and this gives us a contradiction. Indeed, each element in $H - L$ has the form $(b^ia^2j) \in b^ia^{2i+1}$ $(i, j$ are integers). If $j = 2$, then

$$(b^ia^{2i+1})^2 = b^2a^{2i+1}b^2a^{2i+1} = b^4(a^2)^{2i+1}a^{2i+1} = (az)^{2i+1}a^{2i+1} = (a^4z)a^{2i+1},$$

which is an element of order $\geq 8$. If $j = \eta = \pm 1$, then

$$(b^\eta a^{2i+1})^2 = b^{2\eta}a^{2i+1}b^{2\eta}a^{2i+1} = b^{2\eta}(a^\eta)^{2i+1}a^{2i+1} = b^2(a^{-1}v)\eta^{2i+1}a^{2i+1} = b^2(v^\eta)v^\eta = b^2z^iv^\eta,$$

which is an element of order 4 since $[b^2, v] = 1$.

We have proved that $K = \langle b^2, a \rangle$ must be abelian of type $(2, 2^n)$, $n \geq 2$, $E_4 \cong \Omega_1(H) = \langle b^2, z \rangle \leq Z(H)$, where we have set $z = a^{2n-1}$. The element $b$ induces on $\langle a \rangle$ an involutory automorphism and so we have either $a^b = az$, $n \geq 3$ or $a^b = a^{-1}z^\epsilon$, $\epsilon = 0, 1$, $n \geq 2$ (and if $\epsilon = 1$, then $n \geq 3$).
First assume \(a^b = az, n \geq 3\), where \(H' = \langle z \rangle\) and so \(H\) is of class 2. In that case, if \(x, y \in H\) with \(o(x) \leq 4\) and \(o(y) \leq 4\), then \((xy)^4 = x^4y^4[y, x]^6 = 1\) and so \(\exp(\Omega_2(H)) = 4\). But \(o(a) = 2^n \geq 8\) and so \(\Omega_2^2(H) < \Omega_2(H) < H\), a contradiction.

We have proved that \(a^b = a^{-1}z^\epsilon, \epsilon = 0, 1, n \geq 2\), and if \(\epsilon = 1\), then \(n \geq 3\). Assume \(n = 2\) so that \(H = \langle a, b | a^4 = b^4 = 1, a^b = a^{-1}\rangle\). By Proposition 1.3(b), \(G\) is isomorphic to the following (uniquely determined) group of order \(2^5\):

\[
G = \langle b, t | b^4 = t^2 = 1, b' = ab, a^4 = 1, a^b = a^{-1}, a^t = a^{-1} \rangle,
\]

where \(\Omega_2^2(G) = \langle a, b \rangle, \Phi(G) = \langle a, b^2 \rangle \cong C_4 \times C_2,\) and \(\Omega_2(G) = G.\)

It remains to study the case \(n \geq 3\), where

\[
H = \langle a, b | a^{2^n} = b^4 = 1, n \geq 3, a^b = a^{-1}z^\epsilon, \epsilon = 0, 1, z = a^{2^n-1} \rangle,
\]

\[
H' = \langle a^2 \rangle \cong C_{2^{n-1}}, \Omega_1(H) = Z(H) = \langle b^2, z \rangle \cong E_4,\) and \(K = \langle b^2, a \rangle\) is the unique abelian maximal subgroup (of type \((2, 2^n)\)) of \(H\).

Let \(t\) be an involution in \(G - H\) and set \(L = H(t)\). Since \(\langle z \rangle = \Omega_1(H')\), \(z \in Z(G)\) and let \((v)\) be the cyclic subgroup of order 4 in \(H'\) so that \((v)\) is normal in \(G\). Note that \(v^4 = v^{-1}\) and \(C_H(v) = K\) so that \(C = C_G(v)\) covers \(G/H\). Set \(C_0 = C_L(v)\) and we see that

\[
|G : C| = |L : C_0| = 2, L = C_0 \langle b \rangle, G = C \langle b \rangle, C \cap H = K.
\]

If \(t\) does not centralize \(Z(H) = \langle b^2, z \rangle\), then \(\langle t, Z(H) \rangle \cong D_8\) and \(tb^2\) is an element of order 4 in \(\langle t, Z(H) \rangle\), a contradiction. Thus \(t\) centralizes \(Z(H)\) and so \(Z(H) \leq Z(L)\). Also, \(t\) does not centralize any element of order 4 in \(H\) and so \(C_H(t) = \langle b^2, z \rangle = \Omega_1(H)\).

Since \((v)\) is central in \(C_0\), there are no involutions in \(C - K\). But there are no elements of order 4 in \(C - K\) and so \(\Omega_2(C) = \Omega_2(K) = \langle b^2 \rangle \times \langle v \rangle \cong C_2 \times C_4\). The fact that \(C_K(t) = Z(H)\) also implies \(C_G(t) = Z(H) = \Omega_1(C)\). Note that \(Z(H) \leq Z(L)\) implies that \(Z(C_0) \geq \langle b^2, z \rangle\) and so \(Z(C_0)\) is noncyclic. By Proposition 1.1, \(C_0\) is abelian of type \((2, 2^{n+1})\).

We act with the involution \(t\) on the abelian group \(C_0\) and apply Proposition 1.7. It follows that \(t\) acts invertingly on \(C_0/\langle b^2, z \rangle\). We get \(a^t = a^{-1}s,\) where \(s \in \langle b^2, z \rangle\). Then \((ta)^2 = tata = a^{-1}sa = s\) and so \(s = 1\) since \(ta \notin H\) and \(ta\) cannot be an element of order 4. We get \(a^t = a^{-1}\) and so \(t\) acts invertingly on \(K\). On the other hand, \(b = tc_0\) with \(c_0 \in K\) and so \(a^b = a^{tc_0} = (a^{-1})^{tc_0} = a^{-1}\) because \(C_0\) is abelian. We have proved that \(\epsilon = 0\) and so \(b\) also acts invertingly on \(K\).

We show that the involution \(b^2z\) is not a square in \(H\). Indeed, for any \(x \in K\), we get

\[
(bx)^2 = bxbx = b^2x^b = b^2x^{-1}x = b^2.
\]
On the other hand, $b^2$ and $z$ are squares in $H$ and so $\langle b^2, z \rangle$ is a characteristic subgroup of $H$ and therefore $b^2 z \in Z(G)$. It follows that $Z(H) = \langle b^2, z \rangle \leq Z(G)$.

We use again Proposition 1.1 and get that $C$ is also abelian (of type $(2, 2^k), k \geq n + 1$). If $C \neq C_0$, then there is an element $d \in C_0 - K$ such that $d \in \mathcal{O}_1(C)$. By Proposition 1.7, $t$ acts invertingly on $\mathcal{O}_1(C)$ and so $d^t = t^{-1}$. But then $t$ inverts each element in $C_0$ which implies that all elements in $tC_0 = L - C_0$ are involutions. This is a contradiction since $b \in L - C_0$ and $o(b) = 4$.

We have proved that $C = C_0$ and so $G = L$. Since $t$ acts invertingly on $K$, all elements in $tK$ are involutions. But $b$ is not an involution and so $b = tc$ with a suitable element $c \in C_0 - K$ so that $o(c) = 2^{n+1}$. Since $C_0$ is abelian, we have $[b^2, c] = 1$. We have obtained the following group of order $2^{n+3}$:

\begin{equation}
G = \langle c, t \mid c^{2^{n+1}} = t^2 = 1, \; n \geq 3, \; tc = b, \; b^4 = [b^2, c] = 1 \rangle,
\end{equation}

where $\Omega_2^\mu(G) = \langle c^2, b \rangle$ with $(c^2)^b = c^{-2}$.

If we set $n = 2$ in (2.2), we get a group $G$ of order $2^5$ with

$$
\Omega_2^\mu(G) = \langle c^2, b \mid (c^2)^4 = b^4 = 1, \; (c^2)^b = c^{-2} \rangle
$$

and $\Omega_2(G) = G$ and so this group $G$ (because Proposition 1.3(b) implies the uniqueness of such a group) must be isomorphic to the group given in (2.1).

We have obtained the groups given in part (iii) of our theorem for all $n \geq 2$.

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