Triangular Patches

1 Introduction

The presented surface definition is based on a classical interpolation method, where the constructed function of two variables has given values on the boundary of a given triangle. The original formulation of the solution of this interpolation problem is the following [1].

If the real-valued function $F(x, y)$ is continuous on the triangle $T$ with vertices $(0,0), (1,0)$ and $(0,1)$ in the xy plane, then the function given by

$$W(x, y) = \frac{1}{2} \left\{ \begin{array}{l}
\frac{1-x-y}{1-y}F(0, y) + \frac{x}{1-y}F(1-y, y) \\
\frac{1-x-y}{1-x}F(x, 0) + \frac{y}{1-x}F(x, 1-x) \\
\frac{x}{x+y}F(x+y, 0) + \frac{y}{x+y}F(0, x+y) \\
- [xF(1,0) + yF(0,1) + (1-x-y)F(0,0)]
\end{array} \right\}$$

is continuous over $T$ and interpolates to the values of $F$ on its boundary, i.e. along the curves $x = 0$, $y = 0$ and $1-x-y = 0$ [3, §8.2].

A geometric interpretation of this interpolation problem is the construction of Gordon–Coons triangular surface patches, which is the triangular version of the well-known construction of rectangular Coons patches [2] extended in [6]. A Gordon–Coons surface patch is generated by the above formula from three continuous curve segments forming a spatial curvilinear triangle, which are the boundary curves of the generated patch. The boolean sum of convex combinations of three pairs of the given curves is corrected with a convex combination of the vertex points (Fig 1).

![Figure 1: The parameter triangle and boundary curves.](image-url)
be transformed affinely, and barycentric coordinates can be used with respect to a base triangle with the vertices
\((0, 0, 1), (1, 0, 0)\) and \((0, 1, 0)\) [4, §18]. The parameter triangle is determined by \(0 \leq u, v, w \leq 1\) and \(u + v + w = 1\). The three continuous input curves are defined on the boundaries of the parameter triangle. Their representing vector functions are expressed with barycentric coordinates written in a symmetric form:
\[ g_1(0, v, 1 - v) \text{ is defined over the edge } u = 0, g_2(u, 0, 1 - u) \text{ over the edge } v = 0 \text{ and } g_3(u, 1 - u, 0) \text{ over the edge } w = 0, \]
which can be written also as \(g_3(1 - v, v, 0)\) substituting \(u = 1 - v\).

If the three curves satisfy the boundary conditions
\[ g_2(1, 0, 0) = g_3(1, 0, 0) = P_1, \]
\[ g_1(0, 1, 0) = g_3(0, 1, 0) = P_2 \text{ and } g_1(0, 0, 1) = g_2(0, 0, 1) = P_3, \]
then the surface patch given by the vector function
\[ r(u, v, w) = \frac{1}{2} \left\{ \begin{array}{l}
\frac{w}{u + w} g_1(0, v, 1 - v) + \frac{u}{u + w} g_3(1 - v, v, 0) \\
+ \frac{w}{v + w} g_2(u, 0, 1 - u) + \frac{v}{v + w} g_3(u, 1 - u, 0) \\
+ \frac{u}{u + v} g_2(u + v, v, 1 - u - v) \\
+ \frac{v}{u + v} g_3(0, u + v, 1 - u - v) \\
- \left[ u g_3(1, 0, 0) + v g_3(0, 1, 0) + w g_2(0, 0, 1) \right] \end{array} \right\}, \]
\[ 0 \leq u, v, w \leq 1, \quad u + v + w = 1 \]
(1)

interpolates the input curves along the edges of the parame-
ter triangle.

The other surface definition, which we use in our surface construction, was given for the construction of a \(C^1\) con-
tinuous triangular interpolant in [5] as follows. If three functions \(F_i, i = 1, 2, 3\) are \(C^2\) differentiable on the triangle \(T\) described with the barycentric coordinates \(u, v\) and \(w\), \(0 \leq u, v, w \leq 1, u + v + w = 1\), and each of them interpolates one vertex of \(T\) and a vector field along its op-
posite side, then the function given by
\[ D = \frac{u^2 w^2 F_1 + v^2 w^2 F_2 + u^2 v^2 F_3}{u^2 w^2 + v^2 w^2 + u^2 v^2} \]
(2)
is differentiable, and interpolates the values and the first partial derivatives of the given “underlying” surfaces \(F_1, F_2\) and \(F_3\) (consequently, also the given vector fields) along the edge \(u = 0, v = 0\) and \(w = 0\) of the triangle, respectively.

This convex combination scheme was applied and inves-
tigated for three differentiable triangular surface patches de-
fin on the same parameter domain in [7]. However, the problem ensuring the compatibility conditions for the input sur-
faces at the corner points is not solved in general.

\[ \lambda_1 = \frac{1 - \lambda_1 u^2}{\lambda_1 u^2 + (1 - \lambda_1) w^2}, \]
\[ \lambda_2 = \frac{1 - \lambda_2 v^2}{\lambda_2 v^2 + (1 - \lambda_2) u^2}, \]
\[ \lambda_3 = \frac{1 - \lambda_3 w^2}{\lambda_3 w^2 + (1 - \lambda_3) v^2}, \]
where \(0 \leq \lambda_1, \lambda_2, \lambda_3 \leq 1\) are shape parameters of values between 0 and 1, and at the corner points
\[ \lambda_1(0, 1, 0) := 1, \quad \lambda_2(0, 0, 1) := 1, \quad \lambda_3(1, 0, 0) := 1 \]
are required.

**Definition 1.** The surface patch is defined by the vector function
\[ f(u, v, w) = \frac{1}{2} [\lambda_1 r_1 + (1 - \lambda_1) r_3 + \mu_2 r_2 + (1 - \mu_2) r_1 + \mu_3 r_3 + (1 - \mu_3) r_2 - q(u, v, w)], \]
(3)
where \[ q(u, v, w) = \frac{u^2 w^2 g_1(0, v, 1 - v) + v^2 w^2 g_2(u, 0, 1 - u) + u^2 v^2 g_3(u, 1 - u, 0)}{u^2 w^2 + v^2 w^2 + u^2 v^2} \]
\[ 0 \leq u, v, w \leq 1, \quad u + v + w = 1 \]
(4)
is a correction term generated from the auxiliary curves \( g_1, g_2 \) and \( g_3 \) over the boundaries of the parameter triangle.

\[
\begin{align*}
g_1(0,v,1-v) &= [\mu_3 r_3 + (1-\mu_3) r_2](0,v,1-v), \\
g_2(u,0,1-u) &= [\mu_1 r_1 + (1-\mu_1) r_3](u,0,1-u), \\
g_3(u,1-u,0) &= [\mu_2 r_2 + (1-\mu_2) r_1](u,1-u,0)
\end{align*}
\]

are blended curves over the sides \( u = 0, v = 0 \) and \( w = 0 \), respectively of the triangular parameter domain. \( \circ \)

The surface \( f(u,v,w) \) matches the boundary curves \( r_1(0,v,1-v), r_2(u,0,1-u) \) and \( r_3(u,1-u,0), 0 \leq u \leq 1, 0 \leq v \leq 1 [8] \).

The structure of this scheme is similar to that of Gordon–Coons’ construction, where the boolean sum of three convex combinations of the given constituents is corrected according to the interpolation condition. Here the correction function has the structure of the scheme in (2) and fits the auxiliary curves:

\[
\begin{align*}
q(0,v,1-v) &= g_1(0,v,1-v), \\
q(u,0,1-u) &= g_2(u,0,1-u), \\
q(1-v,v,0) &= q(u,1-u,0) = g_3(1-v,v,0) = g_3(u,1-u,0).
\end{align*}
\]

The connection between the resulting surface and the input surface constituents is \( C^0 \) along the common boundary curves.

The shape parameters \( (\lambda_1, \lambda_2, \lambda_3) = \lambda \) are either specified by the user, or can be determined from a fairing condition. We have used the linearized thin plate energy function with

\[
E(\lambda) = \int_A \left( \frac{\partial^2}{\partial u^2} + 2 \frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial u v} \right) dA, \quad A = [0,1] \times [0,1]. \tag{5}
\]

The optimal values of \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are computed by minimizing \( E(\lambda) \). (In the equations the indices \( u \) and \( v \) denote the partial derivatives with respect to \( u \) and \( v \), respectively.) The integral has been approximated by an integral sum computed at 9 inner points, and the numerical minimization has been carried out by the symbolic algebraic program package Mathematica.

For drawing triangular patches with Mathematica the parameter triangle had to be transformed into a rectangle by substituting \( u = t - st, v = st, s,t \in [0,1] \). Therefore, the patches appear in the figures with \( s \) and \( t \) parameter lines.

2 Examples

In Fig 3 three triangular surface patches are shown, which are defined as quadratic Bézier surfaces. One auxiliary curve and the correction term \( q(u,v,w) \) is shown in Fig 4 and in Fig 5, respectively. The resulting surface defined in (3) is shown in Fig 6. It joins to the input surfaces with \( C^0 \) continuity along their connection curves.
3 $C^1$ continuous blending surface constructed from differentiable patches

In this chapter a new definition of a triangular Gordon–Coons-type surface patch will be given. It is determined by three differenctiable triangular patches, where three boundary curves, one of each patch, form a curvilinear triangle. The resulting patch fits these boundary curves and has a $C^1$-continuous connection to the given constituents along them.

Now we investigate the partial derivatives of the vector function defined in (3) along the edges of the parameter triangle $T$. Computing the partial derivatives with barycentric coordinates we get the following.

Along the edge $u = 0$ $f_u = r_{1u}$, $f_w = r_{1w}$ and

$$f_u = \left[ r_{1u} + \frac{1}{2} (\mu_1 r_{3u} + (1-\mu_1) r_{2u}) \right] \bigg|_{u=0}. \quad (6)$$

Along the edge $v = 0$ $f_u = r_{2v}$, $f_w = r_{2w}$ and

$$f_v = \left[ r_{2v} + \frac{1}{2} (\mu_1 r_{1v} + (1-\mu_1) r_{3v}) \right] \bigg|_{v=0}. \quad (7)$$

Along the edge $w = 0$ $f_u = r_{3u}$, $f_v = r_{3v}$ and

$$f_w = \left[ r_{3w} + \frac{1}{2} (\mu_2 r_{2w} + (1-\mu_2) r_{1w}) \right] \bigg|_{w=0}. \quad (8)$$

In order to get $C^1$ continuous connection between the resulting surface represented by $f(u,v,w)$ and the constituents

$$f_u \big|_{u=0} = r_{1u} \big|_{u=0}, \quad f_v \big|_{v=0} = r_{2v} \big|_{v=0}, \quad f_w \big|_{w=0} = r_{3w} \big|_{w=0},$$

must be ensured. For this an additional correction term is needed in Definition 1. Its value has to be zero along the boundary curves, and its partial derivatives have to annihilate the second terms of the partial derivatives in the expressions (6), (7) and (8). The following vector function satisfies these requirements

$$s(u,v,w) = \frac{1}{2} \left[ \kappa_1 (\mu_2 r_{3u} + (1-\mu_3) r_{2u}) \bigg|_{u=0} + \kappa_2 (\mu_1 r_{1v} + (1-\mu_1) r_{3v}) \bigg|_{v=0} + \kappa_3 (\mu_2 r_{2w} + (1-\mu_2) r_{1w}) \bigg|_{w=0} \right]. \quad (9)$$

with the blending functions

$$\kappa_1 = \frac{u^2 v^2}{\Sigma}, \quad \kappa_2 = \frac{v^2 w^2}{\Sigma}, \quad \kappa_3 = \frac{w^2 u^2}{\Sigma}, \quad \Sigma = u^2 w^2 + v^2 w^2 + u^2 v^2. \quad (10)$$

Obviously,

$$\kappa_i \big|_{u=0} = \kappa_i \big|_{v=0} = \kappa_i \big|_{w=0} = 0, \quad i = 1, 2, 3,$$

$$\kappa_1 u \big|_{u=0} = 1, \quad \kappa_1 v \big|_{v=0} = 0, \quad \kappa_1 w \big|_{w=0} = 0,$$

$$\kappa_2 u \big|_{u=0} = 0, \quad \kappa_2 v \big|_{v=0} = 1, \quad \kappa_2 w \big|_{w=0} = 0,$$

$$\kappa_3 u \big|_{u=0} = 0, \quad \kappa_3 v \big|_{v=0} = 0, \quad \kappa_3 w \big|_{w=0} = 1.$$

The required surface is defined by extending Definition 1 in the following way.

**Definition 2.**

$$s(u,v,w) = \frac{1}{2} \left[ \mu_1 r_1 + (1-\mu_1) r_3 + \mu_2 r_2 + (1-\mu_2) r_1 \right. \left. + \mu_3 r_3 + (1-\mu_3) r_2 - q(u,v,w) - s(u,v,w), \right.$$ \[0 \leq u, v, w \leq 1, \quad u + v + w = 1, \quad (11)

where $q(u,v,w)$ is defined in (4), $s(u,v,w)$ in (9) with the weighting functions in (10).

Considering the computed derivatives, we have obtained the following theorem.

**Theorem 1.** Assume that three surface patches are given by the differentiable vector functions $r_1(u,v,w)$, $r_2(u,v,w)$ and $r_3(u,v,w)$ on the parameter triangle $0 \leq u, v, w \leq 1$ with common corner points, i.e.

$$r_1(0,0,1) = r_2(0,0,1),$$

$$r_1(0,1,0) = r_3(0,1,0),$$

$$r_2(1,0,0) = r_3(1,0,0).$$

Then the surface represented by the vector function in Definition 2 interpolates the boundary curves $r_1 \big|_{u=0}$, $r_2 \big|_{v=0}$ and $r_3 \big|_{w=0}$ and joins to the corresponding surface patch $C^1$ continuously along the common boundary with the exception of the corner points.

**Proof.** The proof follows from the computations above. However, the compatibility conditions of the differentiability of the resulting surface at the vertices require further investigations. $\square$

### 4 Examples

Figure 7: $C^1$ continuous surface defined from the constituents in Fig 3.
In Fig 7 the surface constructed by Definition 2 is shown. It is generated from the same quadratic Bézier patches as the surface in Fig 6. There is a visible difference between the $C^0$ and $C^1$ results. While the $C^0$ surface is rather round, and intersects the constituents, the $C^1$ result has common tangent planes with them along the common boundary curves. The next two figures illustrate the effect of the shape parameters $\lambda_i$ included in the blending coefficients $\mu_i$, $i = 1, 2, 3$. In the equation of the resulting surface in Fig 7 the shape parameters have been determined from the fairing condition by minimizing the energy function in (5). The same surface is shown from a side view in Fig 8.

In Fig 9 the surface is generated from the same constituents, but the shape parameters have been given as user inputs. The value of $\lambda_3$ influencing the weight of the given patch on the right hand side has been raised. Consequently, the result is less concave in the middle.

In Fig 10 an open corner on a prism is shown modelled with triangular Bézier patches. The boundary of the triangular hole is drawn with heavy lines. The constituents in the surface definition are in the inside of this triangle. The neighbouring triangles are coplanar extensions of them.

In Fig 12 the same surface is shown from a side view in order to make the comparison with the next examples easier. The next figures show the shaping effect of the constituents. In Fig 13 different constituents with the same boundary curves and tangent planes are shown, the resulting $C^1$ surface is shown in Fig 14.
M. Szilvási-Nagy, I. Szabó: $C^1$-Continuous Coons-type Blending of Triangular Patches

5 Conclusions

We have presented a new surface definition, which generates a triangular patch from three triangular surface patches. Novel in this definition is that the inner shape of the resulting surface can be modified by changing the input surface patches while keeping the boundary conditions fixed. Moreover, new is the introduction of shape parameters in the blending functions. This surface construction can be applied for filling triangular holes which occur in modelling of composite surfaces.

References


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