A strong form of $\beta-I$-continuous functions

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Abstract. In this paper, $\beta-I$-open sets are used to define and investigate a new class of functions called strongly $\beta-I$-continuous functions in ideal topological spaces.

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1. Introduction

The subject of ideals in topological spaces has been introduced and studied by Kuratowski [5] and Vaidyanathasamy [6]. An ideal $I$ on a topological space $(X, \tau)$ is a nonempty collection of subsets of $X$ which satisfies (i) $A \in I$ and $B \subset A$ implies $B \in I$ and (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given a topological space $(X, \tau)$ with an ideal $I$ on $X$ and if $\mathcal{P}(X)$ is the set of all subsets of $X$, a set operator $(\cdot)^*$: $\mathcal{P}(X) \rightarrow \mathcal{P}(X)$, called the local function [6] of $A$ with respect to $\tau$ and $I$, is defined as follows: for $A \subset X$, $A^*(\tau, I) = \{x \in X \mid U \cap A \notin I$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. A Kuratowski closure operator $\text{Cl}^*(\cdot)$ for a topology $\tau^*(\tau, I)$ called the $\tau$-topology, which is finer than $\tau$ is defined by $\text{Cl}^*(A) = A \cup A^*(\tau, I)$. When there is no chance of confusion, $A^*(I)$ is denoted by $A^*$. If $I$ is an ideal on $X$, then $(X, \tau, I)$ is called an ideal topological space. By a space, we always mean a topological space $(X, \tau)$ with no separation properties assumed. If $A \subset X$, $\text{Cl}(A)$ and $\text{Int}(A)$ will denote the closure and interior of $A$ in $(X, \tau)$, respectively.

A point $x \in X$ is called a $\theta$-cluster point of $A$ if $\text{Cl}(V) \cap A \neq \emptyset$ for every open set $V$ of $X$ containing $x$. The set of all $\theta$-cluster points of $A$ is said to be the $\theta$-closure of $A$ [7] and is denoted by $\text{Cl}_\theta(A)$. If $A = \text{Cl}_\theta(A)$, then the set $A$ is said to be $\theta$-closed [7]. The complement of a $\theta$-closed set is said to be $\theta$-open [7]. The union of all $\theta$-open sets contained in a subset $A$ is called the $\theta$-interior of $A$ and is denoted by $\text{Int}_\theta(A)$. It follows from [7] that the collection of $\theta$-open sets in a topological space $(X, \tau)$ forms a topology $\tau_\theta$ on $X$. In this paper, the concept of strongly $\beta-I$-continuous functions is introduced and studied. Some of their characteristic properties are investigated.

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2. Preliminaries

A subset $S$ of an ideal topological space $(X, \tau, I)$ is said to be $\beta$-$I$-open [4] (resp. $\alpha$-$I$-open [4]) if $S \subset \text{Cl}(\text{Int}(\text{Cl}^*(S)))$ (resp. $S \subset \text{Int}(\text{Cl}^*(S)))$. The complement of a $\beta$-$I$-open set is called $\beta$-$I$-closed [4]. The intersection of all $\beta$-$I$-closed sets containing $S$ is called the $\beta$-$I$-closure of $S$ and is denoted by $\beta\text{Cl}(S)$. The $\beta$-$I$-interior of $S$ is defined by the union of all $\beta$-$I$-open sets contained in $S$ and is denoted by $\beta\text{Int}(S)$. A subset $S$ of an ideal topological space $(X, \tau, I)$ is said to be $\beta$-$I$-regular [8] if it is both $\beta$-$I$-open and $\beta$-$I$-closed. The family of all $\beta$-$I$-regular (resp. $\beta$-$I$-open, $\beta$-$I$-closed, $\alpha$-$I$-open) sets of $(X, \tau, I)$ is denoted by $\beta\text{IR}(X)$ (resp. $\beta\text{IO}(X)$, $\beta\text{IC}(X)$, $\alpha\text{IO}(X)$). The family of all $\beta$-$I$-regular (resp. $\beta$-$I$-open, $\beta$-$I$-closed) sets of $(X, \tau, I)$ containing a point $x \in X$ is denoted by $\beta\text{IR}(X, x)$ (resp. $\beta\text{IO}(X, x)$, $\beta\text{IC}(X, x)$). A point $x \in X$ is called the $\beta$-$I$-$\theta$-cluster point of $S$ if $\beta\text{Cl}(U) \cap S \neq \emptyset$ for every $\beta$-$I$-open set $U$ of $(X, \tau, I)$ containing $x$. The set of all $\beta$-$I$-$\theta$-cluster points of $S$ is called the $\beta$-$I$-closure of $S$ and is denoted by $\beta\text{Cl}_\theta(S)$.

A subset $S$ is said to be $\beta$-$I$-$\theta$-open if its complement is $\beta$-$I$-$\theta$-closed. A point $x \in X$ is called the $\beta$-$I$-$\theta$-interior point of $S$ if there exists a $\beta$-$I$-regular set $U$ of $X$ containing $x$ such that $x \in U \subset S$. The set of all $\beta$-$I$-$\theta$-interior points of $S$ and is denoted by $\beta\text{Int}_\theta(S)$.

**Definition 1.** A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be $\beta$-$I$-continuous (see [4]) if $f^{-1}(V) \in \beta\text{IO}(X)$ for every $V \in \sigma$, or equivalently, $f^{-1}(V) \in \beta\text{IC}(X)$ for every closed set $V$ of $Y$.

**Theorem 1** (see [4]). A function $f : (X, \tau, I) \to (Y, \sigma)$ is $\beta$-$I$-continuous if and only if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$ there exists $U \in \beta\text{IO}(X, x)$ such that $f(U) \subset V$.

3. Strongly $\beta$-$I$-continuous functions

We have introduced the following definition

**Definition 2.** A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be strongly $\beta$-$I$-continuous if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists $U \in \beta\text{IO}(X, x)$ such that $f(\beta\text{Cl}(U)) \subset \text{Cl}(V)$.

**Theorem 2.** Every $\beta$-$I$-continuous function is strongly $\beta$-$I$-continuous.

**Proof.** Suppose that $x \in X$ and $V$ is any open set of $Y$ containing $f(x)$. Since $f$ is $\beta$-$I$-continuous and $\text{Cl}(V)$ is closed in $Y$, $f^{-1}(V)$ is $\beta$-$I$-open and $f^{-1}(\text{Cl}(V))$ is $\beta$-$I$-closed in $X$. Now, put $U = f^{-1}(V)$. Then we have $U \in \beta\text{IO}(X, x)$ and $\beta\text{Cl}(U) \subset f^{-1}(\text{Cl}(V))$. Therefore, we obtain $f(\beta\text{Cl}(U)) \subset \text{Cl}(V)$. This shows that $f$ is strongly $\beta$-$I$-continuous.

The converse of Theorem 2 is not true as it can been seen from the following example.

**Example 1.** Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{b\}, X\}$ and $I = \{\emptyset, \{a\}\}$. Then the identity function $f : (X, \tau, I) \to (X, \sigma)$ is strongly $\beta$-$I$-continuous but not $\beta$-$I$-continuous.
Theorem 3. For a function $f : (X,\tau,I) \to (Y,\sigma)$ the following properties are equivalent:

(i) $f$ is strongly $\beta$-$I$-continuous;

(ii) $\beta I \text{Cl}_\theta(f^{-1}(B)) \subset f^{-1}(\text{Cl}_\theta(B))$ for every subset $B$ of $Y$;

(iii) $f(\beta I \text{Cl}_\theta(A)) \subset \text{Cl}_\theta(f(A))$ for every subset $A$ of $X$.

Proof. (i)$\Rightarrow$(iii): Let $B$ be any subset of $Y$. Suppose that $x \notin f^{-1}(\text{Cl}_\theta(B))$. Then $f(x) \notin \text{Cl}_\theta(B)$ and there exists an open set $V$ of $Y$ containing $f(x)$ such that $\text{Cl}(V) \cap B = \varnothing$. Since $f$ is strongly $\beta$-$I$-continuous, there exists $U \in \beta I O(X,x)$ such that $f(\beta I \text{Cl}(U)) \subset \text{Cl}(V)$. Therefore, we have $f(\beta I \text{Cl}(U)) \cap B = \emptyset$ and $\beta I \text{Cl}(U) \cap f^{-1}(B) = \emptyset$. This shows that $x \notin \beta I \text{Cl}_\theta(f^{-1}(B))$. Hence, we obtain $f(\beta I \text{Cl}_\theta(f^{-1}(B)) \subset f^{-1}(\text{Cl}_\theta(B))$.

(ii)$\Rightarrow$(iii): Let $A$ be any subset of $X$. Then we have

$$\beta I \text{Cl}_\theta(A) \subset \beta I \text{Cl}_\theta(f^{-1}(f(A))) \subset f^{-1}(\text{Cl}_\theta(f(A)))$$

and hence $f(\beta I \text{Cl}_\theta(A)) \subset \text{Cl}_\theta(f(A))$.

(iii)$\Rightarrow$(ii): Let $B$ be a subset of $Y$. We have $f(\beta I \text{Cl}_\theta(f^{-1}(B))) \subset \text{Cl}_\theta(f(f^{-1}(B))) \subset \text{Cl}_\theta(B)$ and hence $\beta I \text{Cl}_\theta(f^{-1}(B)) \subset f^{-1}(\text{Cl}_\theta(B))$.

(ii)$\Rightarrow$(i): Let $x \in X$ and $V$ be an open set of $Y$ containing $f(x)$. Then we have

$$\text{Cl}(V) \cap (Y - \text{Cl}(V)) = \emptyset$$

and hence $x \notin \beta I \text{Cl}_\theta(Y - \text{Cl}(V))$. Hence, $x \notin f^{-1}(\text{Cl}_\theta(Y - \text{Cl}(V)))$ and $x \notin \beta I \text{Cl}(U) \cap f^{-1}(f^{-1}(B)) = \emptyset$. Therefore, we obtain $f(\beta I \text{Cl}_\theta(f^{-1}(B)) \subset f^{-1}(\text{Cl}_\theta(B))$.

Theorem 4. For a function $f : (X,\tau,I) \to (Y,\sigma)$ the following properties are equivalent:

(i) $f$ is strongly $\beta$-$I$-continuous;

(ii) $f^{-1}(V) \subset \beta I \text{Int}_\theta(f^{-1}(\text{Cl}(V)))$ for every open set $V$ of $Y$;

(iii) $\beta I \text{Cl}_\theta(f^{-1}(V)) \subset f^{-1}(\text{Cl}(V))$ for every open set $V$ of $Y$.

Proof. (i)$\Rightarrow$(ii): Suppose that $V$ is any open set of $Y$ and $x \notin f^{-1}(f^{-1}(V))$. Then $f(x) \notin V$ and there exists $U \in \beta I O(X,x)$ such that $f(\beta I \text{Cl}(U)) \subset \text{Cl}(U)$. Therefore, $x \notin U \subset \beta I \text{Cl}(U) \subset f^{-1}(\text{Cl}(V))$. This shows that $x \notin \beta I \text{Int}_\theta(f^{-1}(\text{Cl}(V)))$ for any open set $V$ of $Y$. In consequence, $f^{-1}(V) \subset \beta I \text{Int}_\theta(f^{-1}(\text{Cl}(V)))$.

(ii)$\Rightarrow$(iii): Suppose that $V$ is any open set of $Y$ and $x \notin f^{-1}(\text{Cl}(V))$. Then $f(x) \notin \text{Cl}(V)$. It follows that there exists an open set $U$ of $Y$ such that $U \cap V = \emptyset$ and hence $\text{Cl}(U) \cap V = \emptyset$. Therefore, we have $f^{-1}(\text{Cl}(U)) \cap f^{-1}(V) = \emptyset$. Since $x \notin f^{-1}(U)$, by (ii), $x \notin \beta I \text{Int}_\theta(f^{-1}(\text{Cl}(V)))$. In consequence, there exists $W \in \beta I O(X,x)$ such that $\beta I \text{Cl}(W) \subset f^{-1}(\text{Cl}(U))$. Thus, we have $\beta I \text{Cl}(W) \cap f^{-1}(V) = \emptyset$ and hence $x \notin \beta I \text{Cl}_\theta(f^{-1}(V))$. This shows that $\beta I \text{Cl}_\theta(f^{-1}(V)) \subset f^{-1}(\text{Cl}(V))$.

(iii)$\Rightarrow$(i): Suppose that $x \in X$ and $V$ is any open set of $Y$ containing $f(x)$. Then, $V \cap (Y - \text{Cl}(V)) = \emptyset$ and $f(x) \notin \text{Cl}(Y - \text{Cl}(V))$. Therefore, $x \notin f^{-1}(\text{Cl}(Y - \text{Cl}(V)))$ and (iii), $x \notin \beta I \text{Cl}_\theta(f^{-1}(Y - \text{Cl}(V)))$. In consequence, there exists $U \in \beta I O(X,x)$ such that $\beta I \text{Cl}(U) \cap f^{-1}(Y - \text{Cl}(V)) = \emptyset$. Therefore, we obtain $f(\beta I \text{Cl}(U)) \subset \text{Cl}(V)$. This shows that $f$ is strongly $\beta$-$I$-continuous.\qed
**Definition 3.** A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be strongly $\theta$-$\beta$-$\mathcal{I}$-continuous (see [9]) if for each point $x \in X$ and any open set $V$ of $Y$ containing $f(x)$, there exists $U \in \beta\mathcal{I}O(X, x)$ such that $f(\beta\mathcal{I}\text{Cl}(U)) \subset V$.

**Theorem 5.** Let $Y$ be a regular space. Then for a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ the following properties are equivalent:

(i) $f$ is strongly $\theta$-$\beta$-$\mathcal{I}$-continuous;

(ii) $f$ is $\beta$-$\mathcal{I}$-continuous;

(iii) $f$ is strongly $\beta$-$\mathcal{I}$-continuous.

**Proof.** (i)$\Rightarrow$(ii): This is obvious.
(iii)$\Rightarrow$(i): Suppose that $x \in X$ and $V$ is any open set of $Y$ containing $f(x)$. Since $Y$ is regular, there exists an open set $W$ of $Y$ such that $f(x) \in W \subset \text{Cl}(W) \subset V$. Since $f$ is strongly $\beta$-$\mathcal{I}$-continuous, there exists $U \in \beta\mathcal{I}O(X, x)$ such that $f(\beta\mathcal{I}\text{Cl}(U)) \subset \text{Cl}(W) \subset V$. This shows that $f$ is strongly $\theta$-$\beta$-$\mathcal{I}$-continuous. \qed

**Definition 4** (see [3]). Let $A$ and $X_0$ be subsets of an ideal topological space $(X, \tau, \mathcal{I})$ such that $A \subset X_0 \subset X$. Then $(X_0, \tau|_{X_0}, \mathcal{I}|_{X_0})$ is an ideal topological space with an ideal $\mathcal{I}|_{X_0} = \{I \in \mathcal{I} | I \subset X_0\} = \{I \cap X_0 | I \in \mathcal{I}\}$.

**Lemma 1** (see [9]). Let $A$ and $X_0$ be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. Then the following properties hold:

(i) If $A \in \beta\mathcal{I}O(X)$ and $X_0 \in \alpha\mathcal{I}O(X)$, then $A \cap X_0 \in \beta\mathcal{I}O(X_0)$;

(ii) If $A \in \beta\mathcal{I}O(X_0)$ and $X_0 \in \alpha\mathcal{I}O(X)$, then $A \in \beta\mathcal{I}O(X)$.

**Lemma 2** (see [9]). Let $A$ and $X_0$ be subsets of an ideal topological space $(X, \tau, \mathcal{I})$ such that $A \subset X_0 \subset X$. Let $\beta\mathcal{I}\text{Cl}_{X_0}(A)$ denote the $\beta$-$\mathcal{I}$-closure of $A$ with respect to the subspace $X_0$. Then

(i) If $X_0$ is $\alpha$-$\mathcal{I}$-open in $X$, then $\beta\mathcal{I}\text{Cl}_{X_0}(A) \subset \beta\mathcal{I}\text{Cl}(A)$;

(ii) If $A \in \beta\mathcal{I}O(X_0)$ and $X_0 \in \alpha\mathcal{I}O(X)$, then $\beta\mathcal{I}\text{Cl}(A) \subset \beta\mathcal{I}\text{Cl}_{X_0}(A)$.

**Theorem 6.** If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is strongly $\beta$-$\mathcal{I}$-continuous and $X_0$ is an $\alpha$-$\mathcal{I}$-open subset of $X$, then the restriction $f|_{X_0} : (X_0, \tau|_{X_0}, \mathcal{I}|_{X_0}) \to (Y, \sigma)$ is strongly $\beta$-$\mathcal{I}|_{X_0}$-continuous.

**Proof.** For any $x \in X_0$ and any open set $V$ of $Y$ containing $f(x)$, there exists $U \in \beta\mathcal{I}O(X, x)$ such that $f(\beta\mathcal{I}\text{Cl}(U)) \subset \text{Cl}(V)$ since $f$ is strongly $\beta$-$\mathcal{I}$-continuous. Let $U_0 = U \cap X_0$, then by Lemmas 1 and 2, $U_0 \in \beta\mathcal{I}O(X_0, x)$ and $\beta\mathcal{I}\text{Cl}_{X_0}(U_0) \subset \beta\mathcal{I}\text{Cl}(U_0)$. Therefore, we obtain

$$(f|_{X_0})(\beta\mathcal{I}\text{Cl}_{X_0}(U_0)) = f(\beta\mathcal{I}\text{Cl}_{X_0}(U_0)) \subset f(\beta\mathcal{I}\text{Cl}(U_0)) \subset f(\beta\mathcal{I}\text{Cl}(U)) \subset \text{Cl}(V).$$

This shows that $f|_{X_0}$ is strongly $\beta$-$\mathcal{I}|_{X_0}$-continuous. \qed
Theorem 7. A function \( f : (X, \tau, I) \rightarrow (Y, \sigma) \) is strongly \( \beta-I \)-continuous if for each \( x \in X \) there exists \( X_0 \in \alpha I O(X, x) \) such that the restriction \( f|_{X_0} : (X_0, \tau|_{X_0}, I|_{X_0}) \rightarrow (Y, \sigma) \) is strongly \( \beta-I \)-continuous.

Proof. Let \( x \in X \) and \( V \) be an open set of \( Y \) containing \( f(x) \). There exists \( X_0 \in \alpha I O(X, x) \) such that \( f|_{X_0} : (X_0, \tau|_{X_0}, I|_{X_0}) \rightarrow (Y, \sigma) \) is strongly \( \beta-I \)-continuous. Thus, there exists \( U \in \beta I O(X_0, x) \) such that \( f|_{X_0} (\beta \text{Cl}(X_0(U))) \subset \text{Cl}(V) \). By Lemmas 1 and 2, \( U \in \beta I O(X, x) \) and \( \beta \text{Cl}(U) \subset \beta \text{Cl}(X_0(U)) \). Hence, we have \( f(\beta \text{Cl}(U)) = (f|_{X_0}) (\beta \text{Cl}(U)) \subset (f|_{X_0}) (\beta \text{Cl}(X_0(U))) \subset \text{Cl}(V) \). This shows that \( f \) is strongly \( \beta-I_{|x_0} \)-continuous.

Corollary 1. Let \( \{U_\lambda : \lambda \in \Omega\} \) be an \( \alpha-I \)-open cover of an ideal topological space \( (X, \tau, I) \). A function \( f : (X, \tau, I) \rightarrow (Y, \sigma) \) is strongly \( \beta-I \)-continuous if and only if the restriction \( f|_{U\lambda} : (U_\lambda, \tau|_{U\lambda}, I|_{U\lambda}) \rightarrow (Y, \sigma) \) is strongly \( \beta-I_{|x_0} \)-continuous for each \( \lambda \in \Omega \).

Proof. The proof follows from Theorems 6 and 7.

Definition 5. An ideal topological space \( (X, \tau, I) \) is said to be:

(i) \( \beta-I \)-closed (resp. \( \beta-I \)-Lindelof) if every cover of \( X \) by \( \beta-I \)-open sets has a finite (resp. countable) subcover whose \( \beta-I \)-closures cover \( X \);

(ii) countably \( \beta-I \)-closed if every countable cover of \( X \) by \( \beta-I \)-open sets has a finite subcover whose \( \beta-I \)-closures cover \( X \).

Definition 6. A topological space \( (X, \tau) \) is said to be:

(i) quasi \( H \)-closed (see [7]) (resp. almost Lindelof [2]) if every cover of \( X \) by open sets has a finite (resp. countable) subfamily whose closures cover \( X \),

(ii) lightly compact (see [1]) if every countable cover of \( X \) by open sets has a finite subfamily whose closures cover \( X \).

Definition 7. A subset \( K \) of an ideal topological space \( (X, \tau, I) \) is said to be \( \beta-I \)-closed relative to \( X \) if for every cover \( \{V_\lambda : \lambda \in \Omega\} \) of \( K \) by \( \beta-I \)-open subsets of \( X \), there exists a finite subset \( \Omega_0 \) of \( \Omega \) such that \( K \subset \bigcup \{\beta \text{Cl}(V_\lambda) : \lambda \in \Omega_0\} \) (resp. \( K \subset U \{\text{Cl}(V_\lambda) : \alpha \in \Omega_0\} \)).

Definition 8. A subset \( K \) of a topological space \( (X, \tau) \) is said to be quasi \( H \)-closed relative to \( X \) (see [7]) if every cover \( \{V_\lambda : \lambda \in \Omega\} \) of \( K \) by open subsets of \( X \), there exists a finite subset \( \Omega_0 \) of \( \Omega \) such that \( K \subset \bigcup \{\text{Cl}(V_\lambda) : \lambda \in \Omega_0\} \).

Theorem 8. If \( f : (X, \tau, I) \rightarrow (Y, \sigma) \) is a strongly \( \beta-I \)-continuous function and \( K \) is \( \beta-I \)-closed relative to \( X \), then \( f(K) \) is quasi-\( H \)-closed relative to \( Y \).

Proof. Suppose that \( f : (X, \tau, I) \rightarrow (Y, \sigma) \) is strongly \( \beta-I \)-continuous and \( K \) is \( \beta-I \)-closed relative to \( X \). Let \( \{V_\lambda : \lambda \in \Omega\} \) be a cover of \( f(K) \) by open sets of \( Y \). For each point \( x \in K \), there exists \( \lambda(x) \in \Omega \) such that \( f(x) \in V_{\lambda(x)} \). Since \( f \) is strongly \( \beta-I \)-continuous, there exists \( U_x \in \beta I O(X, x) \) such that \( f(\beta \text{Cl}(U_x)) \subset \text{Cl}(V_{\lambda(x)}) \).

The family \( \{U_x : x \in K\} \) is a cover of \( K \) by \( \beta-I \)-open sets of \( X \) and hence there
exists a finite subset $K_1$ of $K$ such that $K \subset \bigcup_{x \in K_1} \beta I \operatorname{Cl}(U_x)$. Therefore, we obtain $f(K) \subset \bigcup_{x \in K_1} \operatorname{Cl}(V_{\lambda(x)})$. This shows that $f(K)$ is quasi-$H$-closed relative to $Y$. □

**Remark 1.** If we change in the above Theorem the condition of a strongly $\beta I$-continuous function by a strongly strongly $\theta I$-continuous function, we obtain that $f(K)$ is a compact subset of $Y$.

**Corollary 2.** If $f : (X, \tau, I) \to (Y, \sigma)$ is a strongly $\beta I$-continuous surjection, then the following properties hold:

(i) If $X$ is $\beta I$-closed, then $Y$ is quasi-$H$-closed;

(ii) If $X$ is countably $\beta I$-closed, then $Y$ is lightly compact;

(iii) If $X$ is $\beta I$-Lindelöf, then $Y$ is almost Lindelöf.

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**References**


