On weakly $BR$-closed functions between topological spaces

Miguel Caldas,∗ Erdal Ekici, Saeid Jafari and Seithuti P. Moshokoa

1 Departamento de Matemática Aplicada, Universidade Federal Fluminense, Rua Mario Santos Braga, s/n, 24 020-140, Niterói, RJ Brasil
2 Department of Mathematics, Canakkale Onsekiz Mart University, Terzioglu Campus 17 020, Canakkale, Turkey
3 College of Vestsjaelland South, Herrestraede 11, 4 200 Slagelse, Denmark
4 Department of Mathematical Sciences, University of South Africa, P.O. Box 392, Pretoria 0003, South Africa

Received August 15, 2008; accepted February 18, 2009

Abstract. In this paper, we offer a new class of functions called weakly $BR$-closed functions. Moreover, we investigate not only some of their basic properties but also their relationships with other types of already well-known functions.

AMS subject classifications: 54B05, 54C08; Secondary 54D05

Key words: topological spaces, $b$-closed sets, $b$-$\theta$-open sets, weakly $BR$-closed functions, $b$-continuous functions

1. Introduction and preliminaries

Recently, Park [9] has introduced a new class of sets called $b$-$\theta$-open sets. He showed that $b$-$\theta$-cluster points can be characterized by $b$-regular sets and that the class of $b$-$\theta$-open sets includes the class of $b$-regular sets. He also introduced the notion of strongly $\theta$-$b$-continuous functions. In 2008, Ekici [4] continued the work of Park and also introduced a new class of functions called weakly $BR$-continuity. In this paper we define the notion of weakly $BR$-closedness as a natural dual to the weakly $BR$-continuity by using the notion of $b$-$\theta$-open and $b$-$\theta$-closed sets. We obtain some characterizations and properties of these functions. Moreover, we also study these functions comparing with other types of already known functions. It turns out that $b$-$\theta$-closedness implies weak $BR$-closedness but not conversely. We show that under a certain condition the converse is also true.

Throughout the present paper, $(X, \tau)$ and $(Y, \sigma)$ (or $X$ and $Y$) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset $A$ of a space $X$, the closure of $A$ and the interior of $A$ are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset $A$ is said to be regular open (resp. regular closed) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). A subset $A$ is said to be preopen [7] (resp. $b$-open [1], $\alpha$-open [8]) if $A \subset Int(Cl(A))$ (resp. $A \subset Int(Cl(A)) \cup Cl(Int(A))$,

∗Corresponding author. Email addresses: gmamccs@vm.uff.br (M. Caldas), eekici@comu.edu.tr (E. Ekici), jafari@stofanet.dk (S. Jafari), moshosp@unisa.ac.za (S. P. Moshokoa)

http://www.mathos.hr/mc ©2009 Department of Mathematics, University of Osijek
A ⊂ Int(Cl(Int(A))). A point $x \in X$ is called a $\theta$-cluster point of $A$ [13] if $A \cap Cl(U) \neq \emptyset$ for each open set $U$ containing $x$. The set of all $\theta$-cluster points of $A$ is called the $\theta$-closure of $A$ and is denoted by $Cl_\theta(A)$. A subset $A$ is called $\theta$-closed [13] if $Cl_\theta(A) = A$. The complement of a $\theta$-closed set is called a $\theta$-open set. The complement of a $b$-open set is said to be $b$-closed. The intersection of all $b$-closed sets of $X$ containing $A$ is called the $b$-closure of $A$ and is denoted by $Cl_b(A)$. The union of all $b$-open sets of $X$ contained in a subset $A$ is called $b$-interior [1] of $A$ and is denoted by $Int_b(A)$. The family of all $b$-open (resp. $b$-regular i.e. $b$-open and $b$-closed) sets is denoted by $BO(X)$ (resp. $BR(X)$). A point $x \in X$ is called a $b$-$\theta$-cluster point of $A$ [9] if $A \cap Cl_b(U) \neq \emptyset$ for each $b$-open set $U$ containing $x$.

The set of all $b$-$\theta$-cluster points of $A$ is called the $b$-$\theta$-closure of $A$ and is denoted by $b$-$\theta$-$Cl(A)$. A subset $A$ is said to be $b$-$\theta$-closed if $b$-$\theta$-$Cl(A) = A$. The complement of a $b$-$\theta$-closed set is called a $b$-$\theta$-open set. The family of all $b$-$\theta$-open (resp. $b$-$\theta$-closed) sets is denoted by $B\theta O(X)$ (resp. $B\theta C(X)$). The set $\{x \in X : x \in U \subseteq A$ for some $b$-regular set $U$ of $X\}$ is called the $b$-$\theta$-Interior of $A$ and is denoted by $b$-$\theta$-$Int(A)$.

Recall that for a subset $U$ of a space $X$ the following implications hold:

- $b$-regular $\Rightarrow b$-$\theta$-open $\Rightarrow b$-open;
- $b$-regular $\Rightarrow b$-$\theta$-closed $\Rightarrow b$-closed and
  $B\theta O(X) \cap B\theta C(X) = BR(X)$ and the converses are not true in general (see [9]).

Observe that $B\theta O(X) \cap B\theta C(X) = BR(X)$ is in fact Theorem 3.8 (b) of [9].

A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called:

(i) weakly closed [11, 12] if $Cl(f(Int(F))) \subseteq f(F)$ for each closed $F \subseteq X$.

(ii) contra $b$-$\theta$-open (resp. contra $b$-$\theta$-closed) if $f(U)$ is $b$-$\theta$-closed (resp. $b$-$\theta$-open) in $Y$ for each open (resp. closed) set $U$ of $X$.

(iii) $BR$-open (resp. $BR$-closed) if $f(U)$ is $b$-regular in $Y$ for each open (resp. closed) set $U$ of $X$.

(iv) strongly continuous [5, 2] if for every subset $A$ of $X$, $f(Cl(A)) \subseteq f(A)$.

2. Weakly $BR$-closed functions

**Definition 1.** A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be weakly $BR$-closed if $b$-$\theta$-$Cl(f(Int(F))) \subseteq f(F)$ for each closed set $F$ of $X$.

**Definition 2.** A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $b$-$\theta$-closed if $f(F)$ is $b$-$\theta$-closed in $Y$ for each closed set $F$ of $X$.

Clearly, every $b$-$\theta$-closed function is weakly $BR$-closed, but the converse is not true in general.

**Example 1.**

(i) A weakly $BR$-closed function need not be $b$-$\theta$-closed.
Let $X = \{a, b\}$, $\tau = \{\emptyset, \{b\}, X\}$, $Y = \{x, y\}$ and $\sigma = \{\emptyset, \{x\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be given by $f(a) = x$ and $f(b) = y$. Then $f$ is clearly weakly $BR$-closed, but it is not $b$-$\theta$-closed since $f(a)$ is not a $b$-$\theta$-closed set in $Y$.

(ii) A weakly closed function need not be weakly $BR$-closed (and also not $b$-$\theta$-closed).
Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$, $\sigma = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$ and
Lemma 1 (see [9]). Let $A$ be a subset of a space $X$. Then:

1. $b$-$\theta$-$\text{Cl}(A) = \cap \{V : A \subset V \text{ and } V \in BR(X)\}$.
2. $x \in b$-$\theta$-$\text{Cl}(A)$ if and only if $A \cap U \neq \emptyset$ for each $b$-regular set $U$ of $X$ containing $x$.
3. $b$-$\theta$-$\text{Cl}(A)$ is $b$-$\theta$-closed.
4. Any intersection of $b$-$\theta$-closed sets is $b$-$\theta$-closed and any union of $b$-$\theta$-open sets is $b$-$\theta$-open.
5. $A$ is $b$-$\theta$-open in $X$ if and only if for each $x \in A$, there exists a $b$-regular set $U$ of $X$ containing $x$ such that $x \in U \subset A$.

Theorem 1. For a function $f : (X, \tau) \to (Y, \sigma)$, the following conditions are equivalent:

1. $f$ is weakly $BR$-closed.
2. $b$-$\theta$-$\text{Cl}(f(U)) \subset f(\text{Cl}(U))$ for every open set $U$ of $X$.
3. $b$-$\theta$-$\text{Cl}(f(U)) \subset f(\text{Cl}(U))$ for each regular open subset $U$ of $X$.
4. For each subset $F$ in $Y$ and each open set $U$ of $X$ with $f^{-1}(F) \subset U$, there exists a $b$-$\theta$-open set $A$ in $Y$ with $F \subset A$ and $f^{-1}(A) \subset \text{Cl}(U)$.
5. For each point $y$ in $Y$ and each open set $U$ in $X$ with $f^{-1}(y) \subset U$, there exists a $b$-regular set $A$ in $Y$ containing $y$ and $f^{-1}(A) \subset \text{Cl}(U)$.
6. $b$-$\theta$-$\text{Cl}(f(\text{Int}(\text{Cl}(U)))) \subset f(\text{Cl}(U))$ for each set $U$ of $X$.
7. $b$-$\theta$-$\text{Cl}(f(U)) \subset f(\text{Cl}(U))$ for each preopen set $U$ of $X$.
8. $b$-$\theta$-$\text{Cl}(f(U)) \subset f(\text{Cl}(U))$ for each $\alpha$-open set $U$ of $X$.
9. $b$-$\theta$-$\text{Cl}(f(\text{Int}(F))) \subset f(F)$ for each preclosed set $F$ of $X$.
10. $b$-$\theta$-$\text{Cl}(f(\text{Int}(F))) \subset f(F)$ for each $\alpha$-closed set $F$ of $X$.

Proof. (1) $\Rightarrow$ (2): Let $U$ be any open subset of $X$. Then $b$-$\theta$-$\text{Cl}(f(U)) = b$-$\theta$-$\text{Cl}(f(\text{Int}(U))) \subset b$-$\theta$-$\text{Cl}(f(\text{Int}(\text{Cl}(U)))) \subset f(\text{Cl}(U))$.
(2) $\Rightarrow$ (1): Let $F$ be any closed subset of $X$. Then $b$-$\theta$-$\text{Cl}(f(\text{Int}(F))) \subset f(\text{Cl}(\text{Int}(F))) \subset f(\text{Cl}(F)) = f(F)$.
It is clear that: (1) $\Rightarrow$ (3), (4) $\Rightarrow$ (5), (1) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (2) $\Rightarrow$ (9) $\Rightarrow$ (10) $\Rightarrow$ (1).
(3) $\Rightarrow$ (4): Let $F$ be a subset of $Y$ and $U$ open in $X$ with $f^{-1}(F) \subset U$. Then $f^{-1}(F) \cap \text{Cl}(X - \text{Cl}(U)) = \emptyset$ and consequently, $F \cap f(\text{Cl}(X - \text{Cl}(U))) = \emptyset$. Since
$X - \text{Cl}(U)$ is regular open, $F \cap b\theta-\text{Cl}(f(X - \text{Cl}(U))) = \emptyset$ by (3). Let $A = Y - b\theta-\text{Cl}(f(X - \text{Cl}(U)))$. Then $A$ is $b\theta$-open with $F \subset A$ and $f^{-1}(A) \subset X - f^{-1}(b\theta-\text{Cl}(f(X - \text{Cl}(U)))) \subset \text{Cl}(U)$.

$(5) \Rightarrow (1)$: Let $F$ be closed of $X$ and $y \in Y - f(F)$. Since $f^{-1}(y) \subset X - F$, there exists a $b$-regular $A$ in $Y$ with $y \in A$ and $f^{-1}(A) \subset \text{Cl}(X - F) = X - \text{Int}(F)$ by (5). Therefore $A \cap f(\text{Int}(F)) = \emptyset$, such that $y \in Y - b\theta-\text{Cl}(f(\text{Int}(F)))$. Thus $(5) \Rightarrow (1)$. \hfill $\square$

Next we investigate conditions under which weakly $BR$-closed functions are $b\theta$-closed.

**Theorem 2.** Let $f : (X, \tau) \to (Y, \sigma)$ be weakly $BR$-closed. If for each closed subset $F$ of $X$ and each fiber $f^{-1}(y) \subset X - F$ there exists an open set $U$ of $X$ such that $f^{-1}(y) \subset U \subset \text{Cl}(U) \subset X - F$, then $f$ is $b\theta$-closed.

**Proof.** Let $F$ be any closed subset of $X$ and $y \in Y - f(F)$. Then $f^{-1}(y) \cap F = \emptyset$ and hence $f^{-1}(y) \subset X - F$. By hypothesis, there exists an open set $U$ of $X$ such that $f^{-1}(y) \subset U \subset \text{Cl}(U) \subset X - F$. Since $f$ is weakly $BR$-closed by Theorem 1, there exists a $b$-regular $V$ in $Y$ with $y \in V$ and $f^{-1}(V) \subset \text{Cl}(U)$. Therefore, we obtain $f^{-1}(V) \cap F = \emptyset$ and hence $V \cap f(F) = \emptyset$. This shows that $y \notin b\theta-\text{Cl}(f(F))$. Therefore, $f(F)$ is $b\theta$-closed in $Y$ and $f$ is a $b\theta$-closed function. \hfill $\square$

**Theorem 3.** If $f : (X, \tau) \to (Y, \sigma)$ is contra $b\theta$-open, then $f$ is weakly $BR$-closed.

**Proof.** Let $F$ be a closed subset of $X$. Then, $b\theta-\text{Cl}(f(\text{Int}(F))) = f(\text{Int}(F)) \subset f(F)$. \hfill $\square$

**Theorem 4.** If $f : (X, \tau) \to (Y, \sigma)$ is strongly continuous, then the following are equivalent:

$1) f$ is weakly $BR$-closed.

$2) f$ is contra $b\theta$-open.

**Proof.** (1) $\Rightarrow$ (2): Let $U$ be an open subset of $X$. By hypothesis and Theorem 1(2), we have $b\theta-\text{Cl}(f(U)) \subset f(\text{Cl}(U)) \subset f(U)$. Hence $f(U)$ is $b\theta$-closed.

(2) $\Rightarrow$ (1): It follows from Theorem 3. \hfill $\square$

**Definition 3.** A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $b\theta$-open if $f(U)$ is $b\theta$-open in $Y$ for each open set $U$ of $X$.

**Theorem 5.** Every weakly $BR$-closed strongly continuous bijection $f : (X, \tau) \to (Y, \sigma)$ is $BR$-open (and $BR$-closed).

**Proof.** Let $U$ be an open subset of $X$. Since $f$ is weakly $BR$-closed $b\theta-\text{Cl}(f(\text{Int}(X - U))) \subset f(X - U)$. Hence, by hypothesis we obtain $f(U) \subset b\theta-\text{Int}(f(\text{Cl}(U))) \subset b\theta-\text{Int}(f(U)) \subset f(U)$. Therefore $f(U)$ is $b\theta$-open. But, $f$ is contra $b\theta$-open by Theorem 4, so $f(U)$ is also $b\theta$-closed. In particular, $f(U)$ is $b\theta$-regular for each open $U \subset X$. \hfill $\square$
Theorem 6. If \( f : (X, \tau) \to (Y, \sigma) \) is a weakly BR-closed bijection, then for every subset \( F \) in \( Y \) and every open set \( U \) in \( X \) with \( f^{-1}(F) \subset U \), there exists a \( b-\theta \)-closed set \( B \) in \( Y \) such that \( F \subset B \) and \( f^{-1}(B) \subset \text{Cl}(U) \).

Proof. Let \( F \) be a subset of \( Y \) and \( U \) an open subset of \( X \) with \( f^{-1}(F) \subset U \). Put \( B = b-\theta-\text{Cl}(f(\text{Int}(\text{Cl}(U)))) \), then \( B \) is a \( b-\theta \)-closed set of \( Y \) such that \( F \subset B \) since \( F \subset f(U) \subset f(\text{Int}(\text{Cl}(U))) \subset b-\theta-\text{Cl}(f(\text{Int}(\text{Cl}(U)))) = B \). And since \( f \) is weakly BR-closed, we have \( f^{-1}(B) \subset \text{Cl}(U) \).

Recall that a set \( F \) in a topological space \( X \) is \( \theta \)-compact [12] if for each cover \( \Omega \) of \( F \) by open sets \( U \) in \( X \), there is a finite family \( U_1, \ldots, U_n \) in \( \Omega \) such that \( F \subset \text{Int}(\bigcup\{\text{Cl}(U_i) : i = 1, 2, \ldots, n\}) \).

Theorem 7. If \( f : (X, \tau) \to (Y, \sigma) \) is weakly BR-closed with all fibers \( \theta \)-closed in \( X \), then \( f(F) \) is \( b-\theta \)-closed for each \( \theta \)-compact set \( F \) in \( X \).

Proof. Let \( F \) be \( \theta \)-compact and \( y \in Y - f(F) \). Then \( f^{-1}(y) \cap F = \emptyset \) and for each \( x \in F \) there is an open \( U_x \) in \( X \) containing \( x \) such that \( \text{Cl}(U_x) \cap f^{-1}(y) = \emptyset \). Clearly \( \Omega = \{U_x : x \in F\} \) is an open cover of \( F \) and since \( F \) is \( \theta \)-compact, there is a finite family \( \{U_{x_1}, \ldots, U_{x_n}\} \) in \( \Omega \) such that \( F \subset \text{Int}(A) \), where \( A = \bigcup\{\text{Cl}(U_{x_i}) : i = 1, \ldots, n\} \). Since \( f \) is weakly BR-closed by Theorem 1, there exists a \( b \)-regular \( B \) in \( Y \) with \( f^{-1}(y) \subset f^{-1}(B) \subset \text{Cl}(X - A) = X - \text{Int}(A) \subset X - F \). Therefore \( y \in B \) and \( B \cap f(F) = \emptyset \). Thus \( y \in Y - b-\theta-\text{Cl}(f(F)) \). This shows that \( f(F) \) is \( b-\theta \)-closed.

Two non-empty subsets \( A \) and \( B \) in \( X \) are strongly separated [12], if there exist open sets \( U \) and \( V \) in \( X \) with \( A \subset U \) and \( B \subset V \) such that \( \text{Cl}(U) \cap \text{Cl}(V) = \emptyset \). If \( A \) and \( B \) are singleton sets we may speak of points being strongly separated. We will use the fact (see [3]) that in a normal space, disjoint closed sets are strongly separated.

Recall that a space \( X \) is said to be \( \text{br-Hausdorff} \) (briefly \( \text{br-T}_2 \)) [4] if for every pair of distinct points \( x \) and \( y \), there exist two \( b \)-regular sets \( U \) and \( V \) such that \( x \in U \) and \( y \in V \) and \( U \cap V = \emptyset \).

Theorem 8. If \( f : (X, \tau) \to (Y, \sigma) \) is a weakly BR-closed surjection and all pairs of disjoint fibers are strongly separated, then \( Y \) is \( \text{br-T}_2 \).

Proof. Let \( y \) and \( z \) be two distinct points in \( Y \). Let \( U \) and \( V \) be open sets in \( X \) such that \( f^{-1}(y) \in U \) and \( f^{-1}(z) \in V \), respectively, with \( \text{Cl}(U) \cap \text{Cl}(V) = \emptyset \). By weak BR-closedness (Theorem 1 (5)), there are \( b \)-regular sets \( F \) and \( B \) in \( Y \) such that \( y \in F \) and \( z \in B \), \( f^{-1}(F) \subset \text{Cl}(U) \) and \( f^{-1}(B) \subset \text{Cl}(V) \). Therefore \( F \cap B = \emptyset \), since \( \text{Cl}(U) \cap \text{Cl}(V) = \emptyset \) and \( f \) is surjective. Then \( Y \) is \( \text{br-T}_2 \).

Corollary 1. If \( f : (X, \tau) \to (Y, \sigma) \) is a weakly BR-closed surjection with all fibers closed and \( X \) is normal, then \( Y \) is \( \text{br-T}_2 \).

Corollary 2. If \( f : (X, \tau) \to (Y, \sigma) \) is a continuous weakly BR-closed surjection with \( X \) a compact \( T_2 \) space and \( Y \) a \( T_1 \) space, then \( Y \) is a compact \( \text{br-T}_2 \) space.

Proof. Since \( f \) is a continuous surjection and \( Y \) is a \( T_1 \) space, \( Y \) is compact and all fibers are closed. Since \( X \) is normal, \( Y \) is also \( \text{br-T}_2 \).
Definition 4. A topological space $X$ is said to be:

(i) quasi $H$-closed [10] if every open cover of $X$ has a finite subfamily whose closures cover $X$. A subset $A$ of a space $X$ is quasi $H$-closed relative to $X$ if every cover of $A$ by open sets of $X$ has a finite subfamily whose closures cover $A$.

(ii) almost $BR$-compact space if every cover of $X$ with $b$-regular sets has a finite subfamily of members whose closures cover $X$. And a subset $A$ of a space $X$ is almost $BR$-compact relative to $X$ if every cover of $A$ with $b$-regular subsets has a finite subfamily of members whose closures cover $A$.

Lemma 2 (see [6]). A function $f: (X, \tau) \to (Y, \sigma)$ is open if and only if for each $B \subset Y$, $f^{-1}(\text{Cl}(B)) \subset \text{Cl}(f^{-1}(B))$.

Recall that a topological space is extremelly disconnected if the closure of every open set is open.

Theorem 9. Let $(X, \tau)$ be an extremelly disconnected space and let $f: (X, \tau) \to (Y, \sigma)$ be an open and weakly $BR$-closed function with quasi $H$-closed fibers. Then $f^{-1}(G)$ is quasi $H$-closed for each almost $BR$-compact set $G \subset Y$.

Proof. Let $\{V_\beta : \beta \in I\}$ be an open cover of $f^{-1}(G)$. Then for each $y \in G$, $f^{-1}(y) \subset \bigcup \{\text{Cl}(V_\beta) : \beta \in I(y)\} = H_y$ for some finite $I(y) \subset I$. Then $H_y$ is closed and open since $X$ is extremelly disconnected. So, by Theorem 1(5), there exists a $b$-regular set $U_y$ containing $y$ such that $f^{-1}(U_y) \subset \text{Cl}(H_y) = H_y$. Then, $\{U_y : y \in G\}$ is a cover of $G$ by $b$-regular sets and $G \subset \bigcup \{\text{Cl}(U_y) : y \in K\}$ for some finite subset $K$ of $G$. Hence, by Lemma 2, $f^{-1}(G) \subset \bigcup \{\text{Cl}(f^{-1}(U_y)) : y \in K\} \subset \bigcup \{H_y : y \in K\}$. Thus $f^{-1}(G) \subset \bigcup \{\text{Cl}(V_\beta) : \beta \in I(y) \text{ and } y \in K\}$. Therefore $f^{-1}(G)$ is quasi $H$-closed. □

Acknowledgement

S. P. Moshokoa acknowledges the support by the South African National Research Foundation under Grant number 2053847. He also thanks Professor Jafari for providing him with an ideal research environment and kind support during his visit in Denmark during June-July 2008.

The authors are very grateful to the referee for his careful work and suggestions that improved this paper.

References


