Notes on Taxicab Geometry

1 Introduction to taxicab geometry

Let’s imagine an ideal city in which the streets run only horizontally and vertically and city blocks have the same size. The intersections of streets correspond to the points in space. Now we would like to take a taxi and drive as fast as possible from the point $A(x_1, y_1)$ to the point $B(x_2, y_2)$. To achieve this goal, it would be reasonable to measure the distance between points $A$ and $B$. We can use Euclidean ("school") way to compute the distance. Then distance be-
between $A$ and $B$ is
\[ d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \] (1)

But this number show us the distance between $A$ and $B$ as a bird flies and it is useless for us if we take a taxi. So we use another measure so called taxicab distance which counts the number of blocks we would have to travel to get from $A$ to $B$. Obviously for taxicab distance between $A$ and $B$ the following formula holds
\[ d_T(A, B) = |x_1 - x_2| + |y_1 - y_2|. \] (2)

According to the above description we could define the taxicab geometry as the metric system $(\mathbb{Z}^2, d_T)$ where points are represented by the ordered pairs of whole numbers. This definition is appropriate for student’s use in a way it will be described in section 3.

For pure mathematical research the taxicab geometry is defined on $\mathbb{R}^2$ and it will be introduced like this in the next section.

As it often happens, the approach which is not interesting for pure mathematical research is more useful.

## 2 Approaches to taxicab geometry

There are three fundamental approaches to geometry:

1. synthetic approach,
2. metric approach,
3. group approach.

### 2.1 Metric approach

In the metric approach the concepts of distance and angle measure are added to an incidence geometry. American mathematician G. D. Birkhoff is founder of this approach (1932).

Let us explain metric approach briefly (it is done in details in [1]).

**Definition 1** An abstract geometry $\mathcal{A}$ consists of a set $S$ (set of points) and a collection $L$ of non-empty subsets of $S$ (collection of lines) such that:

1. $\forall A, B \in S \implies \exists l \in L$ with $A \in l$ and $B \in l$.
2. Every line has at least two points.

**Proposition 2** The Cartesian plane is a model $\mathcal{C} = (\mathbb{R}^2, \mathcal{L})$, where $\mathcal{L}$ is a collection of vertical $(x = a)$ and non-vertical $(y = ax + b)$ lines of $\mathbb{R}^2$. $\mathcal{C}$ is an abstract geometry.

**Definition 3** An abstract geometry $\{S, L\}$ is an incidence geometry if

1. Every two distinct points in $S$ lie on a unique line.
2. There exits three points which do not lie on one line.

**Proposition 4** The Cartesian plane is an incidence geometry.

**Definition 5** A distance function on a set $S$ is a function $d : S \times S \to \mathbb{R}$ such that for all $A, B \in S$

1. $d(A, B) \geq 0$;
2. $d(A, B) = 0 \iff A = B$;
3. $d(A, B) = d(B, A)$.

**Proposition 6** The taxicab distance $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by
\[ d_T(A, B) = |x_1 - x_2| + |y_1 - y_2| \] (3)

where $A = (x_1, y_2)$ and $B = (x_1, y_2)$, is a distance function on $\mathbb{R}^2$.

**Definition 7** Let $l$ be a line in an incidence geometry $\{S, L\}$. Assume that there is a distance function $d$ on $S$. A function $f : l \to \mathbb{R}$ is a ruler (or coordinate system) for $l$ if

1. $f$ is a bijection;
2. $\forall A, B \in l$
\[ |f(A) - f(B)| = d(A, B) \] (4)

(Ruler equation).

**Definition 8** An incidence geometry $\{S, L\}$ together with a distance function $d$ satisfies the ruler postulate if every line $l \in \mathcal{L} S$ has a ruler. In this case we say $\mathcal{M} = \{S, L, d\}$ is a metric geometry.

**Proposition 9** The Cartesian plane with the taxicab distance is a metric geometry.
Proof. Ruler for vertical line \( x = 0 \) is \( f((a,y)) = y \).
Ruler for nonvertical line \( y = ax + b \) is \( f((x,y)) = (1+|a|)(x) \).

Definition 10 The model \( \mathcal{T} = \{ \mathbb{R}^2, L, d_T \} \) is called the taxicab plane.

By using the distance function we define betweenness and betweenness allow us to define elementary figures such as segments, rays, angles and triangles.

Definition 11 \( B \) is between \( A \) and \( C \) if \( A, B \) and \( C \) are distinct collinear points in the metric geometry \( \{ S, L, d \} \) and if
\[
d(A, B) + d(B, C) = d(A, C).
\]

Remark 12 In the taxicab plane we can find three points which are not collinear but which satisfy the equation (5). For example if we take \( A(-1,-1), B(0,0), C(2,1) \) the equation (5) is satisfied but these three point are not collinear. This shows why the definition of between requires collinear points.

Definition 13 Let \( \{ S, L, d \} \) be a metric geometry and let \( S_1 \subset S \). \( S_1 \) is said to be convex if for every two points \( P, Q \in S_1 \), the segment \( PQ \) is a subset of \( S_1 \).

Definition 14 We say that a metric geometry \( \{ S, L, d \} \) satisfies the plane separation axiom (PSA) if for every \( l \in L \), there are two sets \( H_1 \) and \( H_2 \) of \( S \) such that
1. \( S-l = H_1 \cup H_2 \);
2. \( H_1 \) and \( H_2 \) are disjoint and each is convex;
3. If \( A \in H_1 \) and \( B \in H_2 \) then \( \overrightarrow{AB} \cap l \neq \emptyset \).

Definition 15 A Pasch geometry is a metric geometry which satisfies PSA.

Proposition 16 The Taxicab geometry is a Pasch geometry.

Definition 17 Let \( r_0 \) be a fixed positive number. In a Pasch geometry, an angle measure (or protractor) based on \( r_0 \) is a function \( m \) from the set \( \mathcal{A} \) of all angles to the set of real numbers such that
1. If \( \angle ABC \in \mathcal{A} \) then \( 0 < m(\angle ABC) < r_0 \);
2. If \( \overrightarrow{AB} \) lies in the edge of the half plane \( H_1 \) and if \( \theta \) is a real number with \( 0 < \theta < r_0 \), then there is a unique ray \( \overrightarrow{BC} \) with \( C \in H_1 \) and \( m(\angle ABC) = \theta \);
3. If \( D \in \text{int}(\angle ABC) \) then
\[
m(\angle ABD) + m(\angle DBC) = m(\angle ABC).
\]

Definition 18 A protractor geometry \( \{ S, L, d, m \} \) is a Pasch geometry \( \{ S, L, d \} \) together with an angle measure \( m \).

Proposition 19 Assume that \( m_e \) is an angle measure for Euclidean metric geometry. Then \( m_e \) is an angle measure for the Taxicab plane \( \{ \mathbb{R}^2, L, d_T \} \). So, \( \{ \mathbb{R}^2, L, d_T, m_e \} \) is a protractor geometry.

There is another concept of the angle measure in Taxicab plane as it is introduced in [2]. We will briefly describe it here.

Definition 20 A t-radian is an angle whose vertex is the center of a unit (taxicab) circle and which intercepts an arc of length 1. The taxicab measure of a taxicab angle \( \theta \) is the number of t-radians subtended by the angle on the unit taxicab circle about the vertex.

It follows that a taxicab unit circle has 8 t-radians and so we can say that in taxicab geometry \( \pi = 4 \). This taxicab angle measure has a nice application to parallax (the apparent shift of an object due to the motion of the observer). The parallax is used for estimating the distance to a nearby object. In the article is discovered that the taxicab method yields the same formula commonly used in the Euclidean case with the exception that the taxicab formula is exact.

Definition 21 A neutral geometry (or absolute geometry) is a protractor geometry which satisfies Side-Angle-Side Axiom (SAS).

The Taxicab plane is not a neutral geometry.

2.2 Synthetic approach

In synthetic (or axiomatic) approach we have to decide what are the important properties of the geometry and then define these properties axiomatically. This approach was first used in Euclid’s famous Elements but was introduced completely by D. Hilbert in 1899, who put several mathematical areas on axiomatic ground.

We have to choose axioms first and in this some criterions should be obeyed (correspondence to an intuitive picture, richness of the theory and consistency). In the synthetic approach, congruence axioms replace distance and angular measure.
2.3 Group approach

The group approach was introduced by F. Klein and A. Cayley and involved transformation group of a geometry. Then geometry studies those properties which are invariant under the group of motions.

The taxicab metric in $\mathbb{R}^2$ is an interesting example of non-Euclidean metric and it is a special case of the more general Minkowski metric where the defining unit circle is a certain symmetric closed convex curve. It has been shown in [6] that the respective isometries in taxicab space form a subgroup of the whole space.

3 Taxicab in classroom

Why to introduce taxicab in classroom?

Let me cite some experts.

1. B. Grünbaum, J. Mycielski: "Undergraduate courses on the foundations of geometry need examples different from the traditional Euclidean plane in order to give students experience in dealing with axiomatic systems." ([7])

2. M. Artigue: "...much of our knowledge remains strongly contextual, that is, dependent on the situations from which it arises."

   "...learning in mathematics is not a continuous process, (it is necessity sometimes to make) breaks with earlier knowledge and models of thought." ([8])

3. F. Beardon: "A city planning makes also attractive use of taxicab geometry. A taxi company has a number of stations across the city. In order to dispatch the calls, the company want to know for any point in the city, which station is the closest, so that the taxi can get there as fast as possible. Obviously, our task is to find the locus of points which are equidistant to their two closest taxi stations. These boundaries will tessellate the plane in regions. In Euclidean geometry, the corresponding graph is called Voronoi diagram and is widely used in computational geometry and pattern recognition (see [10]). This task can be a interesting diploma work on mathematical or technical oriented faculty (a Java program can be demanded)."

4 Applications of the taxicab geometry

At the end, let us say a few facts about applications of the taxicab geometry in a real world problems.

In the real world the shortest distance between two places for trains, planes and even for plains, is not as a bird flies (Euclidean distance). For example, trains want to take the shortest route possible from one place to another. The lattice points with respect to taxicab geometry would be the depots or turns of the track. We want to consider another variable to travel along taxicab distance, so called taxicab time. But sometimes it is not good enough, because the shortest route is selected thus minimizing taxicab distance and minimizing costs. The minimal taxicab distance, time or costs can be determined by the same method described in [9].

Let us explain another natural problem for taxicab geometry. A taxi company has a number of stations across the city. In order to dispatch the calls, the company want to know for any point in the city, which station is the closest, so that the taxi can get there as fast as possible. Obviously, our task is to find the locus of points which are equidistant to their two closest taxi stations. These boundaries will tessellate the plane in regions. In Euclidean geometry, the corresponding graph is called Voronoi diagram and is widely used in computational geometry and pattern recognition (see [10]). This task can be a interesting diploma work on mathematical or technical oriented faculty (a Java program can be demanded).

A city planning makes also attractive use of taxicab geometry. From the historical point of view there are two types of cities: organically grown cities and planned cities (see Fig. 1 and Fig. 2). Organically grown cities develop slowly, tend to plan around preserving their natural beauty and have unstructured design with curved shapes and sporadic placements of streets, buildings and landscape (early European cities).
The growth and plans of planned cities are on "high levels" coordinated with a population growth (predicted mathematically on the basis of cohort survival method). The patterns in the planned cities are more vertical and horizontal (and more suitable for application of taxicab geometry). In the 1960s the central focus of city planning moved from artistic beauty of the city towards location and efficiency. But even for the curved streets we can set up models such as the shortest rout problem or Voronoi diagrams. The curved streets can be represent as a straight edge form with the distance used as edge weighs and the street intersections as the nodes. Besides a distance we have to consider traffic volume, speed limits, traffic signals and so on.

Additionally, the street patterns tells a lot about the city’s history.

We can conclude by noticing that parts of city planning involve the use of geometric models and mathematical models in general. Taxicab geometry is used in many ways for planning city and the traversed routes between and through cities. "Taxicab geometry is a great model for the artificially grid planned world that man has created" (cite [4]).

References


Dr. sc. Blaženka Divjak
University of Zagreb
Faculty of Organization and Informatics
e-mail: bdivjak@foi.hr