Čebiševljeva nejednakost

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Sažetak. *U radu je dokazana Čebiševljeva nejednakost. Na velikom broju zadataka koji su prilagođeni učenicima srednjih škola pokazane su različite primjene te nejednakosti.*

Ključne riječi: *nejednakost, Čebiševljeva nejednakost*

Chebishev's inequality

Abstract. *In the paper Chebishev's inequality is proved. Various applications of the inequality in question are shown by numerous tasks adapted for high-school pupils.*

Key words: *inequality, Chebishev's inequality*

Čebiševljeva nejednakost se rjeđe primjenjuje u zadaćama elementarne matematike, no neke se zadaće mogu riješiti jedino pomoću ove nejednakosti. Iskazat ćemo Čebiševljevu nejednakost, dokazati ju i pokazati njenu primjenu na nekoliko zadataka.

Teorem 1 [Čebiševljeva nejednakost]. *Neka su* $(a_1, \ldots, a_n)$ *i* $(b_1, \ldots, b_n)$ *dvije* n*-torke realnih brojeva.*

(i) *Ako je ili* $a_1 \le a_2 \le \cdots \le a_n$ *i* $b_1 \le b_2 \le \cdots \le b_n$ *ili* $a_1 \ge a_2 \ge \cdots \ge a_n$ *i* $b_1 \ge b_2 \ge \cdots \ge b_n$*, onda vrijedi nejednakost*

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right) \le \frac{1}{n}\sum_{i=1}^{n} a_i b_i. \tag{1}$$

Jednakost u (1) *vrijedi ako i samo ako je* $a_1 = a_2 = \cdots = a_n$ *ili* $b_1 = b_2 = \cdots = b_n$.

(ii) *Ako je ili* $a_1 \le a_2 \le \cdots \le a_n$ *i* $b_1 \ge b_2 \ge \cdots \ge b_n$ *ili* $a_1 \ge a_2 \ge \cdots \ge a_n$ *i* $b_1 \le b_2 \le \cdots \le b_n$*, onda vrijedi nejednakost*

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right) \ge \frac{1}{n}\sum_{i=1}^{n} a_i b_i. \tag{2}$$

Jednakost u (2) *vrijedi ako i samo ako je* $a_1 = a_2 = \cdots = a_n$ *ili* $b_1 = b_2 = \cdots = b_n$.

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Dokaz.

(i) Vrijedi

$$\begin{aligned}\sum_{k=1}^{n}\sum_{l=1}^{n}(a_kb_k-a_kb_l) &= \sum_{k=1}^{n}\Big(na_kb_k-a_k\sum_{l=1}^{n}b_l\Big)\\ &= n\sum_{k=1}^{n}a_kb_k-\sum_{k=1}^{n}a_k\sum_{l=1}^{n}b_l.\end{aligned}\tag{3}$$

U prvoj od tih dvostrukih suma zamijenimo indekse k i l i promijenimo redoslijed sumacija pa dobivamo

$$\sum_{k=1}^{n}\sum_{l=1}^{n}(a_lb_l-a_lb_k)=n\sum_{k=1}^{n}a_kb_k-\sum_{k=1}^{n}a_k\sum_{l=1}^{n}b_l.\tag{4}$$

Zbrojimo li (3) i (4) te podijelimo s 2, dobivamo

$$\begin{aligned}&n\sum_{k=1}^{n}a_kb_k-\sum_{k=1}^{n}a_k\sum_{l=1}^{n}b_l\\ =\ &\frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}(a_kb_k-a_kb_l+a_lb_l-a_lb_k)\\ =\ &\frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}(a_k-a_l)(b_k-b_l)\ge 0,\end{aligned}\tag{5}$$

jer je

$$(a_k-a_l)(b_k-b_l)\ge 0,\qquad k,l=1,2,\ldots,n,$$

odakle dijeljenjem s n^2 dobivamo nejednakost (1).

Ako je $a_1=a_2=\cdots=a_n$ ili $b_1=b_2=\cdots=b_n$, tada u (1) vrijedi jednakost. Obratno, ako u (1) vrijedi jednakost, tada jednakost vrijedi i u (5), pa je $(a_k-a_l)(b_k-b_l)=0$ za sve $k,l=1,2,\ldots,n$. Posebno je $(a_1-a_n)(b_1-b_n)=0$, odakle slijedi $a_1=a_n$ ili $b_1=b_n$. Ako je $a_1=a_n$, tada zbog uvjeta koji zadovoljava n-torka $(a_1,a_2,\ldots,a_n)$ vrijedi $a_1=a_2=\cdots=a_n$, a ako je $b_1=b_n$, tada zbog uvjeta koji zadovoljava n-torka $(b_1,b_2,\ldots,b_n)$ vrijedi $b_1=b_2=\cdots=b_n$.

(ii) Dokazuje se analogno kao (i), uzevši u obzir da je u ovom slučaju

$$(a_k-a_l)(b_k-b_l)\le 0,\qquad k,l=1,2,\ldots,n.$$

□

Nejednakost (1) često pišemo u obliku

$$\Big(\sum_{i=1}^{n}a_i\Big)\Big(\sum_{i=1}^{n}b_i\Big)\le n\sum_{i=1}^{n}a_ib_i,$$

a nejednakost (2) u obliku

$$\Big(\sum_{i=1}^{n} a_i\Big)\Big(\sum_{i=1}^{n} b_i\Big) \geq n\sum_{i=1}^{n} a_i b_i.$$

Sada se najprije prisjetimo nekih činjenica o nejednakostima.

Neka je $a = (a_1, \dots, a_n)$ dana n-torka pozitivnih brojeva. Tada su harmonijska, geometrijska, aritmetička i kvadratna sredina definirane redom sa

$$H_n(a) = \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}},$$
$$G_n(a) = \sqrt[n]{a_1 \dots a_n},$$
$$A_n(a) = \frac{a_1 + \cdots + a_n}{n},$$
$$K_n(a) = \sqrt{\frac{a_1^2 + \cdots + a_n^2}{n}}.$$

Poznato je da vrijedi

$$H_n(a) \leq G_n(a) \leq A_n(a) \leq K_n(a).$$

Pri tom jednakosti vrijede ako i samo ako je $a_1 = a_2 = \cdots = a_n$.

Nejednakosti $G_n(a) \geq H_n(a)$, $A_n(a) \geq H_n(a)$, $A_n(a) \geq G_n(a)$, $K_n(a) \geq A_n(a)$, nazivamo redom GH, AH, AG, KA nejednakost.

Neka je dana n-torka $a = (a_1, \dots, a_n)$ pozitivnih brojeva i realan broj $r \neq 0$. Sredina reda r definirana je izrazom

$$M_n^{[r]}(a) = \left(\frac{a_1^r + \cdots + a_n^r}{n}\right)^{\frac{1}{r}}.$$

Uočimo da je sredina reda 1 upravo aritmetička sredina tj. $M_n^{[1]}(a) = A_n(a)$. Vrijedi nejednakost

$$M_n^{[s]}(a) \leq M_n^{[t]}(a) \quad \text{za} \quad -\infty < s < t < +\infty.$$

Dokaz navedene tvrdnje može se vidjeti u [2].

Zadatak 1. *Neka su a, b, c duljine stranica, a α, β, γ mjere njima nasuprotnih kutova (u radijanima) trokuta ABC. Dokažite da vrijedi*

$$\frac{\alpha a + \beta b + \gamma c}{a + b + c} \geq \frac{\pi}{3}.$$

Rješenje. Možemo pretpostaviti da je $a \leq b \leq c$. Tada je $\alpha \leq \beta \leq \gamma$. Prema Čebiševljevoj nejednakosti je

$$(a + b + c)(\alpha + \beta + \gamma) \leq 3(\alpha a + \beta b + \gamma c),$$

tj.

$$(a + b + c)\pi \leq 3(\alpha a + \beta b + \gamma c),$$

odnosno

$$\frac{\alpha a+\beta b+\gamma c}{a+b+c}\geq \frac{\pi}{3}.$$

Zadatak 2. *Neka su x_i, $i=1,2,\ldots,n$, pozitivni realni brojevi. Dokažite da vrijedi*

$$(x_1x_2\ldots x_n)^{\frac{1}{n}(x_1+x_2+\cdots+x_n)}\leq x_1^{x_1}x_2^{x_2}\ldots x_n^{x_n}.$$

Rješenje. Bez smanjenja općenitosti možemo pretpostaviti da je $x_1\leq x_2\leq\cdots\leq x_n$. Tada je i $\ln x_1\leq \ln x_2\leq\cdots\leq \ln x_n$, pa primjenom Čebiševljeve nejednakosti imamo redom

$$(x_1+x_2+\cdots+x_n)(\ln x_1+\ln x_2+\cdots+\ln x_n)\leq n(x_1\ln x_1+x_2\ln x_2+\cdots+x_n\ln x_n),$$

$$(x_1+x_2+\cdots+x_n)\ln(x_1x_2\ldots x_n)\leq n(x_1\ln x_1+x_2\ln x_2+\cdots+x_n\ln x_n),$$

$$\ln(x_1x_2\ldots x_n)\leq \frac{n(\ln x_1^{x_1}+\ln x_2^{x_2}+\cdots+\ln x_n^{x_n})}{x_1+x_2+\cdots+x_n},$$

$$\ln(x_1x_2\ldots x_n)\leq \frac{n\ln(x_1^{x_1}x_2^{x_2}\ldots x_n^{x_n})}{x_1+x_2+\cdots+x_n},$$

$$\frac{x_1+x_2+\cdots+x_n}{n}\ln(x_1x_2\ldots x_n)\leq \ln(x_1^{x_1}x_2^{x_2}\ldots x_n^{x_n}),$$

$$\ln\left((x_1x_2\ldots x_n)^{\frac{x_1+x_2+\cdots+x_n}{n}}\right)\leq \ln(x_1^{x_1}x_2^{x_2}\ldots x_n^{x_n}),$$

$$(x_1x_2\ldots x_n)^{\frac{x_1+x_2+\cdots+x_n}{n}}\leq x_1^{x_1}x_2^{x_2}\ldots x_n^{x_n}.$$

Zadatak 3. *Neka su a,b,c duljine stranica, t_a,t_b,t_c duljine težišnica, v_a,v_b,v_c duljine visina i P površina trokuta ABC. Dokažite da vrijedi*

$$(t_a^2+t_b^2+t_c^2)(v_a^2+v_b^2+v_c^2)\geq 27P^2.$$

Rješenje. Dokažimo najprije da za trokut ABC vrijedi jednakost

$$t_a^2+t_b^2+t_c^2=\frac{3}{4}(a^2+b^2+c^2).$$

Neka su A_1,B_1,C_1 redom polovišta stranica $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ trokuta ABC i neka je $|BC|=a$, $|CA|=b$, $|AB|=c$, $|AA_1|=t_a$, $|BB_1|=t_b$, $|CC_1|=t_c$. Primijenimo redom poučak o kosinusu na trokute ABC i ABB_1. Dobivamo:

$$\cos\alpha=\frac{b^2+c^2-a^2}{2bc},\qquad \cos\alpha=\frac{c^2+\frac{b^2}{4}-t_b^2}{2\cdot c\cdot\frac{b}{2}}.$$

Izjednačavanjem desnih strana i sređivanjem dobivamo

$$4t_b^2=2(a^2+c^2)-b^2.$$

Analogno bismo dobili

$$4t_a^2 = 2(b^2+c^2)-a^2, \qquad 4t_c^2 = 2(a^2+b^2)-c^2.$$

Zbrajanjem ovih triju jednakosti i dijeljenjem sa 4 dobivamo

$$t_a^2+t_b^2+t_c^2 = \frac{3}{4}(a^2+b^2+c^2).$$

Stoga je nejednakost koju treba dokazati ekvivalentna nejednakosti

$$\frac{3}{4}(a^2+b^2+c^2)(v_a^2+v_b^2+v_c^2) \geq 27P^2,$$

a ova ekvivalentna nejednakosti

$$(a^2+b^2+c^2)(v_a^2+v_b^2+v_c^2) \geq 36P^2.$$

Dokažimo posljednju nejednakost. Neka je $a \leq b \leq c$. Tada je $v_a \geq v_b \geq v_c$ pa je prema Čebiševljevoj nejednakosti

$$\begin{aligned}(a^2+b^2+c^2)(v_a^2+v_b^2+v_c^2) &\geq 3(a^2v_a^2+b^2v_b^2+c^2v_c^2)\\ &= 3(4P^2+4P^2+4P^2) = 36P^2.\end{aligned}$$

Jednakost vrijedi ako i samo ako je $a=b=c$.

Zadatak 4. *Neka su a, b, c duljine stranica trokuta ABC, s njegov poluopseg, a n prirodan broj. Dokažite da vrijedi*

$$\frac{a^n}{b+c}+\frac{b^n}{c+a}+\frac{c^n}{a+b} \geq \left(\frac{2}{3}\right)^{n-2} s^{n-1}.$$

Rješenje. Pretpostavimo da je $a \leq b \leq c$. Tada je $a^n \leq b^n \leq c^n$ i $\frac{1}{b+c} \leq \frac{1}{c+a} \leq \frac{1}{a+b}$. Primijenimo Čebiševljevu nejednakost na trojke (a^n, b^n, c^n) i $(\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b})$, pa dobivamo redom

$$(a^n+b^n+c^n)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \leq 3\left(a^n\cdot\frac{1}{b+c}+b^n\cdot\frac{1}{c+a}+c^n\cdot\frac{1}{a+b}\right),$$

$$\frac{a^n}{b+c}+\frac{b^n}{c+a}+\frac{c^n}{a+b} \geq \frac{a^n+b^n+c^n}{3}\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right). \tag{6}$$

Primijenimo AH nejednakost na brojeve $a+b$, $b+c$, $c+a$:

$$\frac{3}{\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}} \leq \frac{(a+b)+(b+c)+(c+a)}{3},$$

$$\frac{3}{\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}} \leq \frac{2(a+b+c)}{3},$$

$$2(a+b+c)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)\geq 9,$$

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{9}{2(a+b+c)}. \tag{7}$$

Prema nejednakosti aritmetičke sredine i sredine reda n je

$$\frac{a^n+b^n+c^n}{3}\geq\left(\frac{a+b+c}{3}\right)^n. \tag{8}$$

Iz (7) i (8) slijedi

$$\begin{aligned}\frac{a^n+b^n+c^n}{3}\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)&\geq\frac{9}{2}\cdot\frac{1}{3^n}\cdot(a+b+c)^{n-1}=\\ =\quad\frac{3^2}{2}\cdot\frac{1}{3^n}\cdot(2s)^{n-1}&=\left(\frac{2}{3}\right)^{n-2}s^{n-1}.\end{aligned}$$

Tada je zbog (6)

$$\frac{a^n}{b+c}+\frac{b^n}{c+a}+\frac{c^n}{a+b}\geq\left(\frac{2}{3}\right)^{n-2}s^{n-1}.$$

Zadatak 5. *Neka su a,b,c pozitivni realni brojevi takvi da je $abc=1$. Dokažite da je*

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$$

Rješenje. Dokažimo općenitije da uz zadane uvjete za svaki $\beta\geq 2$ vrijedi nejednakost

$$\frac{1}{a^\beta(b+c)}+\frac{1}{b^\beta(c+a)}+\frac{1}{c^\beta(a+b)}\geq\frac{3}{2}. \tag{9}$$

Uvedimo zamjenu:

$$x=\frac{1}{a},\qquad y=\frac{1}{b},\qquad z=\frac{1}{c}.$$

Nejednakost (9) tada poprima oblik

$$\frac{x^{\beta-1}}{y+z}+\frac{y^{\beta-1}}{z+x}+\frac{z^{\beta-1}}{x+y}\geq\frac{3}{2}.$$

Jednostavnosti radi stavimo $\alpha=\beta-1$. Tada je $\alpha\geq 1$ i prethodna nejednakost prelazi u nejednakost

$$\frac{x^\alpha}{y+z}+\frac{y^\alpha}{z+x}+\frac{z^\alpha}{x+y}\geq\frac{3}{2},\qquad \alpha\geq 1. \tag{10}$$

Za $\alpha = 1$ ova nejednakost vrijedi što dokazujemo na sljedeći način:

$$\begin{aligned}
&\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\\
=\ &\frac{x+y+z}{y+z}-1+\frac{y+z+x}{z+x}-1+\frac{z+x+y}{x+y}-1\\
=\ &(x+y+z)\left(\frac{1}{y+z}+\frac{1}{z+x}+\frac{1}{x+y}\right)-3\\
=\ &3(x+y+z)\cdot\frac{\frac{1}{y+z}+\frac{1}{z+x}+\frac{1}{x+y}}{3}-3.
\end{aligned}$$

Kako je prema AH nejednakosti

$$\frac{(x+y)+(z+x)+(y+z)}{3}\geq\frac{3}{\frac{1}{x+y}+\frac{1}{z+x}+\frac{1}{y+z}},$$

tj.

$$2(x+y+z)\left(\frac{1}{y+z}+\frac{1}{z+x}+\frac{1}{x+y}\right)\geq 9,$$

odnosno

$$(x+y+z)\cdot\frac{\frac{1}{y+z}+\frac{1}{z+x}+\frac{1}{x+y}}{3}\geq\frac{3}{2},$$

to je

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq 3\cdot\frac{3}{2}-3=\frac{3}{2}.$$

Neka je sada $x \geq y \geq z$. Tada je $x^{\alpha-1} \geq y^{\alpha-1} \geq z^{\alpha-1}$ i $\frac{x}{y+z} \geq \frac{y}{z+x} \geq \frac{z}{x+y}$. (Primjerice, iz $x \geq y$ slijedi $y+z \leq x+z$ pa je $\frac{1}{y+z} \geq \frac{1}{z+x}$ i stoga $\frac{x}{y+z} \geq \frac{y}{z+x}$.) Primijenimo sada Čebiševljevu nejednakost na trojke $(x^{\alpha-1}, y^{\alpha-1}, z^{\alpha-1})$ i $\left(\frac{x}{y+z}, \frac{y}{z+x}, \frac{z}{x+y}\right)$. Imamo redom

$$(x^{\alpha-1}+y^{\alpha-1}+z^{\alpha-1})\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)\leq 3\left(\frac{x^\alpha}{y+z}+\frac{y^\alpha}{z+x}+\frac{z^\alpha}{x+y}\right),$$

$$\frac{x^\alpha}{y+z}+\frac{y^\alpha}{z+x}+\frac{z^\alpha}{x+y}\geq\frac{3}{2}\cdot\frac{x^{\alpha-1}+y^{\alpha-1}+z^{\alpha-1}}{3}.$$

Prema AG nejednakosti i iz uvjeta $xyz = 1$ slijedi

$$\frac{x^\alpha}{y+z}+\frac{y^\alpha}{z+x}+\frac{z^\alpha}{x+y}\geq\frac{3}{2}\sqrt[3]{x^{\alpha-1}y^{\alpha-1}z^{\alpha-1}}=\frac{3}{2}\sqrt[3]{(xyz)^{\alpha-1}}=\frac{3}{2}.$$

Kako smo dokazali da vrijedi nejednakost (10), to vrijedi i nejednakost (9), a onda vrijedi i polazna nejednakost. Jednakost vrijedi ako i samo ako je $x = y = z$, tj. $a = b = c$.

Zadatak 6. *Neka su a, b, c duljine stranica, a α, β, γ mjere njima nasuprotnih kutova u radijanima. Dokažite da vrijedi nejednakost*

$$\frac{b+c}{\alpha}+\frac{c+a}{\beta}+\frac{a+b}{\gamma}\geq\frac{12s}{\pi},$$

gdje je s poluopseg trokuta.

Rješenje. Bez smanjenja općenitosti možemo uzeti da je $a \leq b \leq c$. Tada je $\alpha \leq \beta \leq \gamma$, $a+b \leq a+c \leq b+c$ i $\frac{1}{\gamma} \leq \frac{1}{\beta} \leq \frac{1}{\alpha}$. Primijenimo Čebiševljevu nejednakost na trojke $(a+b, a+c, b+c)$ i $\left(\frac{1}{\gamma}, \frac{1}{\beta}, \frac{1}{\alpha}\right)$ pa dobivamo redom

$$\big((a+b)+(a+c)+(b+c)\big)\left(\frac{1}{\gamma}+\frac{1}{\beta}+\frac{1}{\alpha}\right) \leq 3\left(\frac{a+b}{\gamma}+\frac{a+c}{\beta}+\frac{b+c}{\alpha}\right),$$

$$2(a+b+c)\left(\frac{1}{\gamma}+\frac{1}{\beta}+\frac{1}{\alpha}\right) \leq 3\left(\frac{a+b}{\gamma}+\frac{a+c}{\beta}+\frac{b+c}{\alpha}\right),$$

$$4s\left(\frac{1}{\gamma}+\frac{1}{\beta}+\frac{1}{\alpha}\right) \leq 3\left(\frac{a+b}{\gamma}+\frac{a+c}{\beta}+\frac{b+c}{\alpha}\right),$$

$$\frac{b+c}{\alpha}+\frac{c+a}{\beta}+\frac{a+b}{\gamma} \geq \frac{4}{3}s\left(\frac{1}{\gamma}+\frac{1}{\beta}+\frac{1}{\alpha}\right).$$

Kako je prema AH nejednakosti

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} \geq 3 \cdot \frac{3}{\alpha+\beta+\gamma} = \frac{9}{\pi},$$

to je

$$\frac{b+c}{\alpha}+\frac{c+a}{\beta}+\frac{a+b}{\gamma} \geq \frac{4}{3}s \cdot \frac{9}{\pi} = \frac{12s}{\pi}.$$

Jednakost vrijedi ako i samo ako je $a=b=c$, tj. za jednakostraničan trokut.

Zadatak 7. *Neka su a, b, c duljine stranica, a α, β, γ mjere njima nasuprotnih kutova (u radijanima) trokuta ABC. Dokažite da vrijedi*

$$\frac{a\cos\alpha + b\cos\beta + c\cos\gamma}{a+b+c} \leq \frac{1}{2}.$$

Rješenje. Pretpostavimo da je $a \leq b \leq c$. Tada je $\alpha \leq \beta \leq \gamma$ i stoga $\cos\alpha \geq \cos\beta \geq \cos\gamma$. Primjenom Čebiševljeve nejednakosti na trojke (a, b, c) i $(\cos\alpha, \cos\beta, \cos\gamma)$ dobivamo

$$(a+b+c)(\cos\alpha+\cos\beta+\cos\gamma) \geq 3(a\cos\alpha + b\cos\beta + c\cos\gamma),$$

odakle je

$$\frac{a\cos\alpha + b\cos\beta + c\cos\gamma}{a+b+c} \leq \frac{\cos\alpha+\cos\beta+\cos\gamma}{3}.$$

Dokažimo sada da je $\cos\alpha+\cos\beta+\cos\gamma \leq \frac{3}{2}$. Imamo

$$\cos\alpha+\cos\beta+\cos\gamma = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + \cos\big(\pi-(\alpha+\beta)\big)$$

$$
\begin{aligned}
&= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} - \cos(\alpha+\beta)\\
&= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} - \left(\cos^2\frac{\alpha+\beta}{2} - \sin^2\frac{\alpha+\beta}{2}\right)\\
&= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 1 - 2\cos^2\frac{\alpha+\beta}{2}\\
&= 2\cos\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}\right) + 1\\
&\leq 2\cos\frac{\alpha+\beta}{2}\left(1 - \cos\frac{\alpha+\beta}{2}\right) + 1\\
&= 2\left(\cos\frac{\alpha+\beta}{2} - \cos^2\frac{\alpha+\beta}{2}\right) + 1\\
&= 2\left(\frac{1}{4} - \left(\cos\frac{\alpha+\beta}{2} - \frac{1}{2}\right)^2\right) + 1 \leq 2\cdot\frac{1}{4} + 1 = \frac{3}{2}.
\end{aligned}
$$

Dakle,

$$\frac{a\cos\alpha + b\cos\beta + c\cos\gamma}{a+b+c} \leq \frac{1}{3}\cdot\frac{3}{2} = \frac{1}{2}.$$

Zadatak 8. *Neka su a, b, c duljine stranica, a α, β, γ mjere njima nasuprotnih kutova (u radijanima) trokuta ABC. Dokažite da vrijedi*

$$\frac{a\,\mathrm{tg}\,\frac{\alpha}{2} + b\,\mathrm{tg}\,\frac{\beta}{2} + c\,\mathrm{tg}\,\frac{\gamma}{2}}{a+b+c} \geq \frac{\sqrt{3}}{3}.$$

Rješenje. Možemo pretpostaviti da je $a \leq b \leq c$. Tada je $\alpha \leq \beta \leq \gamma$ i stoga $\mathrm{tg}\,\frac{\alpha}{2} \leq \mathrm{tg}\,\frac{\beta}{2} \leq \mathrm{tg}\,\frac{\gamma}{2}$. Primjenom Čebiševljeve nejednakosti na trojke (a, b, c) i $\left(\mathrm{tg}\,\frac{\alpha}{2}, \mathrm{tg}\,\frac{\beta}{2}, \mathrm{tg}\,\frac{\gamma}{2}\right)$ dobivamo

$$(a+b+c)\left(\mathrm{tg}\,\frac{\alpha}{2} + \mathrm{tg}\,\frac{\beta}{2} + \mathrm{tg}\,\frac{\gamma}{2}\right) \leq 3\left(a\,\mathrm{tg}\,\frac{\alpha}{2} + b\,\mathrm{tg}\,\frac{\beta}{2} + c\,\mathrm{tg}\,\frac{\gamma}{2}\right),$$

tj.

$$\frac{a\,\mathrm{tg}\,\frac{\alpha}{2} + b\,\mathrm{tg}\,\frac{\beta}{2} + c\,\mathrm{tg}\,\frac{\gamma}{2}}{a+b+c} \geq \frac{\mathrm{tg}\,\frac{\alpha}{2} + \mathrm{tg}\,\frac{\beta}{2} + \mathrm{tg}\,\frac{\gamma}{2}}{3}.$$

Da bismo dokazali da je $\mathrm{tg}\,\frac{\alpha}{2} + \mathrm{tg}\,\frac{\beta}{2} + \mathrm{tg}\,\frac{\gamma}{2} \geq \sqrt{3}$, dokažimo najprije da je $\mathrm{tg}\,\frac{\alpha}{2}\,\mathrm{tg}\,\frac{\beta}{2} + \mathrm{tg}\,\frac{\alpha}{2}\,\mathrm{tg}\,\frac{\gamma}{2} + \mathrm{tg}\,\frac{\beta}{2}\,\mathrm{tg}\,\frac{\gamma}{2} = 1$. Iz $\frac{\gamma}{2} = \frac{\pi}{2} - \left(\frac{\alpha}{2} + \frac{\beta}{2}\right)$ slijedi $\mathrm{tg}\,\frac{\gamma}{2} = \mathrm{ctg}\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \frac{1}{\mathrm{tg}\left(\frac{\alpha}{2} + \frac{\beta}{2}\right)}$, pa je

$$\begin{aligned}
&\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\gamma}{2}+\operatorname{tg}\frac{\beta}{2}\operatorname{tg}\frac{\gamma}{2}\\
= \quad &\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\gamma}{2}\left(\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}\right)\\
= \quad &\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}+\frac{1}{\operatorname{tg}(\frac{\alpha}{2}+\frac{\beta}{2})}\left(\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}\right)\\
= \quad &\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}+\frac{1}{\frac{\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}}{1-\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}}}\left(\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}\right)\\
= \quad &\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}+1-\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}=1.
\end{aligned}$$

Dalje je

$$\begin{aligned}
&\left(\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\gamma}{2}\right)^2\\
= \quad &\operatorname{tg}^2\frac{\alpha}{2}+\operatorname{tg}^2\frac{\beta}{2}+\operatorname{tg}^2\frac{\gamma}{2}+2\left(\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\gamma}{2}+\operatorname{tg}\frac{\beta}{2}\operatorname{tg}\frac{\gamma}{2}\right)\\
= \quad &\operatorname{tg}^2\frac{\alpha}{2}+\operatorname{tg}^2\frac{\beta}{2}+\operatorname{tg}^2\frac{\gamma}{2}+2
\end{aligned}$$

odakle je

$$\operatorname{tg}^2\frac{\alpha}{2}+\operatorname{tg}^2\frac{\beta}{2}+\operatorname{tg}^2\frac{\gamma}{2}=\left(\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\gamma}{2}\right)^2-2.$$

Prema KA nejednakosti imamo redom

$$\frac{\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\gamma}{2}}{3}\le\sqrt{\frac{\operatorname{tg}^2\frac{\alpha}{2}+\operatorname{tg}^2\frac{\beta}{2}+\operatorname{tg}^2\frac{\gamma}{2}}{3}},$$

$$\frac{\left(\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\gamma}{2}\right)^2}{9}\le\frac{\operatorname{tg}^2\frac{\alpha}{2}+\operatorname{tg}^2\frac{\beta}{2}+\operatorname{tg}^2\frac{\gamma}{2}}{3},$$

$$\left(\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\gamma}{2}\right)^2\le 3\left(\left(\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\gamma}{2}\right)^2-2\right),$$

$$-2\left(\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\gamma}{2}\right)^2\le -6,$$

$$\left(\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\gamma}{2}\right)^2\ge 3,$$

$$\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}\frac{\beta}{2}+\operatorname{tg}\frac{\gamma}{2}\ge\sqrt{3}.$$

Literatura

[1] D. S. Mitrinović, D. Mihailović, P. M. Vasić, *Linearna algebra. Polinomi. Analitička geometrija*, Građevinska knjiga, Beograd, 1971.

[2] D. S. Mitrinović, P. M. Vasić, *Sredine*, Zavod za izdavanje udžbenika, Beograd, 1969.

[3] J. Pečarić, *Nejednakosti*, Hrvatsko matematičko društvo, Zagreb, 1996.

[4] J. Reiman, S. Dobos, *Nemzetközi matematikai diákolimpiák 1959–2003*, Typotex, Budapest, 2003.