Uniform density \( u \) and corresponding \( I_u \) convergence*

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Abstract. The concept of a uniform density of subsets \( A \) of the set \( N \) of positive integers was introduced in [1] and [2]. Corresponding \( I_u \) - convergence to the notion of uniform density \( u \) can be found in [8]. This paper studies \( I_u \) - convergence in detail.

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We recall some known notions. Let \( A \subseteq N \). If \( m, n \in N \), by \( A(m, n) \) we denote the cardinality of the set \( A \cap [m,n] \). Numbers

\[
\underline{d}(A) = \lim_{n \to \infty} \inf_{n} \frac{A(1,n)}{n}, \quad \bar{d}(A) = \lim_{n \to \infty} \sup_{n} \frac{A(1,n)}{n}
\]

are called the lower and the upper asymptotic density of the set \( A \), respectively. If there exists the limit

\[
\lim_{n \to \infty} \sup_{n} \frac{A(1,n)}{n},
\]

then \( d(A) = d(A) = \bar{d}(A) \) is said to be the asymptotic density of \( A \). The uniform density of \( A \subseteq N \) was introduced in [1] and [2] as follows: Put

\[
a_n = \min_{m \geq 0} A(m + 1, m + n), \quad a^n = \max_{m \geq 0} A(m + 1, m + n)
\]

It can be shown (see [2]) that the following limits exist

\[
\underline{u}(A) = \lim_{n \to \infty} \frac{a_n}{n}, \quad \bar{u}(A) = \lim_{n \to \infty} \frac{a^n}{n}
\]

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and they are called the lower and the upper uniform density of the set $A$, respectively. If $\underline{u}(A) = \overline{u}(A)$, then $u(A) = \underline{u}(A)$ is called the uniform density of $A$. It is clear that for each $A \subseteq N$ we have

$$\underline{u}(A) \leq d(A) \leq \overline{d}(A) \leq \overline{u}(A).$$

(1) Hence if there exists $u(A)$, then there also exists $d(A)$ and $u(A) = d(A)$. The converse is not true (see Example 1).

The concept of statistical convergence was introduced in [4] (see also [3], [5], [10], [11]) as follows: Let $x = (x_n)^\infty_1$ be a sequence of complex numbers. The sequence $x$ is said to be statistically convergent to a complex number $L$ if for every $\epsilon > 0$ we have $d(A_\epsilon) = 0$, where $A_\epsilon = \{n \in N : |x_n - L| \geq \epsilon\}$. If $x = (x_n)^\infty_1$ converges statistically to $L$, then we write $\lim \text{- stat } x_n = L$.

A generalized approach to convergence is done in [6] by means of the notion of an ideal $I$ of subsets of $N$ (i.e. $I$ is an additive and hereditary class of sets).

A sequence $x$ is said to be $I$-convergent to $L$ provided that for every $\epsilon > 0$ the set $A_\epsilon$ belongs to $I$, we write $\lim I - \text{ lim } x_n = L$. Put $I = I_d = \{A \subseteq N : d(A) = 0\}$, then $I_d$-convergence coincides with statistical convergence. Hence $\lim I - \text{ stat } x_n = L = I_d - \text{ lim } x_n$. In the case $I = I_u = \{A \subseteq N : u(A) = 0\}$ we obtain $I_u$-convergence. If $x = (x_n)^\infty_1$ is $I_u$-convergent to $L$, we write $\lim I_u - \text{ lim } x_n = L$.

We can easily verify that if $I_u - \text{ lim } x_n = L_1$, then $I_u - \text{ lim } y_n = L_2$ and if $a$ is constant, then $I_u - \text{ lim } ax_n = aL_1$. By $M_1$ we denote the set of all $I_u$-convergent sequences; $M_1$ is a linear space. Analogously, we have for $M_0$, the set of all statistically convergent sequences (see [11]). Let $c$ be the set of all convergent sequences. By (1) we have $c \subseteq M_1 \subseteq M_0$.

The following examples show that $c \neq M$ and $M_1 \neq M_0$ even in case of bounded sequences.

**Example 1.** Let $P$ be the set of all primes. Define $x_k = 1$ for $k \in P$ and $x_k = 0$ otherwise. Because of $u(P) = 0$ (see [2]), we have that $x = (x_k)^\infty_1$ is $I_u$-convergent to 0, but not convergent.

**Example 2.** It is easy to see that for the set

$$A = \bigcup_{k=1}^\infty \{10^k + 1, 10^k + 2, \ldots, 10^k + k\}$$

we have $d(A) = 0$, $\underline{u}(A) = 0$, $\overline{u}(A) = 1$. Put $x_k = 1$ for $k \in A$ and $x_k = 0$ for $k \notin A$. Then $I_d - \text{ lim } x_k = 0$, but $x = (x_k)^\infty_1$ is not $I_u$-convergent.

We recall the notion of strong $p$-Cesàro convergence and almost convergence. A sequence $x = (x_k)^\infty_1$ is said to be strong $p$-Cesàro convergent ($0 < p < \infty$) to a number $L$ if

$$\lim_{n \to \infty} \frac{1}{pn} \sum_{k=1}^n |x_k - L|^p = 0$$

(see [3]). By $\mathcal{W}_p$ denote the set of all strong $p$-Cesàro convergent sequences. A bounded sequence $x = (x_k)^\infty_1$ is almost convergent to a number $L$ if every Banach limit of $x$ is equal to $L$, which is equivalent to the condition

$$\lim_{p \to \infty} \frac{1}{p} \sum_{i=1}^p x_{n+i} = L$$
uniformly in \( n \) (see [9], [10], p 59-62). By \( F \) we denote the set of all almost convergent sequences.

It is shown in [9] that almost convergence and statistical convergence are not compatible even in the case of bounded sequences.

The following Theorem 1 shows that in the case of bounded sequences \( I_u \)-convergence and almost convergence can be compared.

**Theorem 1.** Suppose \( x = (x_k)_1^\infty \) is a bounded sequence. If \( x \) is \( I_u \)-convergent to \( L \), then \( x \) is almost convergent to \( L \).

**Proof.** Let \( p, n \in N \) be arbitrary. We estimate

\[
S(n, p) = \left| \frac{x_{n+1} + x_{n+2} + \ldots + x_{n+p}}{p} - L \right|
\]

We have

\[
S(n, p) \leq S^{(1)}(n, p) + S^{(2)}(n, p),
\]
where

\[
S^{(1)}(n, p) = \frac{1}{p} \sum_{1 \leq j \leq p, n+j \in A_\epsilon} |x_{n+j} - L|,
\]

\[
S^{(2)}(n, p) = \frac{1}{p} \sum_{1 \leq j \leq p, n+j \notin A_\epsilon} |x_{n+j} - L|.
\]

By using the definition of \( A_\epsilon = \{ n \in N : |x_n - L| \geq \epsilon \} \) we have

\[
S^{(2)}(n, p) < \epsilon \quad \text{for every} \quad n = 1, 2, \ldots
\]

The boundedness of \( x = (x_k)_1^\infty \) implies that there exists \( M > 0 \) such that

\[
|x_k - L| \leq M \quad (k = 1, 2, \ldots).
\]

Then (4) implies

\[
S^{(1)}(n, p) \leq M \frac{A_\epsilon(n+1, n+p)}{p} \leq M \frac{\max A_\epsilon(m+1, n+p)}{p} = M a^p
\]

Using the last estimation which holds for every \( n = 1, 2, \ldots \) and (2), (3) we obtain the assertion of Theorem 1. \( \square \)

**Remark 1.** The converse of the previous theorem does not hold. For instance, let \( y = (y_k)_1^\infty \) be the sequence defined by \( y_k = 1 \) if \( n \) is even and \( y_k = 0 \) if \( n \) is odd. The sequence \( y \) is almost convergent to \( 1/2 \) but it is not \( I_u \)-convergent.

In [3] a connection between strong \( p \)-Cesàro convergence and statistical convergence is articulated. In the case of bounded sequences both of these kinds of convergence are equivalent. A similar result can be obtained for \( I_u \)-convergence. First of all, we define a new kind of convergence, so-called uniformly strong \( p \)-Cesàro
convergence, which is a generalization of the notion of strong almost convergence (see [8]).

**Definition 1.** The sequence $x = (x_k)_1^\infty$ is said to be uniformly strong $p$-Cesàro convergent $(0 < p < \infty)$ to a number $L$ if

$$
\lim_{k \to \infty} \frac{1}{k} \sum_{i=n+1}^{n+k} |x_i - L|^p = 0
$$

uniformly in $n$.

By $uw_p$ denote the set of all uniformly strong $p$-Cesàro convergent sequences. It is immediate that $uw_p \subset w_p$ $(0 < p < \infty)$. Example 2 shows that the inclusion is strict.

**Theorem 2.**

a) If $0 < p < \infty$ and a sequence $x = (x_k)_1^\infty$ is uniformly strong $p$-Cesàro convergent to $L$, then it is $I_u$-convergent to $L$.

b) If $x = (x_k)_1^\infty$ is bounded and $I_u$-convergent to $L$, then it is uniformly strong $p$-Cesàro convergent to $L$ for every $p$, $0 < p < \infty$.

**Proof.**

a) Let $x$ be uniformly strong $p$-Cesàro convergent to $L$, $0 < p < \infty$. Suppose $\epsilon > 0$. Then, for every $n \in N$ we have

$$
\frac{1}{k} \sum_{j=1}^{k} |x_{n+j} - L|^p \geq \max_{1 \leq j \leq k, |x_{n+j} - L| \geq \epsilon} |x_{n+j} - L|^p \geq \epsilon^p \frac{A_c(n+1, n+k)}{k},
$$

and further,

$$
\frac{1}{k} \sum_{j=1}^{k} |x_{n+j} - L|^p \geq \epsilon^p \frac{\max_{m \geq 0} A_c(m+1, m+k)}{k} = \epsilon^p \frac{A_c}{k}
$$

for every $n = 1, 2, 3 \ldots$. This implies $\lim_{k \to \infty} \frac{A_c}{k} = 0$, and $u(A_c) = 0$, so that $I_u$-lim $x_n = L$.

b) Now, suppose that $x$ is a bounded sequence and $I_u$-lim $x_n = L$. Let $0 < p < \infty$ and $\epsilon > 0$. According to the assumption, we have

$$
u(A_c) = 0. \quad (5)$$

The boundedness of $x = (x_k)_1^\infty$ implies that there exists $M > 0$ such that $|x_k - L| \leq M$ $(k = 1, 2, \ldots)$. Observe that for every $n \in N$, we have that

$$
\frac{1}{k} \sum_{j=1}^{k} |x_{n+j} - L|^p = \frac{1}{k} \sum_{1 \leq j \leq k, n+j \in A_c} |x_{n+j} - L|^p + \frac{1}{k} \sum_{1 \leq j \leq k, n+j \notin A_c} |x_{n+j} - L|^p
$$

$$
\leq M \frac{\max_{m \geq 0} A_c(m+1, m+k)}{k} + \epsilon^p \leq \epsilon^p + M \frac{A_c}{k} \quad (6)
$$
Using (5) and (6) we obtain \( \lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} |x_{n+j} - L|^p = 0 \), uniformly in \( n \).

**Corollary 1.** If \( x = (x_k)^\infty_1 \) is a bounded sequence, then \( x \) is \( I_u \)-convergent to \( L \) if and only if \( x \) is uniformly strong \( p \)- Cesàro convergent to \( L \) for every \( p \), \( 0 < p < \infty \).

In [3], [5] and [11] it is shown that statistical convergence can be characterized by the convergence in the usual sense along a great set of indexes, great in the sense of asymptotic density. The following theorem shows that the \( I_u \)-convergence can be characterized by the convergence along a great set of indexes, great now being in the sense of uniform density. In [6] it is shown that a similar statement is not true for the \( I \)-convergence where \( I \) is an arbitrary ideal.

**Theorem 3.** A sequence \( x = (x_k)^\infty_1 \) is \( I_u \)-convergent to \( L \) if and only if there exists a set

\[
K = \{k_1 < k_2 < \ldots < k_n < \ldots \} \subseteq \mathbb{N}
\]

such that \( u(K) = 1 \) and \( \lim_{n \to \infty} x_{k_n} = L \).

**Proof.** If there exists a set with the mentioned properties and \( \epsilon \) is an arbitrary positive number, we can choose a number \( m \in \mathbb{N} \) such that for each \( n > m \) we have

\[
|x_{k_n} - L| < \epsilon.  \tag{7}
\]

Let \( A_\epsilon = \{n \in \mathbb{N} : |x_{k_n} - L| \geq \epsilon\} \). Then, on the basis of (7), we have

\[
A_\epsilon \subseteq \mathbb{N} - \{k_{m+1}, k_{m+2}, \ldots \}
\]

where on the right-hand side there is a set with the uniform density 0. Therefore, \( u(A_\epsilon) = 0 \); hence \( I_u \lim x_k = L \).

Now suppose that a sequence \( x = (x_k)^\infty_1 \) is \( I_u \)-convergent to \( L \). Let \( K_j \) be the complement of the set \( A_{1/j} \) for \( j = 1, 2, \ldots \),

\[
K_j = \mathbb{N} - \{n \in \mathbb{N} : |x_{k_n} - L| \geq \frac{1}{j}\}
\]

Then, by the definition of \( I_u \)-convergence, we have

\[
u(K_j) = 1 \quad \text{for} \quad j = 1, 2, \ldots .
\]

By the definition of \( K_j \) we have

\[
K_1 \supseteq K_2 \supseteq \ldots \supseteq K_j \supseteq K_{j+1} \supseteq \ldots \tag{8}
\]

Let us choose an arbitrary number \( s_1 \in K_1 \). By the definition of \( K_j \) there exists a number \( s_2 > s_1, s_2 \in K_2 \) such that for each \( n \geq s_2 \) we have

\[
\min_{m \geq 0} K_2(m + 1, m + n) \frac{K_2(m + 1, m + n)}{n} > \frac{1}{2}.
\]

Again on the basis of the definition of \( K_j \) there exists a number \( s_3 > s_2, s_3 \in K_3 \), such that for each \( n \geq s_3 \) we have
\[
\min_{m \geq 0} K_3(m + 1, m + n) \geq \frac{2}{3}. \]

In this manner we can construct an increasing sequence of positive integers

\[ s_1 < s_2 < \ldots < s_j < \ldots \]

such that \( s_j \in K_j \) and that for each \( n \geq s_j \) we have

\[
\min_{m \geq 0} K_j(m + 1, m + n) \geq 1 - \frac{1}{j} \quad \text{for} \quad j = 1, 2, \ldots. \tag{9}
\]

Define \( K \) as follows:

if \( 1 \leq k \leq s_1 \), then \( k \in K \); suppose that \( j \geq 1 \) and that \( s_j < k \leq s_{j+1} \), then \( k \in K \) if and only if \( k \in K_j \). Let \( K = \{ k_1 < k_2 < \ldots < k_n < \ldots \} \). According to (8) and (9), for each \( n \), \( s_j \leq n < s_{j+1} \) we have

\[
\min_{m \geq 0} K(m + 1, m + n) \geq \frac{\min_{m \geq 0} K_j(m + 1, m + n)}{n} > 1 - \frac{1}{j}.
\]

From this it is obvious that \( u(K) = 1 \).

Let \( \epsilon > 0 \) be given and select \( j \) such that \( 1/j < \epsilon \). Let \( n \geq s_j, n \in K \). Then there exists a number \( r \geq j \) such that \( s_r \leq n < s_{r+1} \). According to the definition of \( K, n \in K_r \), we have

\[
|x_n - L| < \frac{1}{r} \leq \frac{1}{j} < \epsilon.
\]

Thus \( |x_n - L| < \epsilon \) for each \( n \geq s_j, n \in K \). Hence \( \lim_{n \to \infty} x_{k_n} = L \). \( \square \)

**Corollary 2.** If a sequence \( x = (x_k)_1^\infty \) is uniformly strong \( p \)-Cesàro convergent \((0 < p < \infty)\) or \( I_u \)-convergent to \( L \), then there exists a sequence \( y = (y_k)_1^\infty \) convergent to \( L \) and a sequence \( z = (z_k)_1^\infty \) \( I_u \)-convergent to \( 0 \), such that \( x = y + z \) and \( u(B) = 0 \), where \( B = \{ n \in N : z_k \neq 0 \} \).

**Proof.** First observe that if \( x \) is uniformly strong \( p \)-Cesàro convergent to \( L \) \((0 < p < \infty)\), then \( x \) is \( I_u \)-convergent to \( L \). From the previous theorem there exists a set

\[ K = \{ k_1 < k_2 < \ldots < k_n < \ldots \} \subseteq N \]

such that \( u(K) = 1 \) and \( \lim_{n \to \infty} x_{k_n} = L \). We define \( y \) and \( z \) as follows: If \( k \in K \), put \( y_k = z_k \) and \( z_k = 0 \), and if \( k \notin K \), we put \( y_k = L \) and \( z_k = x_k - L \). \( \square \)

**Remark 2.** If a sequence \( x = (x_k)_1^\infty \) is uniformly strong \( p \)-Cesàro convergent \((0 < p < \infty)\) or \( I_u \)-convergent to \( L \), then \( x \) has a subsequence which converges to \( L \).

**References**

Uniform density $u$ and corresponding $I_u$ - convergence


