Better butterfly theorem in the isotropic plane

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Abstract. A real affine plane $A_2$ is called an isotropic plane $I_2$, if in $A_2$ a metric is induced by an absolute $\{f, F\}$, consisting of the line at infinity $f$ of $A_2$ and a point $F \in f$.

Better butterfly theorem is one of the generalisations of the well-known butterfly theorem ([1],[4]). In this paper the better butterfly theorem has been adapted for the isotropic plane and its validity in $I_2$ has been proved.

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1. Isotropic plane

Let $P_2(\mathbb{R})$ be a real projective plane, $f$ a real line in $P_2$, and $A_2 = P_2 \setminus f$ the associated affine plane. The isotropic plane $I_2(\mathbb{R})$ is a real affine plane $A_2$ where the metric is introduced with a real line $f \subset P_2$ and a real point $F$ incidental with it. The ordered pair $\{f, F\}$, $F \in f$ is called the absolute figure of the isotropic plane $I_2(\mathbb{R})$ ([2], [3]). In the affine model, where

$$x = x_1/x_0, \quad y = x_2/x_0,$$  

(1)

the absolute figure is determined by the absolute line $f \equiv x_0 = 0$, and the absolute point $F$ (0:0:1).

We will first define some terms and point out some properties of triangles and circles in $I_2$ that are going to be used further on. The geometry of $I_2$ could be seen for example in Sachs [2], or Strubecker [3].

All straight lines through the point $F$ are called isotropic straight lines. A triangle in $I_2$ is called allowable if none of its sides is isotropic.

An isotropic circle (parabolic circle or simply circle) is a regular 2nd order curve in $P_2(\mathbb{R})$ which touches the absolute line $f$ in the absolute point $F$. In $I_2$ there exists a three parametric family of circles, given by $y = Rx^2 + \alpha x + \beta$, $R \neq 0$, $\alpha, \beta \in \mathbb{R}$. Each circle can be reduced to the normal form $y = Rx^2$. Two circles $k_i \equiv y = R_ix^2 + \alpha_ix + \beta_i$, ($i = 1, 2$) are called congruent if $R_1 = R_2$; they are called concentric if $\alpha_1 = \alpha_2$.

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2. Better butterfly theorem

Euclidean version
Let there be 2 concentric circles with the common centre $O$. A line crosses the two circles at points $P, Q$ and $P', Q'$; $M$ being the common midpoint of $PQ$ and $P'Q'$. Through $M$, draw two lines $AA'B'B$ and $CC'D'D$ and connect $AD', AD, BC'$, $B'C$. Let $X,Y,Z,W$ be the points of intersection of $PP'Q'Q$ with $AD', B'C, AD$, and $BC'$, respectively. Then

$$\frac{1}{MX} + \frac{1}{MZ} = \frac{1}{MY} + \frac{1}{MW}$$

The proof is to be found in [4].

Isotropic version
This statement remains valid in the isotropic plane provided concentric circles are replaced by congruent and concentric circles and the corresponding equation for the signed lengths reads:

$$\frac{1}{d(M,X)} + \frac{1}{d(M,Z)} = -\frac{1}{d(M,Y)} - \frac{1}{d(M,W)}$$

(2)
The proof depends on the following lemma:

**Lemma 1.** In the allowable triangle \( \triangle RST \) let \( RU \) be a non-isotropic straight line connecting the vertex \( R \) with some point \( U \) on the opposite side \( ST \) of \( R \). Let's introduce angles \( \alpha = \angle (UR, RS) \), and \( \beta = \angle (TR, RU) \). Then

\[
\alpha + \beta = \frac{\alpha}{d(U, R)} - \frac{\beta}{d(R, S)}
\]  

(3)

**Proof.** Without loss of generality, we can assume that the vertex coordinates are as follows: \( S(0, 0) \), \( T(t_1, 0) \), \( R(r_1, r_2) \), and \( U(u_1, 0) \) (see Figure 3).

![Figure 3](image)

For angles \( \alpha \), \( \beta \) and \( \alpha + \beta \) we have:

\[
\alpha = \angle (UR, RS) = u(RS) - u(UR) = \frac{s_2 - r_2}{s_1 - r_1} - \frac{r_2 - u_2}{r_1 - u_1},
\]  

(4)

\[
\beta = \angle (TR, RU) = u(RU) - u(TR) = \frac{u_2 - r_2}{u_1 - r_1} - \frac{r_2 - t_2}{r_1 - t_1},
\]  

(5)

and

\[
\alpha + \beta = \angle (TR, RS) = u(RS) - u(TR) = \frac{s_2 - r_2}{s_1 - r_1} - \frac{r_2 - t_2}{r_1 - t_1}.
\]  

(6)

Inserting (4), (5), and (6) in (3), together with \( d(U, R) = r_1 - u_1 \), \( d(T, R) = r_1 - t_1 \), \( d(R, S) = -r_1 \) an equality is obtained. \( \square \)

**Proof of the theorem.** Let \( k \) and \( k' \) be two congruent and concentric circles in \( I_2 \), \( k' \equiv y = Rx^2 \), \( k' \equiv y = Rx^2 + s \), \( s \neq 0 \) and let \( M \) be the midpoint of the chord \( \overline{PQ} \) of \( k \). Let us choose the coordinate system as shown (in the affine model) in Figure 2, i.e. the tangent on the circle \( k \) parallel to the chord \( \overline{PQ} \) as the \( x \)-axis, and the isotropic straight line through \( M \) as the \( y \)-axis.

Choosing \( M(0, m) \), for the chord \( \overline{PQ} \) we have \( \overline{PQ} \equiv y = m \), and \( P(p_1, m) \), \( Q(q_1, m) \), \( P'(p_1', m) \), \( Q'(q_1', m) \). Note that \( p_1^2 = q_1^2 = \frac{m}{R} \), and \( p_1' = q_1' = \frac{m-s}{R} \).
Let \( A(a_1, Ra_1^2) \), \( B(b_1, Rb_1^2) \), with \( a_1 \neq b_1 \), and \( C(c_1, Rc_1^2) \), \( D(d_1, Rd_1^2) \), with \( c_1 \neq d_1 \), be the four points on the circle \( k \), and \( A'(a_1, Ra_1^2 + s) \), \( B'(b_1, Rb_1^2 + s) \), with \( a_1' \neq b_1' \), \( C'(c_1, Rc_1^2 + s) \), \( D'(d_1, Rd_1^2 + s) \), with \( c_1' \neq d_1' \), the four points on the circle \( k' \). Let us introduce angles \( \alpha = \angle(PM, MA) = \angle(QM, MB) \) and \( \beta = \angle(DM, MP) = \angle(CM, MQ) \). Applying Lemma 2 to allowable triangles \( \triangle AD'M, \triangle A'D'M, \triangle B'CM, \) and \( \triangle BC'M \) successively one gets

\[
\frac{\alpha + \beta}{d(X, M)} = \frac{\alpha}{d(D', M)} + \frac{\beta}{d(M, A)} \quad (7)_1, \\
\frac{\alpha + \beta}{d(Y, M)} = \frac{\alpha}{d(C, M)} + \frac{\beta}{d(M, B')} \quad (7)_3, \\
\frac{\alpha + \beta}{d(W, M)} = \frac{\alpha}{d(C', M)} - \frac{\beta}{d(M, B')} \quad (7)_4.
\]

From \((7)_1\) and \((7)_2\) we obtain:

\[
(\alpha + \beta) \left( \frac{1}{d(X, M)} + \frac{1}{d(Z, M)} \right) = \alpha \left( \frac{1}{d(D', M)} + \frac{1}{d(D, M)} \right) - \beta \left( \frac{1}{d(M, A)} + \frac{1}{d(M, A')} \right). \tag{8}
\]

Analogously, \((7)_3\) and \((7)_4\) yield that

\[
(\alpha + \beta) \left( \frac{1}{d(Y, M)} + \frac{1}{d(W, M)} \right) = \alpha \left( \frac{1}{d(C', M)} + \frac{1}{d(C, M)} \right) - \beta \left( \frac{1}{d(M, B)} + \frac{1}{d(M, B')} \right). \tag{9}
\]

Using \( d(Y, M) = -d(M, Y) \) and \( d(W, M) = -d(M, W) \), the latter becomes

\[
(\alpha + \beta) \left( \frac{1}{d(M, Y)} + \frac{1}{d(M, W)} \right) = -\alpha \left( \frac{1}{d(C', M)} + \frac{1}{d(C, M)} \right) + \beta \left( \frac{1}{d(M, B)} + \frac{1}{d(M, B')} \right). \tag{10}
\]

Showing that the right-hand sides in \((8)\) and \((10)\) are equal, i.e.

\[
\alpha \left( \frac{1}{d(D', M)} + \frac{1}{d(D, M)} \right) - \beta \left( \frac{1}{d(M, A)} + \frac{1}{d(M, A')} \right) = -\alpha \left( \frac{1}{d(C', M)} + \frac{1}{d(C, M)} \right) + \beta \left( \frac{1}{d(M, B)} + \frac{1}{d(M, B')} \right), \tag{11}
\]

the theorem will be proved.

Using the point coordinates we can rewrite the identity given in \((11)\) to the following form

\[
\beta \left( \frac{1}{a_1} + \frac{1}{a_1'} \right) + \beta \left( \frac{1}{b_1} + \frac{1}{b_1'} \right) = -\alpha \left( \frac{1}{c_1} + \frac{1}{c_1'} \right) - \alpha \left( \frac{1}{d_1} + \frac{1}{d_1'} \right),
\]

which is equivalent to

\[
\beta \left( \frac{a_1 + b_1}{a_1b_1} \right) + \beta \left( \frac{a_1' + b_1'}{a_1'b_1'} \right) = -\alpha \left( \frac{c_1 + d_1}{c_1d_1} \right) - \alpha \left( \frac{c_1'} + \frac{d_1'}{c_1'd_1'} \right). \tag{12}
\]
Besides, knowing that $\overrightarrow{AB}$ is a chord through $M$, the following relations are obtained:

$$M, A, B \text{ collinear points } \Leftrightarrow \det \begin{pmatrix} 0 & m & 1 \\ a_1 & Ra_1^2 & 1 \\ b_1 & Rb_1^2 & 1 \end{pmatrix} = 0$$

$$\Leftrightarrow -m(a_1 - b_1) - Ra_1b_1(a_1 - b_1) = 0$$

$$\Leftrightarrow a_1b_1 = -\frac{m}{R}. \quad (13)$$

Analogously, for $\overrightarrow{CD}$ being a chord through $M$, we get that

$$c_1d_1 = -\frac{m}{R}. \quad (14)$$

Relations given in (13) and (14) can be reached using the following lemma:

**Lemma 2.** Let $k$ be a circle in $l_2$, a point $P \in l_2$, $P \notin k$, and $S_1, S_2$ two points of intersection of a non-isotropic straight line $g$ through $P$ with $k$. The product $f(P) := d(P, S_1) \cdot d(P, S_2)$ does not depend on the line $g$, but only on $k$ and $P$.

The proof is given in [2, p. 32].

So,

$$a_1b_1 = d(M, A) \cdot d(M, B) = d(M, P) \cdot d(M, Q) = p_1q_1 = p_1(-p_1) = -p_1^2 = -\frac{m}{R},$$

and

$$c_1d_1 = d(M, C) \cdot d(M, D) = d(M, P) \cdot d(M, Q) = p_1q_1 = p_1(-p_1) = -p_1^2 = -\frac{m}{R}.$$

Analogously,

$$a_1'b_1' = d(M, A') \cdot d(M, B') = d(M, P') \cdot d(M, Q')$$

$$= p_1'q_1 = p_1'(-p_1') = -p_1'^2 = -\frac{m - s}{R} \quad (15)$$

$$c_1'd_1 = d(M, C') \cdot d(M, D') = d(M, P') \cdot d(M, Q')$$

$$= p_1'q_1 = p_1'(-p_1') = -p_1'^2 = -\frac{m - s}{R}. \quad (16)$$

Since $A, A'$, and $M$ as well as $A, M,$ and $B'$ are collinear points the relations

$$-m(a_1 - a_1') - a_1a_1'R(a_1 - a_1') + a_1s = 0, \quad -m(a_1 - b_1') - a_1b_1'R(a_1 - b_1') + a_1s = 0$$

respectively, are valid. Subtracting these relations we get

$$m(a_1' - b_1') + a_1R(a_1'^2 - b_1'^2) - a_1^2R(a_1' - b_1') = 0.$$

The chord $\overrightarrow{A'B'}$ being a non-isotropic line allows us to rewrite the latter equation as $m + a_1R(a_1' + b_1') - a_1^2R = 0$, wherefrom, using (13), we finally obtain that

$$a_1 + b_1 = a_1' + b_1'. \quad (17)$$
Following the similar procedure, it can be shown that
\[ c_1 + d_1 = c'_1 + d'_1 \]  
holds as well. For the oriented angles \( \alpha, \beta \), introduced at the beginning, we have as follows:
\[
\alpha = \angle(PM, MA) = u(MA) - u(PM) = \frac{q_2 - m_2}{a_1 - m_1} - \frac{m_2 - p_2}{m_1 - p_1} = \frac{R a_1^2 - m}{a_1}, \tag{19}
\]
\[
\beta = \angle(CM, MQ) = u(MQ) - u(CM) = \frac{q_2 - m_2}{q_1 - m_1} - \frac{m_2 - c_2}{m_1 - c_1} = \frac{m - R c_1^2}{c_1}. \tag{20}
\]
Finally, using the relations given in (13), (14),..., and (20) in (12) one gets that
\[
(12) \iff \beta(a_1 + b_1) = -\alpha(c_1 + d_1) \\
\iff \left( \frac{m - R c_1^2}{c_1} \right) (a_1 - \frac{m}{R a_1}) = \left( \frac{R a_1^2 - m}{a_1} \right) (c_1 - \frac{m}{R c_1}) \\
\iff \left( \frac{m - R c_1^2}{R a_1 c_1} \right) (R a_1^2 - m) = \left( \frac{m - R c_1^2}{R a_1 c_1} \right) (R a_1^2 - m).
\]

\[ \qed \]

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References


