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The Paradox of Knowability from a Russellian Perspective

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ABSTRACT: The paradox of knowability and the debate about it are shortly presented. Some assumptions which appear more or less tacitly involved in its discussion are made explicit. They are embedded and integrated in a Russellian framework, where a formal paradox, very similar to the Russell-Myhill paradox, is derived. Its solution is provided within a Russellian formal logic introduced by A. Church. It follows that knowledge should be typed. Some relevant aspects of the typing of knowledge are pointed out.

KEYWORDS: Church, Fitch, paradox of knowability, Russell

In 1963 Frederic Fitch published Theorem 1, claiming that no truth is known as an unknown truth. It follows that if we state that there are unknown truths, the principle of knowability, which claims all truths are knowable, must be rejected. If, on the other hand, the principle of knowability is endorsed, then it must be denied that there are unknown truths and accepted that all truths are known. Many people have analysed and discussed this puzzling result, which has been called the "paradox of knowability", but very few people tried to explain it by making use of the idea of logical types of propositions. There might be many reasons for that, perhaps also the opinion that the ontological legitimacy of propositions is uncertain. However, it is *prima facie* quite natural to think that knowledge concerns propositions, as Fitch did in his paper. This may be a sufficient reason for explicitly speaking of propositions, admitting propositional quantification, and looking for what is more or less tacitly assumed about them.[1]

[1] Such an attitude is not meant to presuppose fully justified acceptance of propositions.

To derive the paradox of knowability an implicit use is made of the comprehension principle for propositions. Moreover, something paradoxical follows only if propositions are intensionally understood. Taking knowledge as a predicate and quantifying over predicative variables are implicitly involved in Fitch's original formulation of the paradox. If comprehension of what Russell took to be propositional functions is endorsed and stated in the context of a simple type theory, a full classical paradox, similar to the Russell-Myhill paradox, can be derived. Its solution can be provided within the Russellian formal logic introduced by A. Church (1984) and discussed by C.A. Anderson (1989).

However, we should acknowledge that the usual formulation of the paradox does not fully exploit all the mentioned ingredients. More support for the analysis in terms of Russellian types is provided by reference to Fitch's original formulation of the paradox. Moreover, further reflection seems to show that a solution for this paradox is only partially provided by the application of type distinctions and that an appeal to some plausible features of knowledge is also required. Finally, it is natural to note that the application of type distinctions raises some well known problems from a Russellian point of view.

1. What the Paradox Consist of

In 1963 Frederic Fitch published the following theorem, numbered as 1:

$$\forall p\, \neg\Diamond \mathrm{K}(p \wedge \neg \mathrm{K}p) \quad \text{(Fitch)}$$

which asserts that no proposition can be known to be an unknown truth. This apparent triviality has some puzzling consequences.

If it is taken as true that there are unknown truths, i.e. $\exists p\, (p \wedge \neg \mathrm{K}p)$ the principle of knowability:

$$\forall p (p \rightarrow \Diamond \mathrm{K}p) \qquad \text{(KP)}$$

which claims *all truths are knowable*, must be rejected, since its application to $p \wedge \neg \mathrm{K}p$ leads to a contradiction with (Fitch). This is essentially theorem 2 in Fitch (1963). Notice that the rejection of (KP) amounts to the assertion of $\exists p\, (p \wedge \neg\Diamond \mathrm{K}p)$. So we have:

$$\exists p\, (p \wedge \neg \mathrm{K}p) \vdash \exists p\, (p \wedge \neg\Diamond \mathrm{K}p)$$

which looks implausible, if not paradoxical, since it appears that the existence of something true which, as a matter of fact, is not known, entails, on the basis of purely logical reasons, the existence of something true which is necessarily not known.

If, on the other hand, the knowability principle is accepted, then it must be denied that there are unknown truths. Denying that there are unknown truths amounts to saying that all truths are known, i.e.:

$\forall p(p \rightarrow \mathrm{K}p)$ (K)

Thus, if all truths are knowable, then all truths are known, i.e.:

(KP) $\vdash$ (K)

which is essentially the content of theorem 5 in Fitch (1963). That too appears, and appeared, implausible, since it turns out, on the basis of purely logical reasons, that the possible knowledge of all truths implies the factual knowledge of all truths.

The proof of theorem 1, here called (Fitch), is well known. It is based on:

Dist	$\mathrm{K}(\alpha \wedge \beta) \vdash \mathrm{K}\alpha \wedge \mathrm{K}\beta$
Fact	$\vdash \mathrm{K}\alpha \rightarrow \alpha$
Nec	If $\vdash \alpha$, then $\vdash \Box\alpha$
ER	$\Box\neg\alpha \vdash \neg\Diamond\alpha$

and runs as follows:

1) $\mathrm{K}(p \wedge \neg\mathrm{K}p)$	assumption
2) $\mathrm{K}p \wedge \mathrm{K}\neg\mathrm{K}p$	from (1) by Dist
3) $\mathrm{K}p \wedge \neg\mathrm{K}p$	from (2) by Fact and standard logic
4) $\neg\mathrm{K}(p \wedge \neg\mathrm{K}p)$	by (1)–(3), by denying (1) because of the contradiction (3)
5) $\Box\neg\mathrm{K}(p \wedge \neg\mathrm{K}p)$	from (4) by Nec
6) $\neg\Diamond\mathrm{K}(p \wedge \neg\mathrm{K}p)$	from (5) by ER

The principle, above called "(Fitch)", follows from line (6) by universal generalization.

2. The Search for the Source of the Paradox

In Fitch (1963) theorem 1 entails theorem 2, which says that if a proposition is true and unknown, the proposition that it is true and unknown is true and cannot be known. It seems to follow that the principle of knowability (KP), which claims *all truths are knowable*, should be rejected. For, as far as we know, we cannot exclude that, as matter of fact, there are propositions p such that:

$\neg\mathrm{K}p$ and $\neg\mathrm{K}\neg p$

If excluded middle $p \vee \neg p$ is accepted, such a fact would show that there is a proposition that it is true and unknown, i.e.:

$\exists p\ (p \wedge \neg Kp)$

whose negation (K) is implied by (KP) and by theorem 1.

It seems to follow that we have factual falsification of (KP), which is a general philosophical principle about knowledge. In my opinion, any such falsification should look suspicious. Thus I think that we should not agree with people like W. D. Hart and C. McGinn (1976), W. D. Hart (1979) and J. L. Mackie (1980), who take the derivation of (K) as a refutation of (KP). It seems right to think that something else is wrong in the derivation of (K). But what? Most people tried to individuate presumptive local defects concerning some principles or rules used in the derivation. Some, like T. Williamson (1982, 1988, 1992), tried to show that intuitionistic logic does not allow the derivation of the paradox. However, this would amount to claiming that a fact, that – as far as we know – we cannot exclude, i.e. that there are propositions p such that $\neg Kp$ and $\neg K\neg p$, provides a sort of refutation of classical logic. Indeed, that would "refute" intuitionistic logic too, since, as P. Percival (1990) remarked, it is not compatible with a conclusion which intuitionistically follows from (KP).

Some others, like J. C. Beall (2000), argue for a partial contradictoriness of knowledge from a paraconsistent point of view. That would logically neutralize the paradox, but this effect cannot be by itself a reason for abandoning classical (or intuitionistic) logic. Other people, like N. Tennant (1997) and M. Dummett (2001), tried to limit the generality of (KP) so that (KP) does not hold for propositions of the form $p \wedge \neg Kp$. Even if some of these proposals have an intrinsic interest, I think that they, in the end, are more or less ad hoc restrictions (see M. Hand and J. L. Kvanvig (1999), B. Brogaard and J. Salerno (2002), S. Rosenkranz (2004), I. Douven (2005)) and, for this reason, should not be accepted as a solution to the paradox. Something similar can be said of other kinds of restrictions, like those proposed by D. Edgington (1985) and others.

B. Linsky (2009) tried to account for it in a more radical way, taking the unacceptable derivation of theorem 1 as arising from the neglect of the idea of logical types of propositions, in accordance with a suggestion from Alonzo Church.[2]

[2] Even if published in 2009, Linsky's paper was accessible, at least accessible to me, before the publication of A. Paseau (2008), which too provides strong motivations for a typing solution of the paradox. V. Halbach (2008) also takes into account this kind of solution, but just to claim that it does not block a diagonalization argument leading to inconsistency. The soundness of this argument is very disputable, as Paseau (2009) pointed out.

In his referee report, referring to the former title of Fitch's paper "A definition of value", Church says:

> Of course the foregoing refutation of Fitch's definition of value is strongly suggestive of the paradox of the liar and other epistemological paradoxes.

and goes on to hint at the possibility of applying "the standard devices for avoiding the epistemological paradoxes".

Bernard Linsky follows Church's suggestion and provides a uniform account of various paradoxical arguments (Fitch, Hintikka, Fitch-B, Preface) by making use of the idea of logical types of propositions.

Linsky's introduction of type distinctions in the core of Fitch's argument, i.e. in the proof of Fitch's theorem 1 as reported above, leads to:

1) $K^{(2)}(p^1 \wedge \neg K^{(1)} p^1)$	assumption
2) $K^{(2)}p^1 \wedge K^{(2)}\neg K^{(1)} p^1$	from (1) by the suitable typed instance of Dist.
3) $K^{(2)}p^1 \wedge \neg K^{(1)}p^1$	by applying the suitable typed instance of Fact to (2)
4) $\neg K^{(2)}(p^1 \wedge \neg K^{(1)}p^1)$	?

where 1, (1), (2) can be taken as abbreviations for 0/1, 0/1/1, 0/2/1 in Church's type notation.[3]

Here Linsky takes knowledge as represented by predicates of various levels so that a higher level knowledge property applies to a proposition involving a knowledge property of a lower level. It turns out that step 3 does not involve any formal contradiction, since $K^{(2)}p^1$ does not imply $K^{(1)}p^1$, and so $\neg K^{(1)}p^1$ does not imply $\neg K^{(2)}p^1$ either. Thus step 4 cannot be deduced by *reductio*.

The non validity of the implications $K^{(2)}p^1 \rightarrow K^{(1)}p^1$ and $\neg K^{(1)}p^1 \rightarrow \neg K^{(2)}p^1$ strictly depends on the nature of $K^{(2)}$ and $K^{(1)}$, since substituting the truth predicates $T^{(2)}$ and $T^{(1)}$ for $K^{(2)}$ and $K^{(1)}$ respectively makes them valid.[4]

Linsky motivates the application of type distinctions at length. Surely a strong reason for it comes from the fact that such an application provides a uniform way of solving paradoxes which involve different notions and different principles. The first goal of this paper is to support Linsky's anal-

[3] Church's type notation is introduced in section 6. The above application of type distinctions is just an example. If *p* has a type different from 0/1, the types for the K predicates should be different in an corresponding similar way.

[4] I will further discuss the specific features of the knowledge predicates of different types in the final section.

ysis, making explicit all that is involved both in the derivation and in the evaluation of the derivation of the so called paradox of knowability and then showing that, in the end, we have all the ingredients, with the possible exception of one step, to form a standard paradox which looks like another version of the Russell-Myhill paradox.

3. Making Explicit Underlying Assumptions

Let us start by observing that the paradox is formulated and derived in a language whose primitives are the propositional variables, the usual connectives, the propositional operator K, and the usual quantifiers. Quantifiers are applied to propositional variables. Formulas are taken as usually defined, adding the clauses for K and the quantification of propositional variables.

The values of propositional variables are meant to be propositions. How should they be conceived? Let us observe that (K) can appear paradoxical only if propositions are intensionally understood, for if all true propositions are to be identified, then all true propositions are known since at least a true proposition is known, as commonly agreed apart of sceptical objections. Thus propositions are not truth values.

There is no need to state what propositions are, but something should be said about how fine-grained they are. It is quite natural to hold that some formulas having different forms are meant to express different propositions, even if they are logically equivalent. For example, let p and q be two different true propositions. Then also p and $(p \wedge q) \vee (p \wedge \neg q)$, which are logically equivalent, appear to be different propositions because of their different contents, besides the fact that only one of them might be believed or acknowledged as true. This point of view, which likely underlies Fitch's paper, implies that propositions are to be very fine-grained.

Hence K is applied to formulas which are meant to express fine-grained propositions. Moreover, in order to derive (K) from (KP), a bound propositional variable has to be replaced by "$p \wedge \neg Kp$". This appears to involve an implicit appeal to the comprehension axiom for propositions. Such an axiom might be (partially) formulated as

$$\exists p\, (p \leftrightarrow A)$$

where p is meant to be simply the proposition expressed by A. We will provide below a more adequate formulation. Alternatively comprehension might be expressed by:

$$\forall p\, (A \rightarrow A(p/B)$$

where A(p/B) is obtained by suitably substituting the formula B for p in A. Intuitively, this amounts to taking the formula B as expressing a value of the propositional variable p, i.e. a proposition.

Since comprehension for propositions is implicitly present and used, it might be natural to introduce also predicative variables, to quantify over them, and to allow comprehension for properties and relations which can be expressed by formulas with free variables.

Quantification of predicative variables is involved in the general way in which Fitch speaks of classes, in particular of what he calls truth classes, even though he does not use formal variables to speak of them. K is just a specific class, intended to be the set of propositions which someone knows (or has known or will know). A possible further motivation for taking quantification of predicative variables as implicitly involved might be the following. Looking at the usual modal semantics and taking K as a predicate, a formula like $p \rightarrow \Diamond \mathrm{K}p$ can be read as saying that if p is true then there is a possible extension of K which p belongs to.

Applying quantifiers to predicative variables enables us to introduce new forms of inference, for example the following, which are valid under certain suitable conditions here omitted:

$$\frac{\mathrm{A(K)}}{\exists P\, \mathrm{A}(P)} \qquad\qquad \frac{\forall P\, \mathrm{A}(P)}{\mathrm{A(K)}}$$

These are, respectively, the rules for existential generalisation and for instantiation of universal generalisation.

Quantification of predicative variables also allows us to formulate a comprehension axiom for properties (and relations). Of course this axiom should be stated in such a way that the Russell paradox is not derivable. To block its derivation, let us assume that comprehension introduces an entity of a higher (simple) type and so should be formulated by resorting to a variable with a type higher than his arguments. In the case of a unary predicative variable, the type of the variable will be immediately higher than his arguments' type. In the language we are considering, only propositional variables have the role of arguments and all of them have the same lowest type, i.e. 0. This implies that a proposition that is stated to exist by comprehension always has the (simple) type 0, however complex it is, and a property that is stated to exist by comprehension always has the (simple) type 1. A (partial) formulation of the comprehension axioms for properties is

$$\exists P\, \forall \mathrm{x}_1 \ldots \forall \mathrm{x}_m\, (P(\mathrm{x}_1,\ldots,\mathrm{x}_m) \leftrightarrow \mathrm{A})$$

where P is a predicative variable of a suitable type which does not occur free in A. We will provide a more adequate formulation below. Or, alternatively, comprehension for properties might be expressed by:

$$\forall P\ (A \rightarrow A(P(x_1,\ldots,x_m)/B)$$

where $A(P(x_1,\ldots,x_m)/B)$ is obtained by suitably substituting the formula B for the free occurrences of P if the required conditions are satisfied.[5]

Comprehension for properties is not at all involved in the paradox of knowability. However, while discussing and analysing the paradox, it is sometimes considered whether it is justified to deny the validity of (KP) for propositions of the form "$p \wedge \neg Kp$". Such a propositional form appears to isolate a class of propositions. In the same connection it has been guessed or asserted that there are other classes of propositions for which it not reasonable to assume that (KP) is valid. All this appears to involve an appeal to the notion of property of propositions, even though it might be held that this notion is not exploited to its full strength and generality.

The appropriate comprehension axiom can be applied to introduce a predicate corresponding to the K operator occurring in the usual formulations of the paradox of knowability. For, by comprehension, there is a property K* such that for every p:

$$K^*(p) \leftrightarrow Kp$$

where p should be taken both as a formula and a term. Then it can appear natural to conceive K as a predicate from the beginning. In any case Kp and $K^*(p)$ have the same type as p, i.e. 0, within the simple type theory.

The choice of taking K as a predicate from the beginning has to cope with at least two objections. The first is the claim, first made by R. Montague and D. Kaplan (1960), R. Montague (1963), and then by others, that the treatment of knowledge and other modalities as predicates generates inconsistency within the context of a formal theory of arithmetic. Two kinds of replies were given to this claim: first, not every treatment of modalities as predicates leads to inconsistency, as various people showed in different ways,[6] and, second, there are inconsistent operator treatments of epistemic modalities within formal arithmetic theories provided with quantification of propositional variables, as P. Grim (1993) proved.

[5] The substitutivity conditions are specified, for example, in Church (1956: 192–193), Church (1976: 750), and Church (1984: 515).

[6] For example B. Skyrms (1978), T. Burge (1978, 1984), C. A. Anderson (1983), J. des Riviéres and H. Levesque (1986).

The other objection to treating K as a predicate is an argument by analogy. If there is no need – and indeed it appears to be wrong – to transform the usual connectives in predicates, why should K be taken as a predicate and then typed? I will deal with this question in the final section.

4. The Ingredients of the Paradox Formalised in an Extended Simple Type Theory

The kind of language, its intended interpretation and associated principles, which have been outlined above, are implicit in Appendix B of Russell's *Principles of Mathematics*. Indeed Appendix B contains something more, i.e. the notion and extensive informal use of identity between propositions.

In *Principia Mathematica* the identity between propositions is implicitly taken into account by defining "x = y" as $\forall f\,(f(x) \rightarrow f(y))$, where x and y are typed and may be propositional variables and f is a suitable typed variable for propositional functions. Church observed, however, that the formal possibility of speaking of identity of propositions is limited due to some features of the language syntax. To express identity among propositions without technical restrictions, Church introduces the new primitive connective $\equiv$ and the clause:

> If A and B are well formed formulas, $(A \equiv B)$ is a well formed formula.

The new connective $\equiv$ intuitively expresses identity among propositions so that, by example, $\forall x\,(f(x) \equiv g(x))$, where f and g are variables for propositional functions, means that for every x, f(x) is the same proposition as g(x).

The new connective $\equiv$ is added to a simple type theory and provided with suitable axioms that are meant to express a strongly intensional conception of propositions. The resulting logic is partially and informally introduced in Appendix B of *Principles of Mathematics*, where it is used to show that a simple type theory does not block the derivation of the paradox later called "Russell-Myhill paradox", which can be shortly presented as follows. Let us consider, for any class *m* of propositions, the proposition saying that every member of *m* is true. Let *w* be the class of all propositions that, for some *m*, say that every member of *m* is true and are such that they do not belong to *m*. Then the proposition that every member of *w* is true belongs to *w* if and only if it does not belong to *w*. The derivation of this paradox was surely a reason for the subsequent introduction of the ramification of types in 1908.

Here the logic is slightly modified, with respect to the formal version supplied by Church, in order to allow an easier derivation of a paradox based on the form of the sentence $\exists p\ (p \wedge \neg Kp)$, where K is taken as a propositional function. The modifications concern some of the axioms for $\equiv$ and depend on the choice of $\wedge$ as primitive instead of $\vee$ and the formulation of 11 for $\exists$ instead of $\forall$. The modified system of axioms is as follows:

1. $p \equiv p$
2. $(p \equiv q) \equiv (p = q)$
3. $(p \equiv q) \rightarrow (p = q)$
4. $\forall x_1,\ldots,\forall x_m (f(x_1,\ldots,x_m) \equiv g(x_1,\ldots,x_m)) \rightarrow f = g$
5. $\exists p\ (p \equiv A)$

where p is a propositional variable not occurring free in A.

6. $\exists f\, \forall x_1,\ldots,\forall x_m (f(x_1,\ldots,x_m) \equiv A)$

where f is a functional variable of a suitable type which does not occur free in A.

7. $\forall x\ f(x) \equiv \forall y\ f(y)$
8. $(\neg p \equiv \neg q) \rightarrow (p \equiv q)$
9. $((p \wedge q) \equiv (r \wedge s)) \rightarrow (p \equiv r)$
10. $((p \wedge q) \equiv (r \wedge s)) \rightarrow (q \equiv s)$
11. $(\exists x\ f(x) \equiv \exists y\ g(y)) \rightarrow \forall x\ (f(x) \equiv g(x))$

Taking $\exists xA$ as defined by $\neg\forall x\neg A$, as implicitly done by Church, this modified axiom 11 can be derived from the original 11, i.e. $\forall x\ f(x) \equiv \forall y\ g(y) \rightarrow \forall x\ (f(x) \equiv g(x))$, axioms 8 and 6.

12. $\neg (\neg p \equiv \forall x\ f(x))$
13. $\neg (\neg p \equiv (q \wedge r))$
14. $\neg ((p \wedge q) \equiv \forall x\ f(x))$
15. $\neg (\forall x\ f(x) \equiv \forall y\ g(y))$

where the variables x and y have different types.

Standard inference rules are allowed. Besides them, rules of substitution for propositional and functional variables are also used. Church says that they follow from the axiom schemata 5 and 6. In fact, after the introduction of ramified types, the axioms of schemas 5 and 6 appear to be satisfied by a model according to which the propositional functions obtained by comprehension are non eliminable constituents of some propositions, where they have a predicative role and some of their arguments are quantified. In these cases functional substitution does not in general preserve the propositional identity in such a way that all of Church's axioms are satisfied. However, the rules of substitution for propositional and functional variables can be derived from the following alternative versions of 5 and 6:

5'. $\forall p\, \mathrm{A} \rightarrow \mathrm{A}(p/\mathrm{B})$

where p is a propositional variable and $\mathrm{A}(p/\mathrm{B})$ is the result of substituting B for p if the standard required conditions are satisfied.

6'. $\forall \mathrm{f}\, \mathrm{A} \rightarrow \mathrm{A}(\mathrm{f}(\mathrm{x}_1,\ldots,\mathrm{x}_\mathrm{m})/\mathrm{B})$

where f is a functional variable, $\mathrm{x}_1,\ldots,\mathrm{x}_\mathrm{m}$ are distinct variables and $\mathrm{A}(\mathrm{f}(\mathrm{x}_1,\ldots,\mathrm{x}_\mathrm{m})/\mathrm{B})$ is the result of suitably substituting B for the free occurrences of f if the standard required conditions are satisfied.

5. The Derivation of a Formal Paradox

Now a standard paradox, similar to the Russell-Myhill paradox, can be devised starting from the proposition which, together with (KP), implies an absurd:

$$\exists p\, (p \wedge \neg \mathrm{K}p)$$

A possible, perhaps not natural, reaction to the paradox of knowability is to take $\exists p\, (p \wedge \neg \mathrm{K}p)$ as unknown, i.e.:

$$\neg \mathrm{K}(\exists p\, (p \wedge \neg \mathrm{K}p))$$

Taking K as a propositional function and making it explicit that $\exists p\, (p \wedge \neg \mathrm{K}p)$ is a proposition, we can write:

$$(p \equiv \exists q\, (q \wedge \neg \mathrm{K}(q))) \wedge \neg \mathrm{K}(p)$$

Then, we can existentially generalize with respect to K and arrive at:

$$\exists \mathrm{g}\, ((p \equiv \exists q\, (q \wedge \neg \mathrm{g}(q)) \wedge \neg \mathrm{g}(p))$$

At this point we have all we need to derive a contradiction, and that should be no surprise since the Russell-Myhill paradox is obtainable by resorting to $\forall q\, (\neg \mathrm{g}(q) \rightarrow \neg q)$, i.e. a formula equivalent to the negation of $\exists q\, (q \wedge \neg \mathrm{g}(q))$. So the derivation of a contradiction is just a little exercise, which, however, it is worth doing.

By virtue of axiom 6,[7] let F be a propositional function such that:

(*) $\forall p\ (F(p) \equiv \exists g\ ((p \equiv \exists q\ (q \wedge \neg g(q)) \wedge \neg g(p))$

Axiom 5 allows the introduction of a proposition s such that

(**) $s \equiv \exists q\ (q \wedge \neg F(q))$

From (*), by universal instantiation:

(***) $F(s) \equiv \exists g\ ((s \equiv \exists q\ (q \wedge \neg g(q)) \wedge \neg g(s))$

Both $F(s)$ and $\neg F(s)$ can be reduced to absurd.

Let us assume:

1. $F(s)$

Since it is provable that $(A \equiv B) \rightarrow (A \leftrightarrow B)$:

2. $\exists g\ ((s \equiv \exists q\ (q \wedge \neg g(q)) \wedge \neg g(s))$

Let C be such that:

3. $s \equiv (\exists q\ (q \wedge \neg C(q))) \wedge \neg C(s)$

Hence:

4. $s \equiv \exists q\ (q \wedge \neg\, C(q))$
5. $\neg C(s)$

From 4 and (**), by the demonstrable transitivity of $\equiv$:

6. $\exists q\ (q \wedge \neg F(q)) \equiv \exists q\ (q \wedge \neg C(q))$

By axiom 6, there are f′ and g′ such that $\forall q\ (f'(q) \equiv (q \wedge \neg F(q)))$ and $\forall q\ (g'(q) \equiv (q \wedge \neg C(q)))$. So, by axiom 11:

7. $(\exists q\ f'(q) \equiv \exists q\ g'(q)) \rightarrow \forall q\ (f'(q) \equiv g'(q))$

By substituting in 7 $(q \wedge \neg F(q))$ for f′ and $(q \wedge \neg C(q))$ for g′, we arrive at:

8. $(\exists q\ (q \wedge \neg F(q)) \equiv \exists q\ (q \wedge \neg C(q))) \rightarrow \forall q\ ((q \wedge \neg F(q) \equiv (q \wedge \neg C(q))$

From 6 and 8:

9. $\forall q\ ((q \wedge \neg F(q) \equiv (q \wedge \neg C(q))$

Then, by universal instantiation and axiom 10:

10. $\neg F(q) \equiv \neg C(q)$

[7] We mean the pertinent instance of the axiom schema 6. Similarly below.

and, by axiom 8:

11. $F(q) \equiv C(q)$

Since 11 holds for every q, by axiom 4:

12. $F = C$

Thus, from 5:

13. $\neg F(s)$

against 1.

On the other hand, a contradiction follows also starting with:

1. $\neg F(s)$

By (***), since it is provable that $A \equiv B \rightarrow A \leftrightarrow B$:

2. $\neg\exists g\ ((s \equiv \exists q\ (q \wedge \neg g(q))) \wedge \neg g(s))$

By standard logic:

3. $\forall g\ ((s \equiv \exists q\ (q \wedge \neg g(q))) \rightarrow g(s))$

In particular:

4. $s \equiv \exists q\ (q \wedge \neg F(q)) \rightarrow F(s)$

From (**) and 4:

5. $F(s)$

6. The Solution Provided by Ramified Types Distinctions

As in the case of the original version of the Russell-Myhill paradox, a solution is provided by the introduction of ramified type distinctions.

Let us adopt the recursive definition of Church's r-types (1976), and, for sake of simplicity, refer to them as types. So we have that i is a type and if $\beta_1,\ldots,\beta_m$ are types, where $m \geq 0$, then $(\beta_1,\ldots,\beta_m)/n$, where $n \geq 1$, are types. An order is assigned to every type in the following way: the order of type i is 0, the order of a type $(\beta_1,\ldots,\beta_m)/n$ is N+n, where N is the greatest of the orders of the types $\beta_1,\ldots,\beta_m$.

It is intended that i is the type of individuals; ()/n (abbreviated as 0/n) is the type of propositions of level n; and $(\beta_1,\ldots,\beta_m)/n$ is the type of m-ary propositional functions of level n having, as arguments, entities of types $\beta_1,\ldots,\beta_m$ respectively. As in Church, let us define $(\alpha_1,\ldots,\alpha_m)/k$ <

$(\beta_1,\ldots,\beta_m)/n$ if $\alpha_1=\beta_1,\ldots,\alpha_m=\beta_m$ and k<n. Types are intended to be cumulative, that is entities of type α include entities of types <α.

Variables and constants of the language should be typed accordingly and formulas should be constructed on the basis of the type distinctions. An atomic formula is a propositional variable (or constant), which has type 0/n for some n, or a sequence of symbols $F(t_1,\ldots,t_m)$, where F is a variable or a primitive constant of a type $(\beta_1,\ldots,\beta_m)/n$, for some n, where $m\geq 1$ and $t_1,\ldots,t_m$ are variables or constants of the appropriate types $\beta_1,\ldots,\beta_m$ respectively. Non-atomic formulas are built in the standard way.

The modifications required for the axioms 1–15 are simple and quite obvious. Some more care is needed in the formulation of the comprehension axioms. To state them I will use the notion of order of a variable or a constant, identified, as usual, with the order of its type, and the notion of order of a formula: the order of a formula A, abbreviated by ord(A), is max(h, k+1), where h is the greatest order of free variables and constants occurring in A and k is the greatest order of bound variables occurring in A.

Let A be a well formed formula. The comprehension axioms are the instances of the following two schemata:

5_t. $\exists p\ (p \equiv \mathrm{A})$

where p is a propositional variable of type 0/n, $\mathrm{ord(A)} \leq \mathrm{n}$ and p does not occur free in A;

6_t. $\exists \mathrm{f}\ \forall \mathrm{x}_1,\ldots, \forall \mathrm{x}_m\ (\mathrm{f}(\mathrm{x}_1,\ldots,\mathrm{x}_m) \equiv \mathrm{A})$

where f is a functional variable of type $\beta_1,\ldots,\beta_m/n$, $x_1,\ldots,x_m$ are distinct variables of types $\beta_1,\ldots,\beta_m$ respectively, $\mathrm{ord(A)} \leq \mathrm{ord(f)}$ and f does not occur free in A.

Analogous modifications are needed for the ramified type formulations of the alternative comprehension axioms 5′ and 6′.

When the above derivation of, say, the Fitch-Russell-Myhill paradox is formulated in the typed language just described, it turns out that some inferential passages are not valid. For:

1) Let us suppose that p has type 0/1 and g has type 0/1/1. Since $s \equiv \exists q\ (q \wedge \neg F(q))$ by comprehension, s should be of type 0/2 and so it could not instantiate p at the step (***).
2) F has a higher order than g because of its introduction by ramified type comprehension, and so F cannot instantiate g at step 4 of the derivation $\neg F(s) \vdash F(s)$.

7. Final Considerations

One could object that even if all the indicated ingredients which allow the derivation of a contradiction are implicit in the paradox of knowability, as formulated by Fitch, or in some of its subsequent analyses and discussions, not all of them are fully exploited in the formal version usually reported. Looking at it, one can only say that the introduction of type distinctions blocks the derivation of a contradiction in the way indicated by Linsky, that is the contradiction turns out to be not derivable only because it is reasonable to assume that more level 1 propositions might be known at level 2 than at level[8] 1. Clearly the supposed truth of this assumption depends on the nature of knowledge, not on the type distinctions. As anticipated above, in section 2, a contradiction would follow were $K^{(2)}$ and $K^{(1)}$ interpreted, respectively, as the truth predicate $T^{(2)}$ and $T^{(1)}$, since these predicates are equivalent when restricted to the level 1 propositions. But there is no reason to assume such an equivalence as concerns knowledge predicates of different level. For knowledge concerning logically more complex propositions might allow knowledge of logically simpler propositions which might not be otherwise known. In other words, some of the simpler propositions might not be knowable when lacking knowledge of more complex propositions.

Russellian type distinctions enable us to differentiate levels of knowledge on the basis of the (internal) logical complexity of propositions and one might object that such differentiation is rather rough. It might appear more appropriate to differentiate levels of knowledge by also taking into account the complexity of the minimal resources involved in getting to know propositions. This means that levels of knowledge might be even more refined. From this point of view typing knowledge does not appear so simple as it might be suggested by Linsky (2009) and should rather follow the more general indications by Paseau (2008, 2009).

Sensitivity to logical complexity, however specified, provides a reason for typing knowledge, independent of the decision to represent it by means of an operator or of a predicate. As far as the represented typed notion is taken as a constituent of the proposition expressed by a sentence in which the operator or the predicate occurs, the order of the proposition expressed is higher than the order of the proposition which the operator or the predicate applies to. But, then, why should something similar not happen also with the truth functions expressed by the usual connectives?

On page 43 of *Principia Mathematica*, Whitehead and Russell say that these functions are "typically ambiguous", since they apply to propo-

[8] I am using Linsky's informal notion of level which does not exactly correspond to Church's technical notion of level.

sitions of different types. According to them, this does not imply that the proposition resulting from their application has an order higher than the order of the proposition, or the propositions, which they are applied to. For on page 133, *9·131, Whitehead and Russell state a definition according to which a proposition resulting from the application of a usual truth function is of the same type as its arguments, so that, for example, p and $\neg p$ are of the same type; and if p and q are of the same type, then p, q and $p \vee q$ are of the same type. It follows that if being a constituent raises the type of the constituted proposition, then the usual propositional truth functions do not constitute propositions or "are" in propositions in a very special way. In this article, I will not deal with this problem, which appears to be related, on one hand, to the question whether propositions are (genuine) entities and, on the other hand, to the Russellian conception of logical form, which notoriously is a debated and intricate topic of Russell's thought. Let us stick to the point that, granting an ontology of propositions, it is quite reasonable, from a Russellian point of view, to take what is expressed by a knowledge operator or a predicate as a typed ingredient which raises the type of a proposition in which it "occurs".

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