Fixed points of strip $\varphi$-contractions

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Abstract. In this paper, we introduce strip $\varphi$-contraction, where $\varphi$ is an altering distance function, and obtain sufficient conditions for the existence of fixed points for such maps. Further, we extend it to a pair of selfmaps. These results improve and generalize the results of Khan, Swaleh and Sessa [1], Sastry and Babu [5] and Park [4] to strip $\varphi$-contractions.

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1. Introduction

Throughout this paper we assume that $(X, d)$ is a metric space denoted simply by $X$ and $T$ a selfmap of $X$, $R^+ = [0, \infty)$, $N$ denotes the set of all natural numbers. For $x \in X$, $O_T(x) = \{x, Tx, T^2x, \ldots\}$ denotes the orbit of $x$ with respect to $T$. We denote the closure of $O_T(x)$ by $\overline{O_T(x)}$.

We say that $T$ is orbitally continuous at a point $z \in X$ with respect to $x \in X$ if for any sequence $\{x_n\} \subset O_T(x)$, with $x_n \to z$ as $n \to \infty$ implies $Tx_n \to Tz$ as $n \to \infty$. Here we note that any continuous selfmap of a metric space is orbitally continuous, but an orbitally continuous map may not be continuous. For more details and examples, see Turkoglu et al. [6].

We write $\Phi = \{\varphi : R^+ \to R^+ : \varphi \text{ is continuous and } \varphi(t) = 0 \text{ if and only if } t = 0\}$.

We call an element $\varphi \in \Phi$ an “altering distance function”.

Park [4] proved the following theorem.

Theorem 1 (see [4]). Let $T$ be a selfmap of $X$.

Suppose that for some $x_0 \in X$, $O_T(x_0)$ has a cluster point $z$ in $X$. (1)

If $T$ is orbitally continuous at $z$ and $Tz$ and $T$ satisfies

$$d(Tx, Ty) < d(x, y)$$

(2)
for each \(x, y \in O_T(x_0), x \neq y, y = T x\), then \(z\) is a fixed point of \(T\).

By using an altering distance function \(\varphi \in \Phi\), Sastry and Babu [5] proved the following theorem.

**Theorem 2** (see [5]). Let \(T\) be a selfmap of \(X\). Suppose that \(T\) satisfies (1). If \(T\) is orbitally continuous at \(z\) and \(Tz\), and if there exists \(\varphi \in \Phi\) such that

\[
\varphi(d(Tx, Ty)) < \varphi(d(x, y))
\]

for each \(x, y \in O_T(x_0), x \neq y, y = T x\), then \(z\) is a fixed point of \(T\).

**Remark 1.** Theorem 1 follows by choosing \(\varphi(t) = t, t \geq 0\), in Theorem 2.

**Theorem 3** (see [4]). Let \(T\) be a selfmap of a metric space \(X\). Assume that for some positive integer \(m\), there exists a point \(x_0 \in X\) such that

\[
O_{T^m}(x_0) \text{ has a cluster point } z \text{ in } X,
\]

and

\[
d(T^m x, T^m y) < d(x, y)
\]

for all \(x, y \in X, x \neq y\). Then \(z\) is a unique fixed point of \(T\) in \(X\).

The study of fixed points of Meir-Keeler type contractions in the presence of an altering distance function is an interesting and open area. Thus the purpose of this paper is to introduce strip \(\varphi\)-contraction for \(\varphi \in \Phi\), which is more general than Meir-Keeler type contraction (Example 1), and obtain sufficient conditions for the existence of fixed points for such maps. Further, it is extended to a pair of selfmaps. These results improve and generalize the theorems of Khan, Swaleh and Sessa [1], Sastry and Babu [5] and Park [4] to strip \(\varphi\)-contractions.

## 2. Preliminaries

Meir and Keeler [3] established a fixed point theorem for selfmaps satisfying the following \((\epsilon, \delta)\) - contraction, which is known as Meir-Keeler type contraction.

**Definition 1.** Given \(\epsilon > 0\), there exists \(\delta > 0\) such that

\[
\epsilon \leq d(x, y) < \epsilon + \delta \implies d(Tx, Ty) < \epsilon
\]

for all \(x, y \in X\).

Maiti and Pal [2] improved condition (6) in the following way and obtained fixed points: given \(\epsilon > 0\), there is a \(\delta > 0\), such that

\[
\epsilon \leq \max\{d(x, y), d(x, T x), d(y, T y)\} < \epsilon + \delta
\]

implies

\[
d(Tx, Ty) < \epsilon
\]

for all \(x, y \in X\).

We now introduce “strip \(\varphi\)-contraction” as follows:
**Definition 2.** Let \((X, d)\) be a metric space and \(T\) a selfmap on \(X\). Let \(\varphi \in \Phi\). We say that \(T\) is a strip \(\varphi\)-contraction if for a given \(\epsilon > 0\), there is a \(\delta > 0\), such that
\[
\epsilon \leq \varphi(d(x, y)) < \epsilon + \delta \text{ implies } \varphi(d(Tx, Ty)) < \epsilon
\] (7)
for all \(x, y\) in \(X\).

Here we observe that every strip \(\varphi\)-contraction is a Meir-Keeler type contraction when \(\varphi\) is the identity map of \(\mathbb{R}^+\). The following example shows that the class of all strip \(\varphi\)-contractions is larger than the class of all Meir-Keeler type contractions.

**Example 1.** Let \(X = \mathbb{N}\) with the usual metric. Define \(T : X \to X\) by \(Tx = x^3\). Define \(\varphi : \mathbb{R}^+ \to \mathbb{R}^+\) by
\[
\varphi(t) = \begin{cases} 
\frac{t^2}{2}, & \text{if } 0 \leq t \leq 1 \\
1 - \frac{t^2}{2}, & \text{if } t \geq 1.
\end{cases}
\]
Then clearly \(\varphi \in \Phi\).
We now show that \(T\) is a strip \(\varphi\)-contraction. Let \(0 < \epsilon < 1\). For any \(l, m \in X\), with \(l \neq m\),
\[
0 < \epsilon = \varphi(|l - m|) = \frac{1}{2(l - m)^2} < \epsilon + \delta \text{ with } \delta = \min\{\epsilon, 1 - \epsilon\}.
\]
Then we have
\[
\varphi(|Tl - Tm|) = \varphi(|l^3 - m^3|) = \frac{1}{2(|l^3 - m^3|^2)} < \frac{1}{4(l - m)^2} < \frac{1}{2}(\epsilon + \delta) \leq \epsilon,
\]
so that \(T\) satisfies the strip \(\varphi\)-contraction condition. The case when \(\epsilon \geq 1\) is trivial.
But for \(x = 1, y = 5\), with \(\varphi\) the identity map of \(\mathbb{R}^+\), choosing \(\epsilon = 4\) and for any \(\delta > 0\), we have
\[
\epsilon \leq |x - y| = 4 < \epsilon + \delta \text{ and } |Tx - Ty| = |T1 - T5| = 124 \leq \epsilon,
\]
so that \(T\) is not a Meir-Keeler type contraction.

The following example shows that the orbital continuity of \(T\) at \(z\) may not imply the orbital continuity of \(T\) at \(Tz\), where \(z\) is as in (1).

**Example 2.** Let \(X = \{\frac{1}{n}, n \in \mathbb{N}\} \cup \{1 - \frac{1}{n}, n \in \mathbb{N}\}\) with the usual metric. We define \(T : X \to X\) by
\[
T(0) = 1, T(1) = 1, T\left(\frac{1}{n}\right) = 1 - \frac{1}{n} \text{ for } n = 2, 3, \ldots
\]
and
\[
T\left(1 - \frac{1}{n}\right) = \frac{1}{n + 1} \text{ for } n = 3, 4, \ldots.
\]
First we show that $T$ is orbitally continuous at $0$. Let $x \in X$. If $\{x_n\} \subseteq O_T(x)$ such that $x_n \to 0$, then $\{x_n\}$ is a subsequence of $\{\frac{1}{k}\}$ and hence $Tx_n = 1 - x_n \to 1 = T(0)$.

But $T$ is not orbitally continuous at $T(0)$, since $1 - \frac{1}{n} \in O_T(\frac{1}{3})$ for $n \geq 3$,

$1 - \frac{1}{n} \to 1 = T(0)$ as $n \to \infty$ and $T(1 - \frac{1}{n}) = \frac{1}{n + 1} \to 0 \neq T(T(0)) = 1$ as $n \to \infty$.

3. Fixed point theorems using strip \(\varphi\)-contractions

**Theorem 4.** Let $T$ be a selfmap of $X$. Suppose that $T$ satisfies (1). Further, assume that

\[
\text{given } \epsilon > 0, \text{ there exist } \varphi \in \Phi \text{ and } \delta > 0, \text{ such that } \epsilon \leq \varphi(d(x, y)) < \epsilon + \delta \implies \varphi(d(Tx, Ty)) < \epsilon \tag{8}
\]

for all $x, y \in O_T(x_0), x \neq y, y = Tx$. Then $z$ is a fixed point of $T$ in $O_T(x_0)$ provided $T$ is orbitally continuous at $z$. This $z$ is unique in the sense that $O_T(x_0)$ contains one and only one fixed point $z$ of $T$.

**Proof.** We define the sequence $\{x_n\} \subseteq X$ by $\{x_n\} = T^n x_0$, for $n = 1, 2, \ldots$. Let

$\alpha_n = \varphi(d(x_n, x_{n+1}))$. If $x_n = x_{n+1}$ for some $n \in N$, then the conclusion of the theorem trivially holds.

Suppose $x_n \neq x_{n+1}$ for all $n$. Then from (8), we have

\[
\alpha_{n+1} = \varphi(d(Tx_n, Tx_{n+1})) < \varphi(d(x_n, x_{n+1})) = \alpha_n.
\]

Similarly $\alpha_n < \alpha_{n-1}$.

Therefore $\{\alpha_n\}$ is a decreasing sequence of non-negative reals and hence it converges to a nonnegative real number $\alpha$ (e.g.).

From (1), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \to z$ as $k \to \infty$. Hence

\[
\alpha = \lim_{k \to \infty} \alpha_{n(k)} = \lim_{k \to \infty} \varphi(d(x_{n(k)}, x_{n(k)+1})) = \varphi\left( \lim_{k \to \infty} d(x_{n(k)}, x_{n(k)+1}) \right) = \varphi(d(z, Tz))
\]

(since $T$ is orbitally continuous at $z$). Now, we claim that $\alpha = 0$. Suppose $\alpha > 0$. Then

\[
\alpha = \inf_{n \geq 1} \varphi(d(x_n, x_{n+1})).
\]

Also, for any $\delta > 0$ there exists $m \in N$ such that

\[
\alpha \leq \varphi(d(x_n, x_{n+1})) \leq \alpha + \delta \text{ for all } n \geq m. \tag{9}
\]
In particular
\[ \alpha \leq \varphi(d(x_m, x_{m+1})) < \alpha + \delta. \]
Hence from (8) and (9), we have
\[ \alpha \leq \varphi(d(x_{m+1}, x_{m+2})) < \alpha, \]
a contradiction.

Therefore \( \alpha = 0 \) so that \( \lim_{n \to \infty} \varphi(d(x_n, x_{n+1})) = 0 \) and since \( \varphi \) is an element of \( \Phi \), it follows that \( \lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \) so that \( d(z, Tz) = 0 \). Hence \( Tz = z \).

**Theorem 5.** Let \( T \) be a selfmap of \( X \). Assume that \( T \) satisfies (1). Further, assume that given \( \epsilon > 0 \) there exist \( \varphi \in \Phi \) and \( \delta > 0 \), such that
\[ \epsilon \leq \max\{\varphi(d(x, y)), \varphi(d(x, Tx)), \varphi(d(y, Ty))\} < \epsilon + \delta \]
implies
\[ \varphi(d(Tx, Ty)) < \epsilon \]
for all \( x, y \) in \( X \). Then \( z \) of (1) is a unique fixed point of \( T \).

**Proof.** Follows as a corollary to Theorem 4, in the sense that condition (10) implies (8).

**Theorem 6.** Let \( T \) be a selfmap of \( X \). Assume that for some \( x_0 \in X \) and for some positive integer \( m \)
\[ O_T(x_0) \text{ has a cluster point } z \text{ in } X. \]
Further, assume that for a given \( \epsilon > 0 \) there exist \( \varphi \in \Phi \) and \( \delta > 0 \) such that
\[ \epsilon \leq \varphi(d(x, y)) < \epsilon + \delta \text{ implies } \varphi(d(T^m x, T^m y)) < \epsilon \]
for all \( x \neq y, x, y \in X \), i.e. \( T^m \) is a strip \( \varphi \)-contraction. Then \( z \) is the unique fixed point of \( T \), provided \( T^m \) is orbitally continuous at \( z \).

**Proof.** By replacing \( T \) by \( T^m \) in Theorem 4, \( T^m \) has a unique fixed point \( z \) in \( X \). Therefore \( T^m z = z \). Now
\[ Tz = T(T^m z) = T^{m+1} z = T^m(Tz). \]
Therefore \( Tz \) is also a fixed point of \( T^m \). We now show that \( Tz = z \). Suppose \( Tz \neq z \). Then from (12) for
\[ \epsilon = \varphi(d(z, Tz)) < \epsilon + \delta \]
implies
\[ \varphi(d(T^m z, T^m(Tz))) = \varphi(d(z, Tz)) < \epsilon = \varphi(d(z, Tz)), \]
a contradiction. Therefore \( Tz = z \).
Remark 2. Strip $\varphi$-contraction is actually stronger than (3), since condition (8) implies (3). Hence some condition(s) in the hypotheses of Theorem 2, namely $T$ is orbitally continuous at $Tz$, may be relaxed under strip $\varphi$-contraction in obtaining fixed points, which is established in our results (Theorem 4 and Theorem 6).

Remark 3. In Theorem 4 we need not assume that strip $\varphi$-contraction condition (8) holds on the whole space $X$. The following example gives its justification.

Example 3. Let $X = N \cup \{0, 2^{-1}, 2^{-2}, \ldots\}$ with the usual metric. We define $T : X \to X$ by

$$T(0) = 0, T(n) = n + 1, T(2^{-n}) = 2^{-(n+1)}, n = 1, 2, 3, \ldots.$$ 

Here $X = O_T(1) \cup O_T(2^{-1}) \cup \{0\}$. At $x = 1$, $y = 2$, condition (7) fails to hold for any $\varphi \in \Phi$, since $\varphi(d(x, y)) = \varphi(d(1, 2)) = \varphi(1)$, and

$$\varphi(d(Tx, Ty)) = \varphi(d(T1, T2)) = \varphi(d(2, 3)) = \varphi(1).$$

Therefore for $\epsilon = \varphi(1)$, strip $\varphi$-contraction condition (8) fails to hold in $O_T(1)$ for any $\varphi \in \Phi$ and has no fixed point in $O_T(1)$.

But strip $\varphi$-contraction holds on the closure of the orbit of $2^{-1}$, where

$$O_T(2^{-1}) = \{0, 2^{-1}, 2^{-2}, \ldots\}$$

with $\varphi(t) = t^2$, $t \geq 0$ and $\delta = \min\{\epsilon, 1 - \epsilon\}$

when $0 < \epsilon < 1$; $T$ satisfies all the hypotheses of Theorem 4, with 0 as the cluster point of $O_T(2^{-1})$; and $T$ has the unique fixed point 0.

Thus, condition (8) is more general than condition (7).

Remark 4. The following two examples show that

(1) every strip $\varphi$-contraction need not be a contraction, and

(2) an operator satisfying strip $\varphi$-contraction may not have a fixed point if $T$ does not satisfy orbital continuity at $z$ of (1) in $X$.

Example 4. Let $X = \{1 + 2^{-n} : n = 1, 2, 3, \ldots\} \cup \{1\}$ with the usual metric. We define $T$ on $X$ by

$$T(1) = 1 + 2^{-1} \text{ and } T(1 + 2^{-n}) = 1 + 2^{-(n+1)}, n = 1, 2, 3, \ldots.$$ 

For $x_0 = 1$, $O_T(x_0) = \{1 + 2^{-n} : n = 1, 2, 3, \ldots\}$, $O_T(x_0) = O_T(1) \cup \{1\}$.

Then $T$ satisfies all the conditions of Theorem 4 with $\varphi$, the identity map of $R^+$ with $\delta = \min\{\epsilon, 1 - \epsilon\}$ for $0 \leq \epsilon < 1$, but $T$ is not orbitally continuous at $z(= 1)$ and it has no fixed point.

Example 5. Let

$$X = \left\{ \sum_{i=0}^{n} 2^{-i} : n \in N \right\} \cup \{1, 2\}.$$
with the usual metric. Define \( T \) on \( X \) by

\[
T_2 = 1, \; T_1 = 1 + 2^{-1}, \; T \left( \sum_{i=0}^{n} 2^{-i} \right) = \sum_{i=0}^{n+1} 2^{-i}, \; \text{for} \; n \in \mathbb{N}.
\]

If \( x_0 = 1 + 2^{-1} \), then

\[
O_T(1 + 2^{-1}) = \left\{ \sum_{i=0}^{n} 2^{-i} : n \in \mathbb{N} \right\} \quad \text{and} \quad O_T(1 + 2^{-1}) = O_T(1 + 2^{-1}) \cup \{2\}.
\]

Also, \( T \) satisfies all the hypotheses of Theorem 4, with \( \varphi(t) = \frac{t^2}{2}, \; t > 0, \) with

\[
\delta = \min\{\epsilon, 1 - \epsilon\}
\]

but \( T \) is not orbitally continuous at \( z(=2) \) and it has no fixed point.

Remark 5. Let us mention:

(i) In Theorem 4 we do not assume the orbital continuity of \( T \) at \( Tz \). Hence Theorem 4 improves the results of Sastry and Babu [5] and hence also Park [4], which in turn improves the results of Khan, Swaleh and Sessa [1].

(ii) By strengthening condition (3) by (8), the orbital continuity at \( Tz \) is relaxed.

4. Common fixed points for a pair of strip \( \varphi \)-contractions

We now extend Theorem 4 and Theorem 5 to a pair of selfmaps.

Theorem 7. Let \( S \) and \( T \) be selfmaps of \( X \) such that for some \( x_0 \in X \) we define the sequence \( \{x_n\} \) by \( x_{2n+1} = Sx_{2n} \) and \( x_{2n+2} = Tx_{2n+1} \), \( n = 0, 1, 2, \ldots \). Assume that either (a) or (b) of the following holds:

(a) \( \{x_{2n}\} \) has a cluster point \( z \) in \( X \), \( S \) and \( TS \) are orbitally continuous at \( z \),

(b) \( \{x_{2n+1}\} \) has a cluster point \( z \) in \( X \), \( T \) and \( ST \) are orbitally continuous at \( z \).

Further, assume that given \( \epsilon > 0 \) there exist \( \varphi \in \Phi \) and \( \delta > 0 \) such that

\[
\epsilon \leq \varphi(d(x, y)) < \epsilon + \delta \text{ implies } \varphi(d(Sx, Ty)) < \epsilon
\]

for all \( x, y \in \overline{\{x_n\}}, x \neq y \) satisfying either \( x = Ty \) or \( y = Sx \). Then either (i) or (ii) of the following is true:

(i) either \( S \) or \( T \) has a fixed point in \( X \),

(ii) \( z \) is a unique common fixed point of \( S \) and \( T \) in \( \overline{\{x_n\}} \).
Proof.

Suppose that \( x_{2n} = x_{2n+1} \) for some \( n \) in \( N \). Then \( S \) has a fixed point in \( X \). (14)

If \( x_{2n+1} = x_{2n+2} \) for some \( n \) in \( N \), then \( T \) has a fixed point in \( X \). (15)

(14) and (15) together imply that conclusion (i) holds. Now assume that \( x_n \neq x_{n+1} \) for all \( n \). Write \( \beta_n = \varphi(d(x_n, x_{n+1})) \). From (13) we have

\[
\beta_{2n} = \varphi(d(x_{2n}, x_{2n+1})) = \varphi(d(Tx_{2n-1}, Sx_{2n})) < \varphi(d(x_{2n-1}, x_{2n})) = \beta_{2n-1}.
\]

Therefore

\[
\beta_{2n} < \beta_{2n-1}. \tag{16}
\]

Similarly,

\[
\beta_{2n+1} < \beta_{2n}. \tag{17}
\]

Hence from (16) and (17) it follows that \( \{\beta_n\} \) is a decreasing sequence of non-negative reals and it converges to a real number \( \beta \) (e.g.).

Now assume (a). Then there exists a sequence \( \{n(k)\} \) of positive integers such that

\[
x_{2n(k)} \to z, \quad Sx_{2n(k)} \to Sz, \quad T(Sx_{2n(k)}) \to TSz. \tag{18}
\]

From the continuity of \( \varphi \), we have

\[
\beta = \lim_{k \to \infty} \beta_{2n(k)} = \lim_{k \to \infty} \varphi(d(x_{2n(k)}, x_{2n(k)+1})) = \varphi(d(z, Sz)).
\]

We now claim that \( \beta = 0 \). Suppose that \( \beta > 0 \), then

\[
\beta = \inf_{n \geq 1} \varphi(d(x_n, x_{n+1})).
\]

Then for any \( \delta > 0 \), there exists \( m \in N \) such that

\[
\beta \leq \varphi(d(x_n, x_{n+1})) < \beta + \delta \text{ for all } n \geq m. \tag{19}
\]

In particular, writing \( n = 2m \) and using (13), we have

\[
\beta \leq \varphi(d(x_{2m}, x_{2m+1})) < \beta + \delta
\]

which implies

\[
\varphi(d(Sx_{2m}, Tx_{2m+1})) = \varphi(d(x_{2m+1}, x_{2m+2})) < \beta,
\]

a contradiction to (19). Hence \( \beta = 0 \) and it implies that \( Sz = z \).

Now we prove that \( Tz = z \). From (13) we have

\[
\varphi(d(Sx_{2n(k)}, Tx_{2n(k)+1})) < \varphi(d(x_{2n(k)}, Sx_{2n(k)})).
\]
Now by taking limits as \( k \to \infty \), by using (18) and continuity of \( \varphi \), it follows that 
\[
\varphi(d(Sz, T(Sz))) \leq \varphi(d(z, Sz)) = 0.
\]
Thus \( TSz = Sz \). Since \( z = Sz \), it follows that 
\( Tz = z \); and hence \( z \) is a common fixed point of \( S \) and \( T \).

Similarly, when (b) holds, then it follows that \( z \) is a common fixed point of \( S \) and \( T \). Uniqueness of a fixed point trivially follows from (13). Thus \( S \) and \( T \) have a unique common fixed point \( z \in \{x_n\} \).

Hence conclusion (ii) follows. \( \square \)

**Theorem 8.** Let \( S \) and \( T \) be selfmaps of \( X \) such that for some \( x_0 \in X \) the sequence \( \{(TS)^nx_0\} \) has a convergent subsequence, which converges to a point \( z \) in \( X \) and \( S \), and let \( TS \) be orbitally continuous at \( z \). Further assume that \( S \) and \( T \) satisfy the following condition: given \( \epsilon > 0 \), there exist \( \varphi \in \Phi \) and \( \delta > 0 \), such that 
\[
\epsilon \leq \max\{\varphi(d(x, y)), \varphi(d(x, Sz)), \varphi(d(y, Ty))\} < \epsilon + \delta
\]
implies
\[
\varphi(d(Sx, Ty)) < \epsilon
\]
for all \( x, y \in X \), \( x \neq y \) satisfying either \( x = Ty \) or \( y = Sx \). Then, either (i) or (ii) of the following is true:

(i) either \( S \) or \( T \) has a fixed point in \( X \),

(ii) \( S \) and \( T \) have a unique common fixed point in \( \overline{(TS)^nx_0} \).

**Proof.** Follows as a corollary to Theorem 7, since (20) implies (13). \( \square \)

The following is an example in support of Theorem 7.

**Example 6.** Let \( X = [0, 2) \) with the usual metric. We define \( S, T : X \to X \) by
\[
Sx = \begin{cases} 
\frac{x}{2}, & \text{if } x \in [0, 1) \\
\frac{x^2}{8}, & \text{if } x \in [1, 2). 
\end{cases}
\]
\[
Tx = \begin{cases} 
\frac{x}{2}, & \text{if } x \in [0, 1) \\
\frac{x^2}{16}, & \text{if } x \in [1, 2). 
\end{cases}
\]

For any \( x_0 \in [0, 1) \), the sequence \( \{x_n\} \) defined in Theorem 7 is given by \( x_n = \frac{x_{n-1}}{n} \), \( n = 0, 1, 2, 3, \ldots \) and \( \overline{x_n} = \{x_n\}_{n=0}^{\infty} \cup \{0\} \). Now for the case when \( x_0 \in [1, 2) \), the sequence \( \{x_n\} \) is given by
\[
\{x_n\} = \{x_0\} \cup \{\frac{x_{n-1}^2}{2^{n+2}} : n = 1, 2, 3, \ldots\} \quad \text{and} \quad \overline{x_n} = \{x_0\} \cup \{x_n : n = 1, 2, 3, \ldots\} \cup \{0\}.
\]

Case (i): Let \( x_0 \in [0, 1) \). Let \( 0 < \epsilon < 1 \) with \( \delta = \min\{\epsilon, 1 - \epsilon\} \). Define \( \varphi \) on \( R^+ \) by \( \varphi(t) = t^2, t \geq 0 \). For \( x = \frac{x_0}{n} \) and \( y = Sx = \frac{x_0}{2n+1}, \ n = 0, 1, 2, \ldots \), we have
\[
\varphi(d(x, y)) = \varphi(\frac{x_0}{2n} - \frac{x_0}{2n+1}) = \varphi(\frac{x_0}{2n+1}) = \left(\frac{x_0}{2n+1}\right)^2 < \epsilon + \delta,
\]
\[
\varphi(d(Sx, Ty)) = \varphi(\frac{x_0}{2n+1} - \frac{x_0}{2n+2}) = \varphi(\frac{x_0}{2n+2}) = \left(\frac{x_0}{2n+2}\right)^2 = \left(\frac{x_0}{2n+1}\right)^2 < \frac{1}{4}(\epsilon + \delta) < \epsilon.
\]
Case (ii): Let $x_0 \in [1, 2)$. Let $0 < \epsilon < 1$, with $\delta = \min\{\epsilon, 1 - \epsilon\}$. When $x = x_0; y = Sx = \frac{x_0^2}{2^4}$, we have

$$\varphi(d(x, y)) = \varphi(|x_0 - \frac{x_0^2}{2^4}|) = \varphi(\frac{8x_0 - x_0^2}{2^4}) = (\frac{8x_0 - x_0^2}{2^4})^2 < \epsilon + \delta$$

and

$$\varphi(d(Sx, Ty)) = \varphi(|\frac{x_0^2}{2^7} - \frac{x_0^2}{2^4}|) = (\frac{x_0^2}{2^4})^2 < \frac{1}{2}(\epsilon + \delta) < \epsilon.$$

In general, when $x = \frac{x_0^2}{2^{n+3}}; y = Sx = \frac{x_0^2}{2^{n+4}}$, we have

$$\varphi(d(x, y)) = \varphi(|\frac{x_0^2}{2^{n+2}} - \frac{x_0^2}{2^{n+3}}|) = \varphi(\frac{x_0^4}{(2^{n+3})^2}) = (\frac{x_0^4}{(2^{n+3})^2})^2 < \epsilon + \delta$$

and

$$\varphi(d(Sx, Ty)) = \varphi(|\frac{x_0^2}{2^{n+3}} - \frac{x_0^2}{2^{n+4}}|) = (\frac{x_0^4}{(2^{n+4})^2})^2 < \frac{1}{4}(\epsilon + \delta) < \epsilon.$$

Thus $S$ and $T$ satisfy condition (13) with $\varphi(t) = t^2$, $t \geq 0$.

Also, in any case $\{x_n\}$ has a convergent subsequence which converges to the point $0$; and satisfy all the hypotheses of Theorem 7; and $0$ is the unique common fixed point of $S$ and $T$.

**Remark 6.** The following is an example to show that conclusion (i) of Theorem 8 is valid.

**Example 7.** Let $X = \{0, 1\}$. Define selfmaps $S$ and $T$ on $X$ by

$$Sx = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x = 1, \end{cases} \quad \text{and} \quad Tx = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x = 1. \end{cases}$$

Then $S$ and $T$ trivially satisfy strip $\varphi$-contraction for any $\varphi \in \Phi$ (in particular, we take $\varphi(t) = t^2$, $t \geq 0$) and they also satisfy all the conditions of Theorem 7. Observe that $S$ has two fixed points 0 and 1 whereas $T$ has no fixed points.

**References**


