On an extension of a quadratic transformation formula due to Kummer

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Abstract. The aim of this research note is to prove the following new transformation formula

\[
(1 - x)^{-2a} \binom{a}{a + \frac{1}{2}, d + 1} \binom{c + \frac{3}{2}, d}{x^2 (1 - x)^2} = \binom{2a, c}{2d + \frac{1}{2} A + \frac{1}{2}, 2d - \frac{1}{2} A + \frac{1}{2} ; 2x}
\]

valid for $|x| < \frac{1}{2}$ and if $|x| = \frac{1}{2}$, then $\text{Re}(c - 2a) > 0$, where $A = \sqrt{16d^2 - 16cd - 8d + 1}$.

For $d = c + \frac{1}{2}$, we get quadratic transformations due to Kummer. The result is derived with the help of the generalized Gauss's summation theorem available in the literature.

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1. Introduction and results required

We start with the following very interesting and useful quadratic transformation formula due to Kummer [3] or [5, eq. 1, p. 65] viz.

\[
(1 - x)^{-2a} \binom{a, a + \frac{1}{2}, d + 1}{c + \frac{3}{2}, d ; x^2 (1 - x)^2} = \binom{2a, c}{2d + \frac{1}{2} A + \frac{1}{2}, 2d - \frac{1}{2} A + \frac{1}{2} ; 2x}
\]

\[
\binom{2a, c}{2d + \frac{1}{2} A - \frac{1}{2}, 2d - \frac{1}{2} A - \frac{1}{2} ; 2x}
\]
The aim of this research note is to provide an extension of (1) by employing the following summation formula [4, eq. 10, p. 534]

\[
\begin{align*}
_3F_2 \left[ \begin{array}{c} f, a, c + 1 \\ b, c 
\end{array} ; 1 \right] &= \frac{(c-a)(\alpha-f)}{c} \frac{\Gamma(b-a-f-1)}{\Gamma(b-a)\Gamma(b-f)} \\
\end{align*}
\]

provided \( \text{Re}(b-a-f) > 1, c \neq 0, -1, -2, \ldots \) and \( \alpha \) is given by

\[
\alpha = \frac{c(1+a-b)}{a-c}.
\]

2. Main result

The extension of the Kummer’s quadratic transformation formula (1) to be established is given by the following theorem.

**Theorem 1.** For \( |x| < \frac{1}{2} \) or if \( |x| = \frac{1}{2} \), then \( \text{Re}(c-2a) > 0 \), we have for \( d \neq 0, -1, -2, \ldots \)

\[
(1-x)^{-2a} _3F_2 \left[ \begin{array}{c} a, a + \frac{1}{2}, d + 1 \\ c + \frac{3}{2}, d 
\end{array} ; \frac{x^2}{(1-x)^2} \right] \]

\[
= _4F_3 \left[ \begin{array}{c} 2a, c, 2d + \frac{1}{2}A + \frac{1}{2}, 2d - \frac{1}{2}A + \frac{1}{2} \\ 2c + 2, 2d + \frac{1}{2}A - \frac{1}{2}, 2d - \frac{1}{2}A - \frac{1}{2} 
\end{array} ; 2x \right],
\]

where

\[
A = \sqrt{16d^2 - 16cd - 8d + 1}.
\]

**Proof.** In order to derive (3), we proceed as follows. Denoting the left-hand side of (3) by \( S(x) \), we have

\[
S(x) = (1-x)^{-2a} _3F_2 \left[ \begin{array}{c} a, a + \frac{1}{2}, d + 1 \\ c + \frac{3}{2}, d 
\end{array} ; \frac{x^2}{(1-x)^2} \right]
\]

\[
= \sum_{m=0}^{\infty} \frac{(a)_m(a + \frac{1}{2})_m(d+1)_m}{(c + \frac{3}{2})_m(d)_m m!} x^{2m} (1-x)^{-(2a+2m)}
\]

\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m(a + \frac{1}{2})_m(d+1)_m}{(c + \frac{3}{2})_m(d)_m m! n!} x^{2m} (2a+2m)_n n! x^n.
\]

Using now Bailey’s transform of the double series, the appropriate Pochhammer
symbol transformation formula and summing up the resulting series, we get

\[
S(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\binom{n}{m}(a + m + 1)_m}{(c + \frac{1}{2})_m(m!)} x^n
\]

\[
= \sum_{n=0}^{\infty} \frac{(2a)_n}{n!} x^n \sum_{m=0}^{\infty} \frac{\binom{-\frac{1}{2}n}{m}(-\frac{1}{2}n + \frac{1}{2})_m}{(c + \frac{3}{2})_m(m!)} x^n
\]

\[
= \sum_{n=0}^{\infty} \frac{(2a)_n}{n!} x^n \, _3F_2 \left[ \begin{array}{c} -\frac{1}{2}n, \frac{1}{2}n + \frac{1}{2}, d+1 \\ c+\frac{3}{2}, \frac{1}{2} \end{array} ; 1 \right].
\]

The inner \(_3F_2\) can be evaluated using (2) by taking \(f = -\frac{1}{2}n, a = -\frac{1}{2}n + \frac{1}{2}, b = c + \frac{3}{2}\) and \(c = d\). After simplification we get

\[
S(x) = \sum_{n=0}^{\infty} \frac{(2a)_n}{(2c + 2)_n n!} x^n \frac{(2d + \frac{3}{2}A + \frac{1}{2})_n(2d - \frac{1}{2}A + \frac{1}{2})_n}{(2d + \frac{3}{2}A - \frac{1}{2})_n(2d - \frac{1}{2}A - \frac{1}{2})_n}
\]

\[
= \quad \frac{\binom{2a}{c}(2d + \frac{3}{2}A + \frac{1}{2}, 2d - \frac{1}{2}A + \frac{1}{2})_n}{(2d + \frac{3}{2}A - \frac{1}{2}, 2d - \frac{1}{2}A - \frac{1}{2})_n}
\]

where \(A\) is the same as in (4). This completes the proof of (3). \(\square\)

**Corollary 1.** In (3), if we take \(d = c + \frac{1}{2}\), then since \(A = 1\), we get at once the Kummer’s result (1) which was rederived by Bailey [1, p. 243] by employing Gauss’s summation theorem [2, eq. 1, p. 2]. Hence (3) can be regarded as an extension of (1).

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**References**