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Pythagorean theorems in the alpha plane

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Abstract. In this paper, we give three α versions of the Pythagorean Theorem and show that the converse statements are not true. Finally, we give a necessary and sufficient condition for a triangle in the α plane to have a right angle.
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1. Introduction

The taxicab metric was given in a family of metrics of the real plane by Minkowski [12]. Later, Chen [3] developed a Chinese checker metric, and Tian [15] gave a family of metrics, α metric for $\alpha \in [0, \pi/4]$, which include the taxicab and Chinese checker metrics as special cases. Then, Gelişgen and Kaya extended an α distance to three and n dimensional spaces in [7] and [8], respectively. Finally, Çolakoğlu [4] extended the α metric to $\alpha \in [0, \pi/2)$ instead of $\alpha \in [0, \pi/4]$.

Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two points in $\mathbb{R}^2$. For each $\alpha \in [0, \pi/2)$, the α (alpha) distance between P and Q is

$$d_\alpha(P,Q) = \max\{|x_1 - x_2|, |y_1 - y_2|\} + (\sec\alpha - \tan\alpha)\min\{|x_1 - x_2|, |y_1 - y_2|\}$$

while the well-known Euclidean distance between P and Q is

$$d_E(P,Q) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}.$$

As usual, the plane constructed on the real plane $\mathbb{R}^2$ using the α metric instead of the well-known Euclidean metric for the distance between any two points is called an α plane. It is almost the same as a Euclidean plane; the points are the same, the lines are the same, and the angles are measured in the same way. However, the distance function is different. Since α geometry has a distance function different from that in Euclidean geometry, it is interesting to study the α analogues of topics that include the distance concept in Euclidean geometry. In this study, we give three α analogues of the well-known Pythagorean theorem. The taxicab and Chinese checker versions of the Pythagorean theorem were already given in [5], [6] and [9]. However, it is shown here that the given versions can be extended to the α plane, and they are just two special cases of α versions of the Pythagorean theorem.

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2. Two α versions of Pythagorean theorem

It is well-known that if ABC is a triangle with right angle A in the Euclidean plane, then $\mathbf{a}^2 = \mathbf{b}^2 + \mathbf{c}^2$, where $\mathbf{a} = d_E(B, C)$, $\mathbf{b} = d_E(A, C)$ and $\mathbf{c} = d_E(A, B)$; this is the Pythagorean Theorem. It is also well-known that its converse is true in the Euclidean plane. An α version of the Pythagorean Theorem for a right triangle ABC would be an equation that relates the three α distances a, b, c between pairs of vertices, where $a = d_\alpha(B, C)$, $b = d_\alpha(A, C)$ and $c = d_\alpha(A, B)$. Here, we give α versions of the Pythagorean Theorem, that relates α distances between the vertices of a right triangle, using the slope of one side of the triangle. We also show that the converses of these α versions of the Pythagorean Theorem are false in the α plane.

We use $\lambda(\alpha) = (\sec\alpha - \tan\alpha)$ throughout the paper to shorten phrases. The following equation, which relates the Euclidean distance to the α distance between two points in the Cartesian coordinate plane, plays an important role in our arguments.

Lemma 1. *For any two points P and Q in the Cartesian plane that do not lie on a vertical line, if m is the slope of the line through P and Q, then*

$$d_E(P, Q) = \rho(m) d_\alpha(P, Q), \tag{1}$$

where $\rho(m) = (1 + m^2)^{1/2} \big/ (\max\{1, |m|\} + \lambda(\alpha) \min\{1, |m|\})$. If P and Q lie on a vertical line, then by definition, $d_E(P, Q) = d_\alpha(P, Q)$.

Proof. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ with $x_1 \neq x_2$; then $m = (y_2 - y_1) \big/ (x_2 - x_1)$. Equation (1) is derived by a straightforward calculation with m and the coordinate definitions of $d_E(P, Q)$ and $d_\alpha(P, Q)$ given in Section 1. □

Another useful fact that can be verified by direct calculation is:

Lemma 2. *For any real number $m \neq 0$*

$$\rho(m) = \rho(-m) = \rho(1/m) = \rho(-1/m). \tag{2}$$

We first note by Lemma 1 and Lemma 2 that the α distance between two points is invariant under all translations, rotations of $\pi/2$, π and $3\pi/2$ radians around a point, and reflections with respect to the lines parallel to $x = 0$, $y = 0$, $y = x$ or $y = -x$ (see also [4]).

In this section, unless otherwise stated, ABC is a triangle in the Cartesian coordinate plane with vertices labeled in a counterclockwise order, with a right angle at A. Euclidean distances between pairs of vertices are denoted by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and the corresponding α distances are a, b, c, as defined earlier. The following Lemma states that although Euclidean distances $\mathbf{b}$ and $\mathbf{c}$ are in general different from the corresponding α distances b and c, corresponding ratios of these distances are equal.

Lemma 3. $\dfrac{\mathbf{b}}{\mathbf{c}} = \dfrac{b}{c}$.

Proof. If the legs AB and AC of ABC are parallel to the coordinate axes, then $\mathbf{b} = b$ and $\mathbf{c} = c$, and the two ratios are equal. If one of the legs of ABC is not parallel to a coordinate axis, then the other leg is not parallel to a coordinate axis either. If the slope of AB is m, then the slope of AC is $m' = -1/m$, since the legs are perpendicular. By equation (1), $\mathbf{c} = \rho(m)c$ and $\mathbf{b} = \rho(m')b$. But then equation (2) implies that $\mathbf{b}/\mathbf{c} = b/c$. □

Our main results in this section give relations between the three α distances a, b, c and the slope of one of the legs or the slope of the hypotenuse of a right triangle ABC. If a leg or the hypotenuse of ABC is parallel to a coordinate axis, then there is a relation between a, b, and c that does not depend on any other parameter. Note that the real number α, and hence $\lambda(\alpha)$ are not parameters in the α plane.

Theorem 1. *The following statements holds:*

(i) If the legs of ABC are parallel to the coordinate axes, then

$$a = \max\{b, c\} + \lambda(\alpha) \min\{b, c\}. \tag{3}$$

(ii) If the legs of ABC are not parallel to the coordinate axes, the hypotenuse BC is not vertical, and m is the slope of one leg, then

$$(\max\{1, |m|\} + \lambda(\alpha) \min\{1, |m|\})a = \max\{|bm + c|, |cm - b|\} + \lambda(\alpha) \min\{|bm + c|, |cm - b|\}. \tag{4}$$

Proof. (i) This follows immediately from the definition of α distance.
(ii) Let θ denote the angle CBA; note that θ is positive and acute by counterclockwise labeling of ABC (see Figure 1). Then $\tan\theta = \mathbf{b}/\mathbf{c} = b/c$ by Lemma 3.

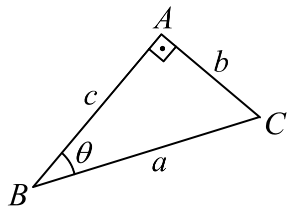

Figure 1.

First, suppose that m is slope of AB; then $m' = -1/m$ is slope of AC. Let m_1 be slope of BC. It is well-known (or easily found, using the identity for the tangent of the difference of two angles) that $\tan\theta = (m - m_1)/(1 + mm_1)$. Thus

$$\frac{b}{c} = \frac{m - m_1}{1 + mm_1}. \tag{5}$$

Solving equation (5) for m_1 yields

$$m_1 = \frac{cm - b}{bm + c}, \tag{6}$$

where $m \neq -c/b$. Applying equation (1) to the Pythagorean theorem $\mathbf{a}^2 = \mathbf{b}^2 + \mathbf{c}^2$, and using Lemma 2 gives

$$\left[\frac{(1+m_1^2)^{1/2}}{(\max\{1,|m_1|\}+\lambda(\alpha)\min\{1,|m_1|\})}\right]^2 a^2 = \left[\frac{(1+m^2)^{1/2}}{(\max\{1,|m|\}+\lambda(\alpha)\min\{1,|m|\})}\right]^2 \times(b^2+c^2) \tag{7}$$

which simplifies to

$$(\max\{1,|m|\}+\lambda(\alpha)\min\{1,|m|\})^2 a^2 = \left(\frac{(\max\{1,|m_1|\}+\lambda(\alpha)\min\{1,|m_1|\})^2}{1+m_1^2}\right) \times(1+m^2)(b^2+c^2). \tag{8}$$

Substituting for m_1 as given in equation (6), the right-hand side of equation (8) can be simplified to $(\max\{|bm+c|\,,|cm-b|\}+\lambda(\alpha)\min\{|bm+c|\,,|cm-b|\})^2$. Finally, taking the square root of both sides of the simplified equation produces equation (4). If the slope of AC is m, then the slope of AB is $m' = -1/m$, and our proof produces equation (4′) which is equation (4) with m' replacing m throughout. But if equation (4′) is multiplied by $|m|$, equation (4) results. Thus equation (4) is true when m is the slope of either AB or AC. □

Corollary 1. *If the hypotenuse BC of ABC is parallel to a coordinate axis, then*

$$(\max\{b,c\}+\lambda(\alpha)\min\{b,c\})a = b^2+c^2. \tag{9}$$

Proof. This is a consequence of equation (7). If BC is parallel to the x-axis, then $m_1 = 0$, and if BC is parallel to the y-axis, then we let $m_1 \to \infty$ in the quotient $[(1+m_1^2)^{1/2}/(\max\{1,|m_1|\}+\lambda(\alpha)\min\{1,|m_1|\})]$. In either case, equation (7) becomes $a^2 = [(1+m^2)^{1/2}/(\max\{1,|m|\}+\lambda(\alpha)\min\{1,|m|\})]^2(b^2+c^2)$, where m is the slope of AB. Let AD be the altitude from A (see Figure 2).

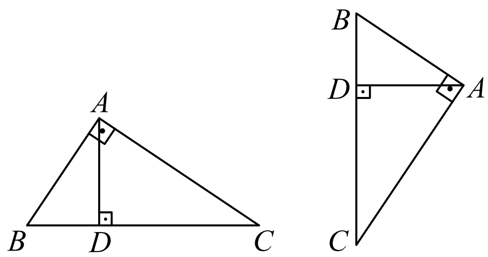

Figure 2.

By similar triangles and Lemma 3, $|m| = |AD/BD| = |AC/AB| = b/c$ when BC is horizontal, and $|m| = c/b$ when BC is vertical. Thus, if BC is horizontal, then $a^2 = [(1+(b/c)^2)^{1/2}/(\max\{1,b/c\}+\lambda(\alpha)\min\{1,b/c\})]^2(b^2+c^2)$; if BC is vertical, then $a^2 = [(1+(c/b)^2)^{1/2}/(\max\{1,c/b\}+\lambda(\alpha)\min\{1,c/b\})]^2(b^2+c^2)$. Each of these equations can be simplified to $(\max\{b,c\}+\lambda(\alpha)\min\{b,c\})^2a^2 = (b^2+c^2)^2$, which is equivalent to equation (9). □

The next corollary also gives an α version of the Pythagorean Theorem, with the slope of the hypotenuse as a parameter, instead of the slope of one of the legs.

Corollary 2. *If no side of ABC is parallel to a coordinate axis, and m_1 is the slope of the hypotenuse BC of ABC, then*

$$a\diagup(\max\{1,|m_1|\}+\lambda(\alpha)\min\{1,|m_1|\}) = (b^2+c^2)\diagup(\max\{|bm_1-c|,|cm_1+b|\} + \lambda(\alpha)\min\{|bm_1-c|,|cm_1+b|\}). \tag{10}$$

Proof. If m is the slope of AB, then we can solve equation (5) for m:

$$m = \frac{b+cm_1}{c-bm_1}, \tag{11}$$

where $m \neq c/b$. Substituting this value for m in equation (4) and simplifying yields equation (10). □

Remark 1. *We note that when AB is parallel to the x-axis, our derivation of equation (4) in the proof of Theorem 1 is still valid, and since $m=0$, equation (4) reduces to equation (3). Similarly, for the case when BC is parallel to the x-axis, equation (10) reduces to equation (9). In addition, equations (3) and (9) for the cases when AB or BC is vertical agree with the limits obtained when $m \to \infty$ in equation (4) or $m_1 \to \infty$ in equation (10), respectively. To see this, first recall that equations (4) and (9) are derived from equation (8). Note that as $m \to \infty$, $[(1+m^2)^{1/2}\diagup(\max\{1,|m|\}+\lambda(\alpha)\min\{1,|m|\})]^2 \to 1$ and $m_1 \to c/b$ (see equation (6)). Thus as $m \to \infty$, equation (8) becomes $a^2(1+c^2/b^2)\diagup(\max\{1,c/b\}+\lambda(\alpha)\min\{1,c/b\})^2 = b^2+c^2$, which simplifies to equation (3). Similarly, as $m_1 \to \infty$, $[(1+m_1^2)^{1/2}\diagup(\max\{1,|m_1|\}+\lambda(\alpha)\min\{1,|m_1|\})]^2 \to 1$ and $m \to -c/b$ (see equation (11)). In this case, as $m_1 \to \infty$, equation (8) becomes equation (9).*

Remark 2. *If ABC with a right angle at A is labeled in a clockwise order, then the roles of b and c are interchanged, and so equation (4) becomes*

$$(\max\{1,|m|\}+\lambda(\alpha)\min\{1,|m|\})a = \max\{|bm-c|,|cm+b|\} + \lambda(\alpha)\min\{|bm-c|,|cm+b|\}$$

and equation (10) becomes

$$a\diagup(\max\{1,|m_1|\}+\lambda(\alpha)\min\{1,|m_1|\}) = (b^2+c^2)\diagup(\max\{|bm_1+c|,|cm_1-b|\} + \lambda(\alpha)\min\{|bm_1+c|,|cm_1-b|\}).$$

We now give an example that shows the converse of Theorem 1, and therefore the converse of Corollary 2, are false. That is, there are triangles ABC for which equation (4) holds, but they have no right angle. The example refers to Figure 3, in which two different α circles are shown. Recall that the α circle with center A and radius r is the set of all points whose α distance to A is r, and if A is the origin and $r=1$, then this locus of points is an octagon with center O and vertices $A_1=(1,0)$, $A_2=(1/k,1/k)$, $A_3=(0,1)$, $A_4=(-1/k,1/k)$, $A_5=(-1,0)$, $A_6=(-1/k,-1/k)$,

$A_7 = (0, -1)$, $A_8 = (1/k, -1/k)$ where $k = [1 + \lambda(\alpha)]$ (see Figure 3) (see also [4]). Vertices and sides of the octagon are called vertices and sides of the α circle, respectively. Just as for a Euclidean circle, the center A and one point at an α distance r from A completely determine the α circle.

Example 1. *Let ABC be a triangle in the α plane, labeled in the counterclockwise order, with right angle A and with AB parallel to the x-axis. Let $d_\alpha(B,C) = a$, $d_\alpha(A,C) = b$, $d_\alpha(A,B) = c$ such that $b > c$. Let C_1 denote the α circle with radius b and center A, and CD a side of C_1 such that DAB is an obtuse angle. Let C' be the intersection point of the line CD and the line through B and parallel to the line AD (see Figure 4). Construct the α circle C_2 with radius $d_\alpha(B,C')$ and center B. Obviously, C' is a vertex of C_2, and C lies on C_2. Since C lies on both C_1 and C_2, we have $d_\alpha(B,C') = a$ and $d_\alpha(A,C') = b$. Applying Theorem 1 to right triangle ABC, one gets that*

$$(\max\{1,|m|\} + \lambda(\alpha)\min\{1,|m|\})a = \max\{|bm+c|, |cm-b|\} + \lambda(\alpha)\min\{|bm+c|, |cm-b|\},$$

where $m = 0$. This equation is also valid for the triangle ABC' which has no right angle. Thus, the converse of Theorem 1 is not true in the α plane. Note that Figure 4 illustrates the example for $\alpha = \pi/4$, but the example is valid in a general case.

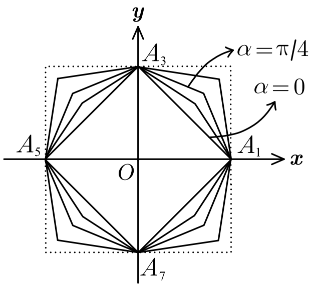

Figure 3.

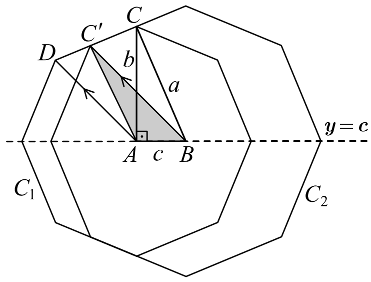

Figure 4.

3. Another α version of Pythagorean theorem

We need the following definitions given in [10] and [11], respectively, to derive another α version of the Pythagorean theorem:

Definition 1. *Let ABC be any triangle in the α plane. Clearly, there exists a pair of lines passing through every vertex of the triangle, each of which is parallel to a coordinate axis. A line l is called a base line of ABC if and only if*

(1) l passes through a vertex,

(2) l is parallel to a coordinate axis,

(3) the side opposite to the vertex described in (1) *(as a line segment) and l have a common point.*

Clearly, at least one of vertices of the triangle always has one or two base lines. Such a vertex of the triangle is called a basic vertex. A base segment is a line segment on a base line, which is bounded by a basic vertex and its opposite side.

Definition 2. *A line with slope m is called a steep line, a gradual line and a separator if $|m| > 1$, $|m| < 1$ and $|m| = 1$, respectively.*

The following theorem gives another α version of the Pythagorean theorem:

Theorem 2. *Let ABC be a triangle with a right angle at A in the α plane, such that BC is a non-gradual (non-steep) line, and B is the vertex whose point of orthogonal projection H to the base line is on the base segment, and let $d_\alpha(B,C) = a$, $d_\alpha(A,C) = b$, $d_\alpha(A,B) = c$, and $\gamma = d_\alpha(A,H)$. Then*

$$a = \begin{cases} \min\{b,c\} + \lambda(\alpha)\max\{b,c\} - \gamma\left(2\lambda(\alpha) - \frac{(1-\lambda^2(\alpha))\max\{b,c\}}{\min\{b,c\}}\right), & \text{if } C_1 \\ \max\{b,c\} + \lambda(\alpha)\min\{b,c\} - \gamma\left(2\lambda(\alpha) - \frac{(1-\lambda^2(\alpha))\min\{b,c\}}{\max\{b,c\}}\right), & \text{if } C_2 \\ \max\{b,c\} + \frac{1}{\lambda(\alpha)}\min\{b,c\} - \gamma\left(\frac{1+\lambda^2(\alpha)}{\lambda(\alpha)}\right), & \text{if } C_3 \\ \max\{b,c\} + \lambda(\alpha)\min\{b,c\} - \gamma\left(1+\lambda^2(\alpha)\right), & \text{if } C_4 \\ \max\{b,c\} + \lambda(\alpha)\min\{b,c\}, & \text{if } C_5 \end{cases} \tag{12}$$

where

C_1: *There exists only one base line through A which is horizontal (vertical), AB is a non-gradual (non-steep) line and $d_\alpha(A,B) \le d_\alpha(A,C)$,*

C_2: *There exists only one base line through A which is horizontal (vertical), AB is a non-gradual (non-steep) line and $d_\alpha(A,B) \ge d_\alpha(A,C)$,*

C_3: *There exists only one base line through A which is horizontal (vertical), and AB is a non-steep (non-gradual) line,*

C_4: *There exists only one base line through A which is vertical (horizontal),*

C_5: *There exist two base lines through A.*

Proof. Without loss of generality, one can take A as the origin and the hypotenuse BC as a non-gradual line since the α distance between two points is invariant under all translations, rotations of $\pi/2$, π and $3\pi/2$ radians around a point, and reflections about the lines parallel to $x = 0$, $y = 0$, $y = x$ or $y = -x$. Besides, while labeling the triangle ABC, B can be chosen as a vertex whose point of orthogonal projection H to the base line is on the base segment. It is clear that A is a basic vertex since triangle ABC has always one or two base lines through A. Let $B = (b_1, b_2)$ and $C = (c_1, c_2)$. Now, we have two possible main cases according to the number of base lines through A:

Case I: There exists only one base line through A. In this case, there are two possible cases according to the position of the base line.

Subcase 1: Let the base line be horizontal. Then we have two cases again:

A: Let AB be a non-gradual line. Then the triangles in Figure 5 and 6 represent all such triangles. Since BC and AB are non-gradual lines and AC is a non-steep line, we have $b_1 = \gamma$, $d_\alpha(A,B) = b_2 + \lambda(\alpha)b_1$, $d_\alpha(A,C) = c_1 + \lambda(\alpha)\,|c_2|$, $d_\alpha(B,C) = \ a = \ b_2 + |c_2| + \lambda(\alpha)(c_1 - b_1)$ and $\frac{|c_2|}{\gamma} = \frac{c_1}{b_2}$. Using d_2 and e_1 values in the last two equations, one gets

$$d_\alpha(B,C) = d_\alpha(A,B) + \lambda(\alpha)d_\alpha(A,C) - 2\lambda(\alpha)b_1 + |c_2|\,(1 - \lambda^2(\alpha)) \qquad (13)$$

and $|c_2| = \gamma\frac{d_\alpha(A,C)}{d_\alpha(A,B)}$. Also, it is clear that if $d_\alpha(A,B) \leq d_\alpha(A,C)$, then

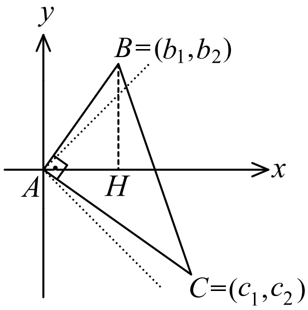

Figure 5.

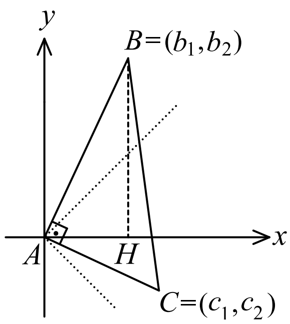

Figure 6.

$d_\alpha(A,B) = \min\{b,c\}$ and $d_\alpha(A,C) = \max\{b,c\}$; if $d_\alpha(A,B) \geq d_\alpha(A,C)$, then $d_\alpha(A,B) = \max\{b,c\}$ and $d_\alpha(A,C) = \min\{b,c\}$. Substituting for b_1 and $|c_2|$, we have

$$a = \min\{b,c\} + \lambda(\alpha)\max\{b,c\} - \gamma\left(2\lambda(\alpha) - (1 - \lambda^2(\alpha)\frac{\max\{b,c\}}{\min\{b,c\}}\right)$$

if C_1 holds, and

$$a = \max\{b,c\} + \lambda(\alpha)\min\{b,c\} - \gamma\left(2\lambda(\alpha) - (1 - \lambda^2(\alpha)\frac{\min\{b,c\}}{\max\{b,c\}}\right)$$

if C_2 holds.

B: Let AB be a non-steep line. Then the triangle in Figure 7 represents all such triangles. Since BC and AC are non-gradual lines and AB is a non-steep line, we have

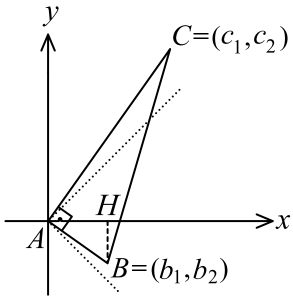

Figure 7.

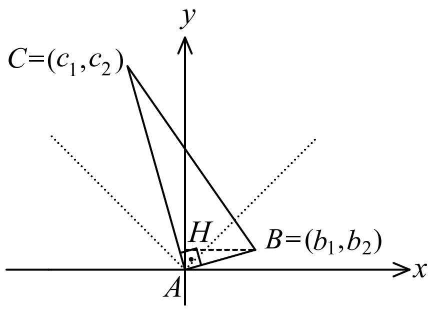

Figure 8.

$b_1 = \gamma$, $d_\alpha(A,B) = \min\{b,c\} = b_1 + \lambda(\alpha)|b_2|$, $d_\alpha(A,C) = \max\{b,c\} = c_2 + \lambda(\alpha)c_1$ and $d_\alpha(B,C) = a = c_2 + |b_2| + \lambda(\alpha)(c_1 - b_1)$. Using c_2 and $|b_2|$ values in the last equation, one gets

$$d_\alpha(B,C) = d_\alpha(A,C) + (\lambda(\alpha))^{-1} d_\alpha(A,B) - b_1(\lambda(\alpha) + (\lambda(\alpha))^{-1}). \tag{14}$$

By calculation we have $a = \max\{b,c\} + \frac{1}{\lambda(\alpha)}\min\{b,c\} - \gamma\left(\frac{1+\lambda^2(\alpha)}{\lambda(\alpha)}\right)$ if C_3 holds.
Subcase 2: Let the base line be vertical. Then the triangle in Figure 8 represents all such triangles. Since BC and AC are non-gradual lines and AB is a non-steep line, we have $b_2 = \gamma$, $d_\alpha(A,B) = \min\{b,c\} = b_1 + \lambda(\alpha)b_2$, $d_\alpha(A,C) = \max\{b,c\} = c_2 + \lambda(\alpha)|c_1|$ and $d_\alpha(B,C) = a = c_2 - b_2 + \lambda(\alpha)(b_1 + |c_1|)$. Using b_1 and c_2 values in the last equation, one gets

$$d_\alpha(B,C) = d_\alpha(A,C) + \lambda(\alpha) d_\alpha(A,B) - b_2(1 + \lambda^2(\alpha)). \tag{15}$$

Thus, we have $a = \max\{b,c\} + \lambda(\alpha)\min\{b,c\} - \gamma\left(1 + \lambda^2(\alpha)\right)$ if C_4 holds.

Case II: There exist two base lines through A. Then the legs of triangle ABC coincide with the base lines as in Figure 9.

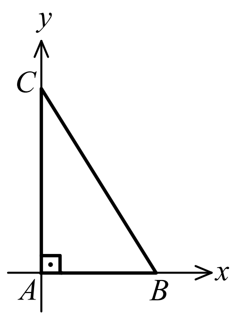

Figure 9.

By the definition of an α distance, it is clear that $a = \max\{b,c\} + \lambda(\alpha)\min\{b,c\}$ if C_5 holds. The proof is completed. □

It is very easy to see that the converse of Theorem 2 is false, and showing this fact is left to the reader.

The following theorem gives a necessary and sufficient condition for a triangle in the α plane to have a right angle. A sufficient condition is in fact a restatement of the converse of the Pythagorean Theorem.

Theorem 3. *Let ABC be a triangle in the α plane with no side parallel to the y-axis. Let $d_\alpha(B,C) = a$, $d_\alpha(A,C) = b$ and $d_\alpha(A,B) = c$, and let m_1, m, and m' denote the slopes of the lines BC, AB and AC, respectively. Then $\angle A$ is a right angle if and only if*

$$\rho(m_1)a^2 = \rho(m)(b^2 + c^2) = \rho(m')(b^2 + c^2) \tag{16}$$

where $\rho(x) = (1 + x^2)^{1/2} \big/ (\max\{1, |x|\} + \lambda(\alpha) \min\{1, |x|\})$.

Proof. If equation (16) holds, then $\rho(m) = \rho(m')$ and

$$\rho(m_1)a^2 = \rho(m)b^2 + \rho(m)c^2 = \rho(m')b^2 + \rho(m')c^2.$$

Therefore

$$\rho(m_1)a^2 = \rho(m)b^2 + \rho(m')c^2. \tag{17}$$

Applying equation (1) to equation (17) gives (by Lemma 1) $\mathbf{a}^2 = \mathbf{b}^2 + \mathbf{c}^2$, where $\mathbf{a} = d_E(B,C)$, $\mathbf{b} = d_E(A,C)$ and $\mathbf{c} = d_E(A,B)$. Since the converse of the Pythagorean Theorem is true, $\angle A$ is a right angle.

Now suppose, conversely, that $\angle A$ is a right angle. Then $m' = -1/m$, and so $\rho(m) = \rho(m')$ by Lemma 2. Equation (16) is just equation (7), derived in the proof of Theorem 1. □

Remark 3. *When $\alpha = 0$ and $\alpha = \pi/4$, the α-distance function is equal to Taxicab and the Chinese checker distance function, respectively. Therefore, if one uses $\alpha = 0$ and $\alpha = \pi/4$ in Theorem 1, Corollary 2 and Theorem 2, then one finds the taxicab and the Chinese checker versions of the Pythagorean theorem, respectively (see [10], [5], [9] and [6] for taxicab and the Chinese checker versions of the Pythagorean theorem).*

In a few words, we give in this study three alpha versions of the Pythagorean theorem, which relates lengths of sides of a right triangle in the alpha plane, considering the Euclidean angle measurement. However, we remark that it can be defined or different angle measurements or orthogonalities can be used in this plane (see [1] and [14] for different taxicab angle measurements, and see [2] for different orthogonalities), and one can derive different alpha versions of the Pythagorean theorem considering them. On the other hand, we have a question waiting to be answered: what is the n-dimensional case of the alpha version of the Pythagorean theorem?

References

[1] Z. Akça, R. Kaya, *On the Taxicab Trigonometry*, Journal of Inst. of Math. and Comp. Sci. **10**(1997), 151–159.

[2] D. Amir, *Characterizations of Inner Product Spaces*, Birkhäuser Verlag, Stuttgart, 1986.
[3] G. Chen, *Lines and Circles in Taxicab Geometry*, Master thesis, University of Central Missouri, 1992.
[4] H. B. Çolakoğlu, *Concerning the Alpha Distance*, Algebras, Groups and Geometries, to appear.
[5] H. B. Çolakoğlu, R. Kaya, *Taxicab Versions of the Pythagorean Theorem*, Pi Mu Epsilon J. **12**(2008), 535–539.
[6] H. B. Çolakoğlu, R. Kaya, *Chinese Checker Versions of the Pythagorean Theorem*, Int. J. Contemp. Math. Sciences **4**(2009), 61–69.
[7] Ö. Gelişgen, R. Kaya, *On α-distance in Three Dimensional Space*, Applied Sciences **8**(2006), 65–69.
[8] Ö. Gelişgen, R. Kaya, *Generalization of α-distance to n-dimensional Space*, Scientific-Professional Information Journal of Croatian Society for Constructive Geometry and Computer Graphics **10**(2006), 33–35.
[9] Ö. Gelişgen, R. Kaya, *CC Analog of the Theorem of Pythagoras*, Algebras, Groups and Geometries **23**(2006), 179–188.
[10] R. Kaya, H. B. Colakoglu, *Taxicab Versions of Some Euclidean Theorems*, Int. Jour. of Pure and Appl. Math. **26**(2006), 69–81.
[11] E. F. Krause, *Taxicab Geometry*, Dover Publications, New York, 1987.
[12] H. Minkowski, *Gesammelte Abhandlungen*, Chelsea Publishing Co., New York, 1967.
[13] A. C. Thompson, *Minkowski Geometry*, Cambridge University Press, Cambridge, 1996.
[14] K. Thompson, T. Dray, *Taxicab Angles and Trigonometry*, Pi Mu Epsilon J. **11**(2000), 87–96.
[15] S. Tian, *Alpha Distance-A Generalization of Chinese Checker Distance and Taxicab Distance*, Missouri J. of Math. Sci. **17**(2005), 35–40.