On certain Durrmeyer type operators

Purshottam Narayan Agrawal\(^1\) and Asha Ram Gairola\(^{1,\dagger}\)

\(^1\) Department of Mathematics, Indian Institute of Technology, Roorkee-247 667, India

Received February 2, 2009; accepted June 9, 2009

Abstract. Deo [5] introduced \(n\)-th Durrmeyer operators defined for functions integrable in some interval \(I\). There are gaps and mistakes in some of his lemmas and theorems. Further, in his paper [4] he did not give results on simultaneous approximation as the title reveals. The purpose of this paper is to correct those mistakes.

AMS subject classifications: 41A25, 41A30

Key words: Durrmeyer type operators, ordinary approximation, simultaneous approximation

1. Introduction

Deo proposed the following operators defined for functions integrable on \(I\) as

\[
(V_n f)(x) = (n - c) \sum_{k=0}^{\infty} p_{n,k}(x) \int_{I} p_{n,k}(t) f(t) \, dt \quad x \in I,
\]

\[
p_{n,k}(x) = \frac{(-x)^k \phi_n(x)}{k!}
\]

whenever the right-hand side makes sense and \(\phi_n(x), I, c, \sum^\oplus\) are given as:

\[
\phi_n(x) = \begin{cases} 
(1 - x)^n, & I = [0, 1], \\
e^{-nx}, & I = [0, \infty), c = 0 ,
\end{cases}
\]

\[
(1 + cx)^{-\frac{n}{2}}, & I = [0, \infty), c > 0.
\]

\[
\sum^\oplus = \sum_{k=0}^{\infty} \quad \text{for } c \geq 0 \quad \text{and} \quad \sum^\oplus = \sum_{k=0}^{n} \quad \text{when } c = -1.
\]

We introduce the class \(\mathcal{H}\) defined by

\[
\mathcal{H} \overset{\text{def}}{=} \left\{ f \left| \int_{I} \frac{|f(t)|}{\beta_n(t)} \, dt < \infty, \text{ for some } n \in \mathbb{N}, t \in I = [0, \infty) \right\} \right.,
\]

where the function \(\beta_n\) is defined as:

\[
\beta_n(t) = \begin{cases} 
(1 + ct)^{n/c}, & c > 0 \\
e^{nt}, & c = 0.
\end{cases}
\]

*This work was supported by the Council of Scientific & Industrial Research, New Delhi, India.
†Corresponding author. Email addresses: pnappfma@iitr.ernet.in (P.N. Agrawal), ashagairola@gmail.com (A.R. Gairola)

http://www.mathos.hr/mc ©2009 Department of Mathematics, University of Osijek
Clearly the class $H$ contains the class $L$ of all Lebesgue integrable functions on $[0, \infty)$. Further, we assume that
\[
\phi_\alpha(t) = \begin{cases} t^\alpha, & c > 0 \\ e^{\alpha t}, & c = 0 \end{cases},
\]
where $\alpha > 0$.

We define the norm $\| \cdot \|_{C_\alpha}$ in the space $H$ by
\[
\| f \|_{C_\alpha} = \sup_{0 \leq t < \infty} |f(t)| \phi_\alpha(t).
\]

Let $d_0, d_1, \ldots, d_k$ be $(k + 1)$ arbitrary but fixed distinct positive integers. We define the linear combination $V_n(f, k, x)$ of the operators $V_n$, as follows:
\[
V_n(f, k, x) = \sum_{j=0}^{k} C(j, k) V_{d_j n}(f, x),
\]
where
\[
C(j, k) = \prod_{i=0, i \neq j}^{k} \frac{d_j}{d_j - d_i}, k \neq 0 \text{ and } C(0, 0) = 1.
\]

The aim of this paper is to correct and improve the results given in [5]. Further, we extend the results proved in [4] to the case of simultaneous approximation.

2. Auxiliary results

Lemma 1 (see [5]). Let $r, m \in \mathbb{N} \cup \{0\}$ and $n > cr$, we define the functions $\mu_{r, n, m}(x)$ as follows
\[
\mu_{r, n, m}(x) = \lfloor n - c(r + 1) \rfloor \sum_{k=0}^{\infty} p_{n + cr, k}(x) \int_I p_{n - cr, k+x}(t)(t-x)^m \, dt, \ x \in I.
\]

Then, there holds the recurrence relation
\[
[n - c(r + m + 2)]\mu_{r, n, m+1}(x) = x(1+cx)\{\mu_{r, n, m}(x) + 2m\mu_{r, n, m-1}(x)\}
\]
\[
+ (r + m + 1)(1 + 2cx)\mu_{r, n, m}(x), \quad n > c(r + m + 2).
\]

Consequently,
\[
\mu_{r, n, 0}(x) = 1, \ \mu_{r, n, 1}(x) = \frac{(r + 1)(1 + 2cx)}{n - c(r + 2)}
\]
and
\[
\mu_{r, n, 2}(x) = \frac{2(n - c)(1 + cx) + (r + 1)(r + 2)(1 + 2cx)^2}{(n - c(r + 2))(n - c(r + 3))}.
\]

For all $x \in I$, $\mu_{r, n, m}(x) = \lfloor n - [(m+1)/2] \rfloor$, where $[\alpha]$ denotes the integer part of $\alpha$. 
In the proof of [5, Lemma 2.2], from the step

\[
(V_n^{(r)} f)(x) = (n-c) \sum_{k=0}^{\infty} \frac{(-1)^k x^k \phi_n^{(k+r)}(x)}{k!} \\
\times \int \sum_{i=0}^{r} \frac{r!}{t^i} \left\{ \frac{(-1)^r (-t)^{k+i} \phi_n^{(k+i)}(t)}{(k+i)!} \right\} f(t) \, dt
\]

the lemma is proved using integration by parts \( r \) times. But for this the expressions of the type \( p_{n-cr,k+r}(t)f^{(r-1)}(t) \), \( j = 1, 2, ..., r \) must be zero; and in order to claim this we must have \( f^{(r-1)}(t) = O(\phi_{\alpha}(t)) \) for some \( \alpha > 0 \) as \( t \to \infty \) and \( n > \alpha + cr \), \( r = 1, 2, 3, ... \) Hence [5, Lemma 2.2] should be stated as follows (the proof remains the same).

**Lemma 2.** Let \( f \) be \( r \) times differentiable on \([0, \infty)\) such that \( f^{(r-1)}(t) = O(\phi_{\alpha}(t)) \) for some \( \alpha > 0 \) as \( t \to \infty \). Then for \( r = 1, 2, ... \) and \( n > \alpha + cr \), we have

\[
(V_n^{(r)} f)(x) = (n-c) \beta(n,r) \sum_{k=0}^{\infty} p_{n-cr,k}(x) \int p_{n-cr,k+r}(t)f^{(r)}(t) \, dt,
\]

where \( \beta(n,r) = \prod_{j=0}^{r-1} \frac{n + cj}{n - c(j + 1)} \).

3. Main result

In [5] Deo has stated the following theorem:

**Theorem 1.** If \( f^{(r)}(t) \geq 0 \) is bounded and integrable in \( I \) and if admits the \((r+2)\)-th derivative at a point \( x \in I \), and \( f^{(r)}(t) = O(t^{\alpha'}) \) as \( t \to \infty \) for some \( \alpha > 0 \), then we get

\[
\lim_{x \to \infty} n \left\{ \frac{n - c(r + 1)}{(n-c)\beta(n,r)} (V_n^{(r)} f)(x) - f^{(r)}(x) \right\} = (r + 1)(1 + 2cx)f^{(r+1)}(x) \\
+ \phi^2(x)f^{(r+2)}(x).
\]

We wish to make the following comment regarding Theorem 1:

(i) in the hypothesis of the theorem, the existence of the \( r \)-th derivative of \( f \) is assumed globally while the conclusion is obtained locally.

So Theorem 1 should be stated as follows:

**Theorem 2.** Let \( f \in H \) be bounded on every finite sub-interval of \([0, \infty)\) admitting a derivative of order \((r+2)\) at a fixed point \( x \in (0, \infty) \). Let \( f(t) = O(\phi_{\alpha}(t)) \) as \( t \to \infty \) for some \( \alpha > 0 \), then we have

\[
\lim_{x \to \infty} n \left\{ \frac{n - c(r + 1)}{(n-c)\beta(n,r)} (V_n^{(r)} f)(x) - f^{(r)}(x) \right\} = (r + 1)(1 + 2cx)f^{(r+1)}(x) \\
+ \phi^2(x)f^{(r+2)}(x).
\]
Proof. By Taylor’s expansion of $f$, we write

$$f(t) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} (t-x)^i + \epsilon(t,x)(t-x)^{r+2},$$

where $\epsilon(t,x) \to 0$ as $t \to x$.

Using [5, Lemma 2.2], we can write

\[
\frac{n}{(n-c)\beta(n,r)} \left( V^{(r)} f(x) - f^{(r)}(x) \right) = n[n-c(r+1)] \left[ \sum_{i=r+1}^{r+2} \frac{f^{(i)}(x)}{i!} \sum_{k=0}^{\infty} p_{n+c,r,k}(x) \times \int_I p_{n-c,r-k}(t) \frac{d^r}{dt^r} (t-x)^i \, dt \right. \\
\left. + n \frac{n-c(r+1)}{\beta(n,r)} \sum_{k=0}^{\infty} p_{n,r,k}(x) \times \int_I p_{n,k}(t) \epsilon(t,x)(t-x)^{r+2} \, dt \right] \\
= n \left[ f^{(r+1)}(x) \mu_{r,n,1}(x) + \frac{1}{2} f^{(r+1)} \mu_{r,n,2}(x) \right] + I_n,
\]

where

\[
I_n = \frac{n[n-c(r+1)]}{\beta(n,r)} \sum_{k=0}^{\infty} p_{n,r,k}(x) \int_I p_{n,k}(t) \epsilon(t,x)(t-x)^{r+2} \, dt.
\]

In order to prove the theorem it is sufficient to show that $I_n \to 0$ as $n \to \infty$. Using Lorentz type lemma, we get

\[
|I_n| \leq \frac{n[n-c(r+1)]}{\beta(n,r)} \sum_{2t+j \leq r} n^i |k-nx|^j \frac{|g_{i,j,r}(x)|}{(x(1+cx))^p} p_{n,r,k}(x) \\
\times \int_I p_{n,k}(t) |\epsilon(t,x)| |(t-x)|^{r+2} \, dt \\
\leq C \frac{n[n-c(r+1)]}{\beta(n,r)} \sum_{2t+j \leq r} n^i \sum_{i,j \geq 0} p_{n,k}(x) |k-nx|^j \\
\times \int_I p_{n,k}(t) |\epsilon(t,x)| |(t-x)|^{r+2} \, dt \\
\leq C \frac{n[n-c(r+1)]}{\beta(n,r)} \sum_{2t+j \leq r} n^i \left( \sum_{i,j \geq 0} p_{n,k}(x) |k-nx|^{2j} \right)^{1/2} \\
\times \left( \sum_{i,j \geq 0} p_{n,k}(x) \left( \int_I p_{n,k}(t) |\epsilon(t,x)| |(t-x)|^{r+2} \, dt \right)^2 \right)^{1/2}
\]
i.e.
\[
\leq C \frac{n[n - c(r + 1)]}{\beta(n, r)} n^{r/2} \left( \sum p_n,k(x) \right) \\
\times \left( \int_I p_n,k(t) |e(t, x)||t - x|^{r+2} dt \right)^{-1/2},
\]
where \( C = C(x) = \sup_{2t + j \leq r} |q_{i,j,r}(x)|. \)

For a given \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that \( |e(t, x)| < \varepsilon \) whenever \( 0 < |t - x| < \delta \). For \( |t - x| \geq \delta \), we have \( |e(t, x)| \leq K|t - x|^{2s} \), for any \( s \geq 0 \).

Therefore, we have
\[
\left( \int_I p_n,k(t) |e(t, x)||t - x|^{r+2} dt \right)^2 \leq \left( \int_I p_n,k(t) dt \right) \left( \int_I p_n,k(t) \left( e(t, x) \right)^2 (t - x)^{2r+4} dt \right) \\
= \frac{1}{(n-c)} \left( \int_{|t - x| < \delta} + \int_{|t - x| \geq \delta} \right) p_n,k(t) \left( e(t, x) \right)^2 \\
\times (t - x)^{2r+4} dt \\
= \frac{1}{(n-c)} \left( \int_{|t - x| < \delta} p_n,k(t) \varepsilon^2 (t - x)^{2r+4} dt \\
+ \int_{|t - x| \geq \delta} p_n,k(t) K^2 (t - x)^{2r+2s+4} dt \right).
\]

In view of [5, Lemma 2.1],
\[
\sum p_n,k(x) \left( \int_I p_n,k(t) |e(t, x)||t - x|^{r+2} dt \right)^2 \leq \frac{(n-c)}{(n-c)^2} \sum p_n,k(x) \\
\times \int_I p_n,k(t) \varepsilon^2 (t - x)^{2r+4} dt \\
+ \frac{K^2(n-c)}{(n-c)^2} \sum p_n,k(x) \\
\times \int_{|t - x| \geq \delta} p_n,k(t) (t - x)^{2r+2s+4} dt \\
= \varepsilon^2 O(n^{-(r+4)}) + K^2 O(n^{-(r+s+4)}) \\
= \varepsilon^2 O(n^{-(r+4)}) + O(n^{-(r+s+4)}).
\]

This in view of [5, Lemma 2.1] gives
\[
|I_n| \leq C \frac{n[n - c(r + 1)]}{\beta(n, r)} n^{r/2} \times \varepsilon^2 O(n^{-(r+4)})^{1/2} + o(1) \\
\leq \varepsilon + o(1), \text{ choosing } s > 0.
\]

Since \( \varepsilon \) is arbitrary, this implies that \( I_n \to 0 \) as \( n \to \infty \).

Finally, taking the limit \( n \to \infty \) and using the values of \( \mu_{r,n,1}(x) \) and \( \mu_{r,n,2}(x) \) the theorem is proved.
Remark 1. If \( f^{(r)}(t) = O(t^s) \) (as \( t \to \infty \)), then \( f(t) \) will be of order \( t^{a+r} \). Moreover, since \( f \) is of order \( O(1) \) while its \( r \)-th derivative \( (r \geq 1) \) is not of \( O(t^a) \). So the hypothesis of Theorem 2 is certainly weaker than the hypothesis of Theorem 1.

Further, in [5] the author gave another theorem as:

**Theorem 3.** Let \( f^{(r+1)} \in C[0, \infty) \) and \( [0, \lambda] \subseteq [0, \infty) \) and let \( \omega(f^{(r+1)}; \cdot) \) be the modulus of continuity of \( f^{(r+1)} \), then for \( r = 0, 1, 2, \ldots \)

\[
\left\| \frac{n-c(r+1)}{(n-c)\beta(n,r)} (V_n^{(r)} f)(x) - f^{(r)}(x) \right\|_{C[0,\lambda]} \leq \frac{(r+1)(1 + 2c\lambda)}{n - c(r + 2)} \left\| f^{(r+1)} \right\| + C(n, r) \left( \sqrt{\eta} + \frac{\eta}{2} \right) + \omega(f^{(r+1)}; C(n, r)),
\]

where the norm is sup-norm over \([0, \lambda] \),

\[
\eta = 2\lambda^2 \{ c^2(2r^2 + 6r + 3) + cn \} + 2\lambda \{ c(r^2 + 3r + 1) + n \} + (r^2 + 3r + 2)
\]

and

\[
C(n, r) = \frac{1}{(n-c(r+2))(n-c(r+3))}.
\]

Regarding this theorem, we wish to make the following comments:

(i) in the hypothesis of the theorem the existence of the \((r+1)\)th derivative of \( f \) is assumed globally while the conclusion is obtained locally.

(ii) in the proof of the theorem, the property \( \omega(f^{(r+1)}; \delta) \to 0 \) as \( \delta \to 0 \) is used which need not be true unless one assumes that \( f^{(r+1)} \) is uniformly continuous on \([0, \infty) \).

For example, consider the function \( g(x) = \cos \pi x^2 \), \( x \in [0, \infty) \).

Clearly, this function is bounded and continuous on \([0, \infty) \). But, \( |g(\sqrt{n+1}) - g(\sqrt{n})| = 2 \), while \( |\sqrt{n+1} - \sqrt{n}| \to 0 \) as \( n \to \infty \), so the function is not uniformly continuous. Hence \( \omega(g; \delta) \) does not tend to zero as \( \delta \) tends to zero.

In the light of above comments, Theorem 3 should be stated as follows:

**Theorem 4.** Let \( f \in H \) be bounded on every finite subinterval of \([0, \infty) \) and \( f(t) = O(\phi(t)) \) as \( t \to \infty \) for some \( \alpha > 0 \). If \( f^{(r+1)} \) exists and if it is continuous on \((a-\delta, b+\delta) \subseteq (0, \infty) \), \( \delta > 0 \), then for sufficiently large \( n \),

\[
\left\| (V_n^{(r)} f)(x) - f^{(r)}(x) \right\| \leq C_1 n^{-1} \left( \|f^{(r)}\| + \|f^{(r+1)}\| \right) + C_2 n^{-1/2} \omega(f^{(r+1)}, n^{-1/2}) + O(n^{-s/2}) \text{ for any } s > 0,
\]

where \( C_1 \) and \( C_2 \) are both independent of \( f \) and \( n \), and \( ||.|| \) is sup-norm on \([a, b] \).

**Proof.** By finite Taylor’s expansion of \( f \) we write

\[
f(t) = \sum_{i=0}^{r+1} \frac{f^{(i)}(x)}{i!} (t-x)^i + \frac{f^{(r+1)}(\xi) - f^{(r+1)}(x)}{(r+1)!} (t-x)^{r+1} \chi(t) + h(t, x) (1 - \chi(t)),
\]
where \( \xi \) lies between \( t \) and \( x \) and \( \chi(t) \) is the characteristic function of \( (a - \delta, b + \delta) \).

For \( t \in (a - \delta, b + \delta) \) and \( x \in [a, b] \) we have

\[
f(t) = \sum_{i=0}^{r+1} \frac{f^{(i)}(x)}{i!} (t - x)^r + \frac{\{f^{(r+1)}(\xi) - f^{(r+1)}(x)\}}{(r+1)!} (t - x)^{r+1}.
\]

For \( t \in [0, \infty) \setminus (a - \delta, b + \delta) \) and \( x \in [a, b] \) we define

\[
h(t, x) = f(t) - \sum_{i=0}^{r+1} \frac{f^{(i)}(x)}{i!} (t - x)^i.
\]

Now

\[
(V_n^{(r)} f)(x) - f^{(r)}(x) = (n - c)\beta(n, r) \left[ \sum_{i=0}^{r+1} \frac{f^{(i)}(x)}{i!} \sum_{k=0}^{\infty} p_{n+r,k}(x) \right. \\
\left. \times \int f_n(x) dx^r (t - x)^i \right] - f^{(r)}(x) \\
+ (n - c) \sum_{k=0}^{\infty} p_n^{(r)}(x) \int I_p f_n(x) \left[ \frac{f^{(r+1)}(\xi) - f^{(r+1)}(x)}{(r+1)!} \right] \\
\times (t - x)^{r+1} \chi(t) + h(t, x) (1 - \chi(t)) \right dt
\]

\[= I_1 + I_2 + I_3, \text{ say.}\]

Using [5, Lemma 2.1], we obtain

\[
I_1 = \left( \frac{(n - c)\beta(n, r)}{|n - c(r + 1)|} \mu_r, n, 0(x) - 1 \right) f^{(r)}(x) + \frac{(n - c)\beta(n, r)}{|n - c(r + 1)|} \mu_r, n, 1(x) f^{(r+1)}(x),
\]

in view of \( \frac{d^r}{dx^r} (t - x)^i = 0 \) for \( i < r \).

Next, using Lorentz type lemma, we get

\[
I_2 \leq (n - c) \sum_{2i+j \leq r \atop i, j \geq 0} n^i |k - nx|^j \frac{|q_{i,j,r}(x)|}{x^r (1 + cx)^r} p_{n,k}(x) \\
\times \int I_p f_n(x) \left[ \frac{f^{(r+1)}(\xi) - f^{(r+1)}(x)}{(r+1)!} \right] (t - x)^{r+1} \chi(t) dt \\
\leq (n - c) \sum_{2i+j \leq r \atop i, j \geq 0} n^i \sum_{k=0}^\infty p_{n,k}(x) |k - nx|^j \\
\times \int I_p f_n(x) \left( 1 + \left| \frac{t - x}{\delta} \right| \right) \omega(f^{(r+1)}, \delta) |t - x|^{r+1} dt, \text{ for all } \delta > 0,
\]
\begin{align*}
&= C(n - c) \omega(f^{(r+1)}, \delta) \sum_{2i + j \leq r \atop i, j \geq 0} n^i \sum_{k} p_{n,k}(x)|k - nx|^j \\
&\times \int I_p n,k(t) \left(|t - x|^{r+1} + \frac{|t - x|^{r+2}}{\delta}\right) dt.
\end{align*}

By induction it can be easily shown that for \( p = 0, 1, 2, \ldots \)
\begin{align*}
&\sum p_{n,k}(x)|k - nx|^j \times \int p_{n,k}(t)|t - x|^p dt = \frac{1}{\sqrt{n - c}} O(n^{(j-p)/2}).
\end{align*}

Hence, choosing \( \delta = n^{-1/2} \) we have
\begin{align*}
|I_2| &\leq C n^{-1/2} \omega(f^{(r+1)}, n^{-1/2}).
\end{align*}

Now, from the definition of \( h(t, x) \), we have \( h(t, x) = O(\phi_{s}(t)) \Rightarrow h(t, x) = O(t-x)^{s} \), for any \( s \in \mathbb{N} \) with \( s \geq \alpha \).
\begin{align*}
|I_3| &\leq M' \sum_{2i + j \leq r \atop i, j \geq 0} n^i |k - nx|^j p_{n,k}(x) \\
&\times \int \left[ M'' |t - x|^s \right] dt.
\end{align*}

Applying Cauchy’s inequality [5, Lemma 2.1] we obtain
\begin{align*}
|I_3| &\leq M'(n - c) \sum \sum_{2i + j \leq r \atop i, j \geq 0} n^i p_{n,k}(x)|k - nx|^j \\
&\times \int |t - x|^{s} dt \\
&\leq C n^{(1+s-r)/2} \omega(f^{(r+1)}, \delta).
\end{align*}

Choosing \( s > r + 1 \), we get the limit \( I_3 \to 0 \) as \( n \to \infty \).

Combining the estimates of \( I_1, I_2 \) and \( I_3 \), we get the required result.

4. Simultaneous approximation

**Theorem 5.** Let \( f \in \mathcal{H} \) and let it be bounded on every finite subinterval of \([0, \infty)\) admitting a derivative of order \( 2k + r + 2 \) at a point \( x \in (0, \infty) \). Let \( f(t) = O(\phi_{s}(t)) \) as \( t \to \infty \) for some \( \alpha > 0 \). Then
\begin{align*}
\lim_{n \to \infty} n^{k+1} \left[ V^{(r)}(f, k, x) - f^{(r)}(x) \right] = \sum_{j=r}^{2k+2+r} \frac{f^{(j)}(x)}{j!} Q(j, k, r, c, x)
\end{align*}
and
\[
\lim_{n \to \infty} n^{k+1} \left[ V_n^{(r)}(f,k+1,x) - f^{(r)}(x) \right] = 0, \tag{2}
\]
where \(Q(j,k,r,c,x)\) are certain polynomials in \(x\). Further, if \(f^{(2k+2+r)}\) exists and if it is continuous on \((a-\eta, b+\eta) \subset (0, \infty), \eta > 0,\) then (1) and (2) hold uniformly on \([a,b]\\).

**Proof.** The proof is similar to [1, Theorem 2].

In our next result we obtain an estimate of the degree of approximation.

**Theorem 6.** Let \(f \in H\) be bounded on every finite subinterval of \([0, \infty)\) and \(f(t) = O(\phi_\alpha(t))\) as \(t \to \infty\) for some \(\alpha > 0\). Further, let \(1 \leq p \leq 2k+2\) and \(r \in \mathbb{N}$. If \(f^{(p+r)}\) exists and if it is continuous on \((a-\delta, b+\delta) \subset (0, \infty), \delta > 0,\) then for sufficiently large \(n,\\)
\[
\|V_n^{(r)}(f,k,\cdot) - f^{(r)}(\cdot)\| \leq \max \left\{ C_1 n^{-p/2} \omega(f^{(p+r)}, n^{-1/2}), C_2 n^{-(k+1)} \right\},
\]
where \(C_1 = C_1(k,p,c,r), C_2 = C_2(k,p,r,c,f)\) and \(\omega(f^{(p+r)}, \cdot)\) denotes the modulus of continuity of \(f^{(p+r)}\) on \((a-\delta; b+\delta).\\)

**Proof.** The proof is similar to [1, Theorem 3] and hence it is omitted.

**Theorem 7.** Let \(f \in H\) be bounded on \([0, \infty)\) and \(f(t) = O(\phi_\alpha(t))\) as \(t \to \infty\) for some \(\alpha > 0\). If \(f^{(r)}\) exists and if it is continuous on \((a-\eta, b+\eta), \eta > 0,\) then for sufficiently large \(n,\\)
\[
\|V_n^{(r)}(f,k,\cdot) - f^{(r)}(\cdot)\|_{C(I)} \leq C n^{-(k+1)}
\times \left\{ \|f\|_{C_\alpha} + \omega_{2k+2}(f^{(r)}; n^{-1/2}; (a-\eta, b+\eta)) \right\},
\]
where \(C\) is independent of \(f\) and \(n.\\)

**Proof.** The proof follows along the lines [3, Theorem 3]

**Acknowledgment**

The authors are highly thankful to the referee for his valuable suggestions leading to a better presentation of the paper. The author Asha Ram Gairola is thankful to the “Council of Scientific and Industrial Research”, New Delhi, India for financial support to carry out the above research work.

**References**


