ON THE SECOND HOMOTOPY GROUP OF \( SC(Z) \)

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Abstract. In our earlier paper we introduced a cone-like space \( SC(Z) \). In the present note we establish some new algebraic properties of \( SC(Z) \).

1. Introduction

In our earlier paper ([1]) we introduced a new, cone-like construction of a space \( SC(Z) \) using the topologist sine curve and proved that \( SC(Z) \) is simply connected for every path-connected space \( Z \). In another paper ([3]) we proved that the singular homology \( H_2(SC(Z);\mathbb{Z}) \) is non-trivial if \( \pi_1(Z) \) is non-trivial. In the present paper we prove its converse, that is:

**Theorem 1.1.** Let \( Z \) be any path-connected space, \( z_0 \in Z \). If \( \pi_1(Z, z_0) \) is trivial, then \( \pi_2(SC(Z), z_0) \) is also trivial.

Consequently, we get the following:

**Corollary 1.2.** For any path-connected space \( Z, z_0 \in Z \), the following statements are equivalent:

1. \( \pi_1(Z, z_0) \) is trivial;
2. \( \pi_2(SC(Z), z_0) \) is trivial; and
3. \( H_2(SC(Z);\mathbb{Z}) \) is trivial.

We also use this opportunity to correct the proof of Lemma 3.2 in [2] (see Section 3).

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2. Proof of Theorem 1.1

In order to describe the homotopies we shall need to introduce some notations. The unit interval \([0,1]\) is denoted by \(I\). For a map \(f : [a_0, b_0] \times [c, d] \to X\), define \(f^- : [a_0, b_0] \times [c, d] \to X\) by:

\[ f^-(x, y) = f(a_0 + b_0 - x, y) \]

and \(f_{[a,b]} : [a, b] \times [c, d] \to X\) by:

\[ f_{[a,b]}(x, y) = f(a_0 + (b_0 - a_0)(x - a)/(b - a), y) \].

We follow the notation in [1] for the space \(SC(Z)\) and the projection \(p : SC(Z) \to I^2\). In particular, a point of the subspace \(I^2\) of \(SC(Z)\) is denoted by \((x; y)\). The following figure denotes the part \(I^2\) of \(SC(Z)\), where the polygonal line \(A_1B_1A_2B_2 \cdots AB\) is the piecewise linear version of the topologist’s sine curve in Figure 1.

**Definition 2.1 ([4, Definition 2.1]).** A continuous map \(f : I^2 \to p^{-1}(I \times \{0\})\) with \(f(\partial I^2) = \{A\}\) is said to be standard, if \(f([1/(m+1), 1/m] \times I) \subset [A, A_m] \cup p^{-1}(\{A_m\})\), and \(f(\partial([1/(m+1), 1/m] \times I)) = \{A\}\) for each \(m < \omega\).

![Figure 1](image)

**Proof of Theorem 1.1.** Recall that \(p : SC(Z) \to I^2\) and \(p(z) = (p_1(z); p_2(z))\). Let \(f : I^2 \to SC(Z)\) with \(f(\partial I^2) = A = (0,0)\). First we claim that \(f\) is homotopic to a map \(f_1\) in \(p^{-1}(I \times \{0\})\) relative to \(\partial I^2\).
Since \( p^{-1}(I \times \{0\}) \) is a retraction of \( SC(Z) \setminus \bigcup \{ p^{-1}(\{B_n\}) : n \in \mathbb{N} \} \), it suffices to show that \( f \) is homotopic to a map in \( SC(Z) \setminus \bigcup \{ p^{-1}(\{B_n\}) : n \in \mathbb{N} \} \) relative to \( \partial\mathbb{I}^2 \). The idea to remove \( p^{-1}(\{B_n\}) \) from the image of \( f \) is basically the same as that in the proof of the simple connectivity of \( SC(Z) \). Since the boundary of an open connected subset of the square is complicated in comparison with those of the interval, some more care is needed.

Here we also use the simple connectivity of \( Z \). Using this property we get the simple connectivity of a certain small open subset containing \( p^{-1}(\{B_n\}) \) in \( SC(Z) \), i.e., \( p^{-1}(U_n) \) according to the following notation. Let \( U_n \) be a square neighborhood of \( B_n \) in \( \mathbb{I}^2 \) which does not contain any \( B_i \) for \( i \neq n \), and choose a point \( u_n \in U_n \cap T \) satisfying \( u_n \neq B_n \).

We have finitely many polygonal connected open sets \( O_i \) in \( \mathbb{I}^2 \) such that \( (p \circ f)^{-1}(\{B_n\}) \subseteq \bigcup_{i} O_i \subseteq (p \circ f)^{-1}(U_n) \), where \( O_i \) may fail to be simply connected. Observe that \( p^{-1}(U_n) \) is homotopy equivalent to \( Z \) and so it is simply connected. Hence, working in each \( O_i \), we have \( g_1 : \mathbb{I}^2 \to SC(Z) \) satisfying:

1. \( g_1 \) is homotopic to \( f \) relative to \( \mathbb{I}^2 \setminus O_i \).
2. there exist finitely many simply connected polygonal pairwise disjoint subsets \( P_{ij} \) of \( O_i \) such that \( (p \circ g_1)^{-1}(\{B_n\}) \) is not intersect with \( \bigcup_{i} P_{ij} \).
3. \( g(\partial P_{ij}) = \{u_n\} \).

We remark that the range of the homotopy between \( f \) and \( g_1 \) is contained in \( p^{-1}(U_n) \). Since \( p^{-1}(\{u_n\}) \) is a strong deformation retract of \( p^{-1}(U_n) \), we have \( g_2 : \mathbb{I}^2 \rightarrow SC(Z) \) such that \( g_2(\bigcup_{i} P_{ij}) \subseteq p^{-1}(\{u_n\}) \cup I \times \{1\} \), \( g_2 \) is homotopic \( g_1 \) relative to \( \mathbb{I}^2 \setminus \bigcup_{i} P_{ij} \), and the range of the homotopy is contained in \( p^{-1}(U_n) \).

Observe that there are only finitely many \( O_i \) for each \( B_n \) and that \( p^{-1}(U_n) \) converge to \( B \). Working on each \( B_n \) successively, we obtain maps homotopic to \( f \). Let \( f_0 \) be the limit of these maps.

Since \( f(O_i) \subseteq p^{-1}(U_n) \) and \( p^{-1}(U_n) \) converge to \( B \), \( f_0 \) is continuous and \( f_0 \) is homotopic to \( f \) relative to \( \partial\mathbb{I}^2 \) and also \( f_0(\mathbb{I}^1) \) does not intersect with any \( p^{-1}(\{B_n\}) \), as desired. Hence we have \( f_1 : \mathbb{I}^2 \rightarrow p^{-1}(I \times \{0\}) \) which is homotopic to \( f \) relative to \( \partial\mathbb{I}^2 \).

The next procedure is similar to the proof of [4, Lemma 2.2], which is rather long but each step is simple. Using the commutativity of \( \pi_2 \) and the simple connectivity of \( Z \) again and also using the fact that \( p^{-1}(I \times \{0\}) \) is locally strongly contractible at points \( (x;0) \) with \( (x;0) \notin \{A_n : n \in \mathbb{N} \} \cup \{A\} \), we get a standard map \( f_2 : \mathbb{I}^2 \rightarrow p^{-1}(I \times \{0\}) \) which is homotopic to \( f_1 \) relative to \( \partial\mathbb{I}^2 \).

The proof that \( f_2 \) is null-homotopic is the 2-dimensional version of procedures II and III in the proof of the simple connectivity of \( SC(Z) \) ([1, Theorem 1.1]). We outline these procedures. We concentrate on description of the null homotopy of \( f_2 | [1/(k+1), 1/k] \times \mathbb{I} \).
Fix $k \in \mathbb{N}$. For $m \in \mathbb{N}$, define $h_m : [1/(k + 1), 1/k] \times I \to SC(Z)$ by:
\[
h_m(x, y) = \begin{cases} (ku/(k + m - 1)); 0 & \text{if } f_2(x, y) = (u; 0) \\ (A_{k+m-1}, z) & \text{if } f_2(x, y) = (A_k, z). \end{cases}
\]
Next define $g_{k,m} : [1/(k + 1) + 1/((m + 1)k(k + 1)), 1/(k + 1) + 1/(mk(k + 1))] \times I \to SC(Z)$ by:
\[
g_{k,2m+1} = (h_m)_{[1/(k+1)+1/(2mk(k+1)),1/(k+1)+1/(2m+1)k(k+1)]}, \\
g_{k,2m} = (h_{m+1})_{[1/(k+1)+1/(2m+1)k(k+1)),1/(k+1)+1/(2mk(k+1))]}.
\]
Let $g_k : [1/(k + 1), 1/k] \times I \to SC(Z)$ be the unique continuous extension of
\[
\bigcup_{m \in \mathbb{N}} g_{k,m},
\]
i.e., $g_k(1/(k + 1), y) = A$.
Since the images of $g_{k,m}$ converge to $A$ and $g_{2m+1}$ is the homotopy inverse of $g_{2m}$ in $p^{-1}(I \times \{0\})$ for each $m$, $g_k$ is continuous and is homotopic relative to the boundary to the restriction $f_2 \mid [1/(k + 1), 1/k] \times I$, and the homotopy can be taken in $p^{-1}([A, A_k])$. Hence $f_2$ is homotopic relative to the boundary to $g : I \times I \to SC(Z)$ which is the unique continuous extension of $\bigcup \{g_k : k \in \mathbb{N}\}$, i.e., $g(0, y) = A$.

For the next step we do not care for the boundary for a while. We push up the ranges of $g_{k,2m-1}$ along $A_{k+m}B_{k+m}$ for $m \geq 0$ and $g_{k,2m}$ along $A_{k+m}B_{k+m-1}$ so that the $y$-coordinates of $p(u)$ for $u$ in each of the ranges are the same. Then the resulting map is defined in $p^{-1}(I \times \{1\})$ and we couple $g_{k,1}$ and $g_{k,2}$, and generally $g_{k,2m-1}$ and $g_{k,2m}$. Since these homotopies of couplings converge to $B$, we see that the resulting map is null-homotopic. We can perform these procedures uniformly in $k$, and we have a homotopy from $f_2$ to the constant map $B$.

In order to obtain the desired homotopy to the constant $A$ relative to the boundary, we can modify the homotopy above to the desired one, because we have homotopies in the pushing up procedure above, so that the $y$-coordinates of $p(u)$ for $u$ in the ranges are the same, even uniformly in $k$.

3. Correction of the proof of Lemma 3.2 of [2]

In our earlier paper ([2, Lemma 3.2]) we used the following auxiliary result:

**Lemma 3.1.** Let $p_0, p_1, p^*$ be distinct points in a Hausdorff space $X$ and let $f$ be a loop in $X$ with the base point $p_1$ such that $f^{-1}(\{p_0\}) = \emptyset$ and $f^{-1}(\{p^*\})$ is a singleton.

If $f$ is null-homotopic relative to end points, then there exists a loop $f'$ in $X$ with the base point $p_1$ in $X \setminus \{p_0, p^*\}$ such that $f$ and $f'$ are homotopic relative to end points in $X \setminus \{p_0\}$.
We use this opportunity to correct our original proof. The assertion “\(G^{-1}([p^*, p_0]) \cap O\) is compact” in [2, p.92 l.5 from the bottom] is wrong. We begin by the following well-known result - see e.g. [5, p.169]:

**Lemma 3.2.** Let \(X\) be a compact space and \(C\) a closed component of \(X\). Then \(C\) is the intersection of clopen sets containing \(C\). \(\blacksquare\)

**Proof of Lemma 3.1.** Since \(f\) is null-homotopic, we have a homotopy \(F : \mathbb{I} \times \mathbb{I} \to X\) from \(f\) to the constant mapping to \(p_1\), i.e.

\[
F(s, 0) = f(s), F(s, 1) = F(0, t) = F(1, t) = p_1 \quad \text{for } s, t \in \mathbb{I}.
\]

Let \(\{s_0\}\) be the singleton \(f^{-1}(\{p^*\})\). Let \(M_0\) be the connectedness component of \(\{p^*\}\) containing \((s_0, 0)\), and \(O\) the connectedness component of \(\mathbb{I} \times \mathbb{I}\) \(\setminus M_0\) containing \(\mathbb{I} \times \{1\}\). Define \(G : \mathbb{I} \times \mathbb{I} \to P^*\) by:

\[
G(s, t) = \begin{cases} F(s, t) & \text{if } (s, t) \in O, \\ p^* & \text{otherwise} \end{cases}
\]

Then \(G\) is also a homotopy from \(f\) to the constant mapping to \(p_1\) and \(G^{-1}(\{p_0\})\) is contained in \(O\) (observe that \(\partial(\mathbb{I} \times \mathbb{I}) \setminus \{(s_0, 0)\} \subseteq O\)).

Put \(M_1 = (\mathbb{I} \times \mathbb{I} \setminus O) \cup \mathbb{I} \times \{0\}\). Since \(G^{-1}(\{p_0\}) \cap O\) is compact and disjoint from \(M_1\), we have a polygonal neighborhood \(U\) of \(M_1\) whose closure is disjoint from \(G^{-1}(\{p_0\})\) and also \(\mathbb{I} \times \{1\}\). The boundary of \(U\) is a piecewise linear arc connecting a point in \((0) \times (0, 1)\) and a point in \((1) \times (0, 1)\). We want to get a piecewise linear injective path \(g : \mathbb{I} \to \mathbb{I} \times \mathbb{I}\) such that

\[
\text{Im}(G \circ g) \subseteq X \setminus \{p_0, p^*\}, \quad g(0) \in \{0\} \times \mathbb{I}, \quad \text{and } g(1) \in \{1\} \times \mathbb{I}
\]

and \(\text{Im}(g)\) divides \(\mathbb{I} \times \mathbb{I}\) into two components, one of which contains \(G^{-1}(\{p_0\})\) and the other contains \(M_1\).

If the boundary of \(U\) is disjoint from \(G^{-1}(\{p^*\})\), then we have such a path \(g\) tracing the boundary. Otherwise, let \(C_0\) be the intersection of the boundary of \(U\) and \(G^{-1}(\{p^*\})\). Since \(M_0\) is contained in a connected component of \(G^{-1}(\{p^*\})\) which is disjoint with \(C_0\), we have clopen sets \(C_1\) and \(C_2\) in \(G^{-1}(\{p^*\}) \cap U\) such that \(C_1 \cap C_2 = \emptyset\), \(C_0 \subseteq C_1\) and \(M_0 \subseteq C_2\) by Lemma 3.2.

Observe that \(C_1\) and \(C_2\) are closed subset of \(U\). Then we can see that \((0) \times (0, 1) \cap U\) and \((1) \times (0, 1) \cap U\) belong to the same component of \(U \setminus G^{-1}(\{p^*\})\). We have a piecewise linear arc between a point in \((0) \times (0, 1)\) and a point in \((1) \times (0, 1)\) in \(U\) which does not intersect with \(G^{-1}(\{p^*\})\) and have a path \(g\) with the required properties. We now see that \(G \circ g\) is the desired loop \(f'\). \(\blacksquare\)

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