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ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR MAXIMUM EQUATIONS

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Abstract. An existence-uniqueness result for the Cauchy problem for a system of ordinary differential equations with maximums is established.

The paper is concerned with the following initial–value problem (IVP):

$$\begin{cases} \dot{x}(t) = f(t, x(t), \|x(t)\|_g), & t > 0 \\ x(t) = \varphi(t), & t \le 0, \end{cases} \tag{1}$$

where

$$\begin{aligned} x(t) &= (x_1(t), \ldots, x_n(t)),\ \dot{x}(t) = (\dot{x}_1(t), \ldots, \dot{x}_n(t)), \\ \|x(t)\|_g &= \max_{g(t) \le s \le t} \|x(s)\|,\ \|x(s)\| = \max_{1 \le i \le n} |x_i(s)|, \end{aligned}$$

$g(t) : [0, \infty) \to \mathbf{R}$ being a prescribed function, such that $-\infty < g(t) \le t$, for every $t \ge 0$.

The mathematical formulation above mentioned arises in automatic regulations, integral electronics and measurement devices. In [2] (p.p. 29, 477, 565) the authors present various relay systems for automatic regulation - for instance, of the temperature in some chamber. For the variation of the temperature $\theta(t)$ the equation

$$T\frac{d\theta}{dt} + \theta = -k\varphi + f,$$

is obtained, where T, k are constants, $f = f(t)$ - external perturbations, and φ is the variation of the regulating device (relay system), which depends on t and $\max\{|\theta(s)| : t_0 \le s \le t\}$.

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The main interest of many authors is the existence of periodic and oscillating solutions of (1) ([8]–[7]). In many cases, however, various conditions are formulated which do not guarantee even an existence of a solution. That is why, here we present existence conditions applying fixed point technics, obtained in a previous paper [1].

As usually, using that

$$x(t) = x(0) + \int_0^t \dot{x}(s)\, ds,$$

for $t > 0$, we reduce the IVP(1) to the following one:

$$(2) \qquad \begin{cases} x(t) = \varphi(0) + \displaystyle\int_0^t f(\tau, x(\tau), \|x(\tau)\|_g)\, d\tau, & t > 0 \\ x(t) = \varphi(t), & t \le 0. \end{cases}$$

First of all we have to investigate the measurability of $\|x(t)\|_g$ on $[0,\infty)$.

Proposition 1. *Let $g(t)$ be defined and measurable on $[0,\infty)$ function, $-\infty < g(t) \le t, \forall t \ge 0$. Then for every $x \in C(\mathbf{R};\mathbf{R}^n)$, $\|x(t)\|_g$ is a measurable locally bounded function on $[0,\infty)$.*

Proof. Inequality

$$\|x(t)\|_g \le \max\{\|x(s)\| : \inf_{\tau\in K} g(\tau) \le s \le \sup K\}$$

for any compact interval $K \subset \mathbf{R}$, shows that $\|x(t)\|_g$ is a bounded function on every compact subset of $\mathbf{R}$.

Let us assume that $\|x(t)\|_g$ is not a measurable function. Then there exists $c \in (-\infty,\infty)$ such that the set $A_c = \{t \ge 0 : \|x(t)\|_g < c\}$ is not measurable.

Consider the function

$$\varphi_\alpha : [0,\infty) \to [0,\infty) : \varphi_\alpha(t) = \|x(\alpha t + (1-\alpha)g(t))\|, \quad 0 \le \alpha \le 1.$$

For any fixed $\alpha \in [0,1]$ the function $\tau_\alpha(t) = \alpha t + (1-\alpha)g(t)$ is measurable on $[0,\infty)$ as a linear combination of measurable functions. Consequently $|x_i(\tau_\alpha)|$ is measurable for every $i = 1,2,...,n$, and so $\varphi_\alpha = \max\{|x_i(\tau_\alpha)| : 1 \le i \le n\}$ is measurable, which means that the set $A_{\alpha,a} = \{t \ge 0 : \varphi_\alpha(t) < a\}$ is measurable for every $a \in \mathbf{R}$, and $\alpha \in [0,1]$.

On the other hand $\|x(t)\|_g = \sup\{\varphi_\alpha(t) : 0 \le \alpha \le 1\} = \varphi_\beta(t)$ is attained for some $\beta \in [0,1]$, and the set $A_{\beta,c}$ is measurable. But $A_{\beta,c} = \{t \ge 0 : \varphi_\beta(t) < c\} = \{t \ge 0 : \|x(t)\|_g < c\} = A_c$ – contradiction, which completes the proof. ∎

We are going to look for a continuous solutions of (2).

Consider the linear space $C(\mathbf{R};\mathbf{R}^n)$ with a saturated family of seminorms

$$p_K(y) = \sup_{t\in K} e^{-\lambda t}\|y(t)\|,$$

where $\lambda > 0$ and K runs over all compact subsets of $\mathbf{R}$. It defines a locally convex Hausdorff topology on $C(\mathbf{R};\mathbf{R}^n)$.

We denote by Ψ the set of all compact subsets of $\mathbf{R}$ and we define the map $j : \Psi \to \Psi$:

$$j(K) = \begin{cases} K, & \sup K \leq 0 \\ [0, \sup K], & \sup K > 0. \end{cases}$$

It is obvious that $j^2(K) = j(j(K)) = j(K)$ and consequently, $j^m(K) = j(K)$ for all $m \in \mathbf{N}$.

Now we make the following assumptions (I):

(i) The function $f(t,u,v) : [0,\infty) \times \mathbf{R}^n \times [0,\infty) \to \mathbf{R}^n$ satisfies the Caratheodory condition (measurable in t and continuous in u, v), $\|f(\cdot,0,0)\| \in L^1_{loc}([0,\infty))$ and

$$\|f(t,u_1,v_1) - f(t,u_2,v_2)\| \leq \Omega(t, \|u_1 - u_2\|, |v_1 - v_2|),$$

where the comparison function $\Omega(t,x,y)$ satisfies the Caratheodory condition. It is non-decreasing in x and y and for any fixed $y \geq 0, \Omega(\cdot,y,y) \leq y\omega(\cdot)$ with some $\omega \in L^p([0,\infty);[0,\infty)), p \geq 1$;

(ii) The initial function $\varphi : (-\infty,0] \to \mathbf{R}^n$ is continuous.

THEOREM 2. *If conditions (I) are fulfilled, then for any measurable function $g(t) : -\infty < g(t) \leq t$ there exists a unique continuous global solution of the IVP(2).*

We shall use the fixed point theorems from [1]. Let X be a Hausdorff sequentially complete uniform space with uniformity defined by a saturated family of pseudometrics $\{\rho_\alpha(x,y)\}_{\alpha\in\mathcal{A}}, \mathcal{A}$ being an index set. Let $\Phi = \{\Phi_\alpha(t) : \alpha \in \mathcal{A}\}$ be a family of functions $\Phi_\alpha(t) : [0,\infty) \to [0,\infty)$ with the properties

1) $\Phi_\alpha(t)$ is monotone non-decreasing and continuous from the right on $[0,\infty)$;
2) $\Phi_\alpha(t) < t, \forall t > 0$,

and $j : \mathcal{A} \to \mathcal{A}$ is a mapping on the index set $\mathcal{A}$ into itself, where $j^0(\alpha) = \alpha, j^k(\alpha) = j(j^{k-1}(\alpha)), k \in \mathbf{N}$.

DEFINITION 3. *The map $T : M \to M$ is said to be a $\Phi-$contraction on M if $\rho_\alpha(Tx,Ty) \leq \Phi_\alpha(\rho_{j(\alpha)}(x,y))$ for every $x,y \in M$ and $\alpha \in \mathcal{A}, M \subset X$.*

THEOREM 4 ([1]). *Let us suppose*

1. *the operator $T : X \to X$ is a Φ-contraction;*
2. *for each $\alpha \in \mathcal{A}$ there exists a Φ-function $\overline{\Phi}_\alpha(t)$ such that*

$$\sup\{\Phi_{j^n(\alpha)}(t) : n = 0,1,2,...\} \leq \overline{\Phi}_\alpha(t)$$

and $\overline{\Phi}_\alpha(t)/t$ is non-decreasing;

3. *there exists an element* $x_0 \in X$ *such that*

$$\rho_{j^n(\alpha)}(x_0, Tx_0) \le p(\alpha) < \infty \; (n = 0, 1, 2, ...).$$

Then T *has at least one fixed point in* X.

THEOREM 5 ([1]). *If, in addition, we suppose that*

4. *the sequence* $\{\rho_{j^k(\alpha)}(x, y)\}_{k=0}^{\infty}$ *is bounded for each* $\alpha \in \mathcal{A}$ *and* $x, y \in X$, *i.e.*

$$\rho_{j^k(\alpha)}(x, y) \le q(x, y, \alpha) < \infty \; (k = 0, 1, 2, ...),$$

then the fixed point of T *is unique.*

PROOF OF THEOREM 2. Let X be the uniform sequentially complete Hausdorff space consisting of all functions, belonging to $C(\mathbf{R}; \mathbf{R}^n)$, which are equal to $\varphi(t) \, \forall t \le 0$, with a saturated family of pseudometrics $\rho_{K}(x, y) = p_{K}(x - y)$, where K runs over all compact subsets of $\mathbf{R}$. The operator $T : X \to X$ is defined by the formula:

$$T(x)(t) = \begin{cases} \varphi(0) + \displaystyle\int_0^t f(\tau, x(\tau), \|x(\tau)\|_g) \, d\tau, & t > 0 \\ \varphi(t), & t \le 0. \end{cases}$$

The function $\tau \to f(\tau, x(\tau), \|x(\tau)\|_g)$ is measurable, since f satisfies the Caratheodory condition, and $\|x(\tau)\|_g$ is a measurable function.

By condition (I)

$$\begin{aligned} \|f(\tau, x(\tau), \|x(\tau)\|_g)\| &\le \|f(\tau, 0, 0)\| + \Omega(\tau, \|x(\tau)\|, \|x(\tau)\|_g) \\ &\le \|f(\tau, 0, 0)\| + \|x(\tau)\|_g \omega(\tau), \end{aligned}$$

which belongs to $L^1_{loc}([0, \infty))$ ($\|x(\cdot)\|_g$ is locally bounded!) Thus $T(x) \in C(\mathbf{R}; \mathbf{R}^n)$. Choosing

$$x_0(t) = \begin{cases} \varphi(0), & t > 0 \\ \varphi(t), & t \le 0 \end{cases}$$

we obtain

$$\rho_{K}(x_0, T(x_0)) \le \rho_{j^m(K)}(x_0, T(x_0)) = \rho_{j(K)}(x_0, T(x_0)) \le c(K, f, \varphi) < \infty,$$

that is condition 3 of Theorem 4 is fulfilled.

The sequence $\{\rho_{j^m(K)}(x, y)\}_{m=0}^{\infty}$ in our case turns into

$$\rho_K(x, y), \rho_{j(K)}(x, y), \ldots, \rho_{j(K)}(x, y), \ldots,$$

$\rho_{K}(x, y) \le \rho_{j(K)}(x, y)$ for every $K \in \Psi$ and $x, y \in X$. Consequently condition 4 of Theorem 5 is also fulfilled.

We need the following

LEMMA 6. *Let* $y(t), x(t) \in C(\mathbf{R}; \mathbf{R}^n), g(t) : [0, \infty) \to \mathbf{R}$ *is a measurable function,* $-\infty < g(t) \le t$. *Then* $|\|x(t)\|_g - \|y(t)\|_g| \le \|x(t) - y(t)\|_g$.

The proof of Lemma 6 is obtained as a consequence of Minkowski's inequality.

Let $p>1$. Define $\Phi_K:[0,\infty)\to[0,\infty)$ by the formula:

$$\Phi_K(y)=\begin{cases}(\lambda q)^{-\frac{1}{q}}y\|\omega\|_{L^p([0,\sup K])}, & \sup K>0\\ 0, & \sup K\le 0,\end{cases}$$

where $1/p+1/q=1$, or $q=1$, if $p=\infty$, and λ is fixed such that $(\lambda q)^{-1/q}\|\omega\|_{L^p([0,\infty))}<1$. Then Φ_K is a continuous, non-decreasing function, $\Phi_K(y)<y$ for every $y>0$ and $\Phi_K(y)/y$ does not depend on y, in particular it is non-decreasing. We have $\Phi_K(y)=\Phi_{j^m(K)}(y)=\overline{\Phi}_K(y)$ for all $m=1,2,\ldots$, consequently $\overline{\Phi}_K(y)/y$ is non-decreasing (i.e. condition 2 of Theorem 4).

We are able to prove that the operator $T:X\to X$ is a Φ-contraction on X, i.e. $\rho_{_K}(T(x),T(y))\le\Phi_K(\rho_{j(K)}(x,y))$ for every $x,y\in X$, and $K\in\Psi$.

If $\sup K\le 0$, then $T(x)(t)-T(y)(t)=\varphi(t)-\varphi(t)=0$ for every $t\in K$.

For $t\in K\cap(0,\infty)\ne\emptyset$, we have

$$\begin{aligned}\|T(x)(t)-T(y)(t)\|&\le\int_0^t\|f(\tau,x(\tau),\|x(\tau)\|_g)-f(\tau,y(\tau),\|y(\tau)\|_g)\|\,d\tau\\
&\le\int_0^t\Omega\left(\tau,\|x(\tau)-y(\tau)\|,|\|x(\tau)\|_g-\|y(\tau)\|_g|\right)d\tau\\
&\le\int_0^t\Omega(\tau,\sup_{0\le s\le\tau}(\|x(s)-y(s)\|),\sup_{0\le s\le\tau}(\|x(s)-y(s)\|))\,d\tau\\
&\le\int_0^t\Omega(\tau,e^{\lambda\tau}\sup_{0\le s\le\tau}(e^{-\lambda s}\|x(s)-y(s)\|),e^{\lambda\tau}\sup_{0\le s\le\tau}(e^{-\lambda s}\|x(s)-y(s)\|))\,d\tau\\
&\le\rho_{j(K)}(x,y)\int_0^t e^{\lambda\tau}\omega(\tau)\,d\tau\le\rho_{j(K)}(x,y)\|\omega\|_{L^p[0,t]}\left(\int_0^t e^{\lambda q\tau}\,d\tau\right)^{\frac{1}{q}}\\
&\le\rho_{j(K)}(x,y)\|\omega\|_{L^p[0,\sup K]}e^{\lambda t}(\lambda q)^{-\frac{1}{q}}=e^{\lambda t}\Phi_K(\rho_{j(K)}(x,y)).\end{aligned}$$

Consequently

$$\begin{aligned}\rho_{_K}(T(x),T(y))&=\sup\{e^{-\lambda t}\|T(x)(t)-T(y)(t)\|:t\in K\}\\
&=\sup\{e^{-\lambda t}\|T(x)(t)-T(y)(t)\|:t\in K\cap(0,\infty)\}\\
&\le\Phi_K(\rho_{j(K)}(x,y))\end{aligned}$$

for every $x,y\in X$. Hence condition 1 of Theorem 4 is fulfilled. Therefore T has a unique fixed point in X, which is a solution of the IVP(2).

Let $p=1$. Extending ω as 0 on $(-\infty,0]$ and denote again by ω the resulting extension, we obtain a function $\omega\in L^1(\mathbf{R})$. Then $\forall\varepsilon>0\ \exists h=h_\varepsilon\in C_0^\infty(\mathbf{R})$ such that ([3], p.71)

$$\int_{-\infty}^{+\infty}|\omega(\tau)-h(\tau)|\,d\tau<\varepsilon.$$

Fixing $\varepsilon \in (0, \frac{1}{2})$ and $\lambda \geq 2 \int_{-\infty}^{+\infty} h^2(\tau)\,d\tau$, we define $\Phi_K : [0, \infty) \to [0, \infty)$ as follows:

$$\Phi_K(y) = \begin{cases} y\left(\varepsilon + \left(\frac{1}{2\lambda}\int_0^{\sup K} h^2(\tau)\,d\tau\right)^{\frac{1}{2}}\right), & \sup K > 0 \\ 0, & \sup K \leq 0. \end{cases}$$

Φ_K is a continuous, non-decreasing function, $\Phi_K(y) \leq y(\varepsilon + \frac{1}{2}) < y$ for every $y > 0$; $\Phi_K(y)/y$ does not depend on y, in particular it is non-decreasing. $\Phi_K(y) = \Phi_{j^m(K)}(y) = \overline{\Phi}_K(y)$ for every $m \in \mathbf{N}$, consequently $\overline{\Phi}_K(y)/y$ is non-decreasing (i.e. condition 2 of Theorem 4).

For $t \in K \cap (0, \infty) \neq \emptyset$, we have

$$\begin{aligned} \|T(x)(t) - T(y)(t)\| &\leq \int_0^t \Omega(\tau, e^{\lambda\tau}\rho_{j(K)}(x,y), e^{\lambda\tau}\rho_{j(K)}(x,y))\,d\tau \\ &\leq \rho_{j(K)}(x,y)\int_0^t e^{\lambda\tau}\omega(\tau)\,d\tau \\ &\leq \rho_{j(K)}(x,y)\left(\int_0^t e^{\lambda\tau}|\omega(\tau) - h(\tau)|\,d\tau + \left(\int_0^t e^{2\lambda\tau}\,d\tau\right)^{\frac{1}{2}}\left(\int_0^t h^2(\tau)\,d\tau\right)^{\frac{1}{2}}\right) \\ &\leq e^{\lambda t}\rho_{j(K)}(x,y)\left(\varepsilon + \left(\frac{1}{2\lambda}\int_0^{\sup K} h^2(\tau)\,d\tau\right)^{\frac{1}{2}}\right) = e^{\lambda t}\Phi_K(\rho_{j(K)}(x,y)). \end{aligned}$$

Thus T is a Φ-contraction on X, which is a condition 1 of Theorem 4.

Therefore T has a unique fixed point in X. The proof of the Theorem 2 is complete. ∎

In what follows we consider a maximum equation

$$L\dot{I}(t) + M\|I(t)\|_h = k\frac{I^3(t)}{1 + I^2(t)},$$

where the unknown function $I(t)$ is electric current, $L \neq 0, M, k$ are constants and $\|I(t)\|_h = \max\{|I(s)| : t - h \leq s \leq t\}$, with some $h > 0$. It is derived treating the original automatic regulation phenomenon ([2]) without linearization. Then we can formulate an initial-value problem for the above equation as follows:

$$\begin{cases} \dot{I}(t) = f(I(t), \|I(t)\|_h), & t > 0 \\ I(t) = \varphi(t), & t \leq 0, \end{cases} \tag{3}$$

where φ is a prescribed initial continuous function, and

$$f(u, v) = L^{-1}(k\frac{u^3}{1 + u^2} - Mv).$$

We check conditions of the Theorem 2: $\varphi : (-\infty, 0] \to \mathbf{R}$ is a continuous function – that is the condition (ii) of the Theorem 2.

$$\begin{aligned} |f(u_1, v_1) - f(u_2, v_2)| &\leq |L|^{-1}\left(\frac{9}{8}|k||u_1 - u_2| + |M||v_1 - v_2|\right) \\ &= \Omega(|u_1 - u_2|, |v_1 - v_2|). \end{aligned}$$

Here $\Omega(u, v) = |L|^{-1}(C_k u + |M|v)$ is a homogeneous polynomial of the non-negative variables u, v. $\Omega(v, v) = |L|^{-1}(C_k + |M|)v = \omega v$, where ω does not depend on t and in particular $\omega \in L^\infty([0, \infty); [0, \infty))$. Thus condition (i) of the Theorem 2 is also fulfilled, which implies an existence of solution of (3).

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