ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR MAXIMUM EQUATIONS

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Abstract. An existence-uniqueness result for the Cauchy problem for a system of ordinary differential equations with maximums is established.

The paper is concerned with the following initial-value problem (IVP):

\[ \begin{aligned}
\dot{x}(t) &= f(t, x(t), \|x(t)\|_g), & t > 0 \\
x(t) &= \varphi(t), & t \leq 0,
\end{aligned} \]

where

\[ x(t) = (x_1(t), \ldots, x_n(t)), \quad \dot{x}(t) = (\dot{x}_1(t), \ldots, \dot{x}_n(t)), \]

\[ \|x(t)\|_g = \max_{g(t) \leq s \leq t} \|x(s)\|, \quad \|x(s)\| = \max_{1 \leq i \leq n} |x_i(s)|, \]

\[ g(t) : [0, \infty) \to \mathbb{R} \] being a prescribed function, such that \(-\infty < g(t) \leq t\), for every \(t \geq 0\).

The mathematical formulation above mentioned arises in automatic regulations, integral electronics and measurement devices. In [2] (p.p. 29, 477, 565) the authors present various relay systems for automatic regulation - for instance, of the temperature in some chamber. For the variation of the temperature \(\theta(t)\) the equation

\[ T \frac{d\theta}{dt} + \theta = -k\varphi + f, \]

is obtained, where \(T, k\) are constants, \(f = f(t)\) - external perturbations, and \(\varphi\) is the variation of the regulating device (relay system), which depends on \(t\) and \(\max \{\|\theta(s)\| : t_0 \leq s \leq t\}\).

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The main interest of many authors is the existence of periodic and oscillating solutions of (1) ([8]-[7]). In many cases, however, various conditions are formulated which do not guarantee even an existence of a solution. That is why, here we present existence conditions applying fixed point technics, obtained in a previous paper [1].

As usually, using that

$$x(t) = x(0) + \int_0^t \dot{x}(s) \, ds,$$

for $t > 0$, we reduce the IVP(1) to the following one:

$$\begin{cases}
  x(t) = \varphi(0) + \int_0^t f(\tau, x(\tau), \|x(\tau)\|_g) \, d\tau, & t > 0 \\
  x(t) = \varphi(t), & t \leq 0.
\end{cases}$$

(2)

First of all we have to investigate the measurability of $x(t)$ on $[0; 1]$.

**Proposition 1.** Let $g(t)$ be defined and measurable on $[0; 1)$ function, $-\infty < g(t) \leq t, \forall t \geq 0$. Then for every $x \in C(\mathbb{R}; \mathbb{R}^n), \|x(t)\|_g$ is a measurable locally bounded function on $[0; \infty)$.

**Proof.**

Inequality

$$\|x(t)\|_g \leq \max\{\|x(s)\| : \inf_{\tau \in K} g(\tau) \leq s \leq \sup K\}$$

for any compact interval $K \subset \mathbb{R}$, shows that $\|x(t)\|_g$ is a bounded function on every compact subset of $\mathbb{R}$.

Let us assume that $\|x(t)\|_g$ is not a measurable function. Then there exists $c \in (-\infty; \infty)$ such that the set $A_c = \{t \geq 0 : \|x(t)\|_g < c\}$ is not measurable.

Consider the function

$$\varphi_\alpha : [0; \infty) \rightarrow [0; \infty) : \varphi_\alpha(t) = \|x(\alpha t + (1 - \alpha)g(t))\|, \quad 0 \leq \alpha \leq 1.$$ 

For any fixed $\alpha \in [0, 1]$ the function $\varphi_\alpha(t) = \alpha t + (1 - \alpha)g(t)$ is measurable on $[0; \infty)$ as a linear combination of measurable functions. Consequently $\|x_\alpha(t)\|$ is measurable for every $i = 1, 2, ..., n$, and so $\varphi_\alpha = \max\{\|x_\alpha(t)\| : 1 \leq i \leq n\}$ is measurable, which means that the set $A_{\alpha, a} = \{t \geq 0 : \varphi_\alpha(t) < a\}$ is measurable for every $a \in \mathbb{R}$, and $\alpha \in [0, 1]$.

On the other hand $\|x(t)\|_g = \sup \{\varphi_\alpha(t) : 0 \leq \alpha \leq 1\} = \varphi_\beta(t)$ is attained for some $\beta \in [0, 1]$, and the set $A_{\beta, c}$ is measurable. But $A_{\beta, c} = \{t \geq 0 : \varphi_\beta(t) < c\} = \{t \geq 0 : \|x(t)\|_g < c\} = A_c$ - contradiction, which completes the proof. \[\square\]

We are going to look for a continuous solutions of (2).

Consider the linear space $C(\mathbb{R}; \mathbb{R}^n)$ with a saturated family of seminorms

$$p_k(y) = \sup_{t \in K} e^{-\lambda t} \|y(t)\|,$$
where $\lambda > 0$ and $K$ runs over all compact subsets of $\mathbb{R}$. It defines a locally convex Hausdorff topology on $C(\mathbb{R}; \mathbb{R}^n)$.

We denote by $\Psi$ the set of all compact subsets of $\mathbb{R}$ and we define the map $j : \Psi \to \Psi$:

$$j(K) = \begin{cases} K, & \sup K \leq 0 \\ [0, \sup K], & \sup K > 0. \end{cases}$$

It is obvious that $j^2(K) = j(j(K)) = j(K)$ and consequently, $j^m(K) = j(K)$ for all $m \in \mathbb{N}$.

Now we make the following assumptions (I):

(i) The function $f(t, u, v) : [0, \infty) \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ satisfies the Caratheodory condition (measurable in $t$ and continuous in $u, v$), $\|f(\cdot, 0, 0)\| \in L_{\text{loc}}^1([0, \infty))$ and

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leq \Omega(t, \|u_1 - u_2\|, \|v_1 - v_2\|),$$

where the comparison function $\Omega(t, x, y)$ satisfies the Caratheodory condition. It is non-decreasing in $x$ and $y$ and for any fixed $y \geq 0, \Omega(\cdot, y, y) \leq g\omega(\cdot)$ with some $\omega \in L^p([0, \infty); [0, \infty]), p \geq 1$;

(ii) The initial function $\varphi : (-\infty, 0] \to \mathbb{R}^n$ is continuous.

**Theorem 2.** If conditions (I) are fulfilled, then for any measurable function $g(t) : -\infty < g(t) \leq t$ there exists a unique continuous global solution of the IVP(2).

We shall use the fixed point theorems from [1]. Let $X$ be a Hausdorff sequentially complete uniform space with uniformity defined by a saturated family of pseudometrics $\{\rho_\alpha(x, y)\}_{\alpha \in A}, A$ being an index set. Let $\Phi = \{\Phi_\alpha(t) : \alpha \in A\}$ be a family of functions $\Phi_\alpha(t) : [0, \infty) \to [0, \infty)$ with the properties

1) $\Phi_\alpha(t)$ is monotone non-decreasing and continuous from the right on $[0, \infty)$;
2) $\Phi_\alpha(t) < t, \forall t > 0$,

and $j : A \to A$ is a mapping on the index set $A$ into itself, where $j^0(\alpha) = \alpha, j^k(\alpha) = j(j^{k-1}(\alpha)), k \in \mathbb{N}$.

**Definition 3.** The map $T : M \to M$ is said to be a $\Phi$-contraction on $M$ if $\rho_\alpha(Tx, Ty) \leq \Phi_\alpha(\rho_{j(\alpha)}(x, y))$ for every $x, y \in M$ and $\alpha \in A, M \subset X$.

**Theorem 4 ([1]).** Let us suppose

1. the operator $T : X \to X$ is a $\Phi$-contraction;
2. for each $\alpha \in A$ there exists a $\Phi$-function $\overline{\Phi}_\alpha(t)$ such that

$$\sup \{\Phi_{j^n(\alpha)}(t) : n = 0, 1, 2, \ldots\} \leq \overline{\Phi}_\alpha(t)$$

and $\overline{\Phi}_\alpha(t)/t$ is non-decreasing;
3. there exists an element \( x_0 \in X \) such that
\[
\rho_{j^n(\alpha)}(x_0, Tx_0) \leq p(\alpha) < \infty \quad (n = 0, 1, 2, ...).
\]
Then \( T \) has at least one fixed point in \( X \).

**Theorem 5** ([1]). If, in addition, we suppose that
4. the sequence \( \{\rho_{j^n(\alpha)}(x, y)\}_{k=0}^{\infty} \) is bounded for each \( \alpha \in A \) and \( x, y \in X \),
i.e.
\[
\rho_{j^n(\alpha)}(x, y) \leq q(x, y, \alpha) < \infty \quad (k = 0, 1, 2, ...),
\]
then the fixed point of \( T \) is unique.

**Proof of Theorem 2.** Let \( X \) be the uniform sequentially complete Hausdorff space consisting of all functions, belonging to \( C(\mathbb{R}; \mathbb{R}^n) \), which are equal to \( \varphi(t) \forall t \leq 0 \), with a saturated family of pseudometrics \( \rho_{\kappa}(x, y) = p_\kappa(x - y) \), where \( \kappa \) runs over all compact subsets of \( \mathbb{R} \). The operator \( T : X \to X \) is defined by the formula:
\[
T(x)(t) = \begin{cases} 
\varphi(0) + \int_0^t f(\tau, x(\tau), \|x(\tau)\|_g) \, d\tau, & t > 0 \\
\varphi(t), & t \leq 0.
\end{cases}
\]
The function \( \tau \to f(\tau, x(\tau), \|x(\tau)\|_g) \) is measurable, since \( f \) satisfies the Caratheodory condition, and \( \|x(\tau)\|_g \) is a measurable function.

By condition (I)
\[
\|f(\tau, x(\tau), \|x(\tau)\|_g)\| \leq \|f(\tau, 0, 0)\| + \Omega(\tau, \|x(\tau)\|, \|x(\tau)\|_g) \\
\leq \|f(\tau, 0, 0)\| + \|x(\tau)\|_g \omega(\tau),
\]
which belongs to \( L^1_{loc}([0, \infty)) \) (\( \|x(\cdot)\|_g \) is locally bounded!) Thus \( T(x) \in C(\mathbb{R}; \mathbb{R}^n) \). Choosing
\[
x_0(t) = \begin{cases} 
\varphi(0), & t > 0 \\
\varphi(t), & t \leq 0
\end{cases}
\]
we obtain
\[
\rho_{\kappa}(x_0, T(x_0)) \leq \rho_{j^n(\kappa)}(x_0, T(x_0)) = \rho_{j^n(\kappa)}(x_0, T(x_0)) \leq c(K, f, \varphi) < \infty,
\]
that is condition 3 of Theorem 4 is fulfilled.

The sequence \( \{\rho_{j^n(\kappa)}(x, y)\}_{m=0}^{\infty} \) in our case turns into
\[
\rho_{\kappa}(x, y), \rho_{j(\kappa)}(x, y), \ldots, \rho_{j^n(\kappa)}(x, y), \ldots,
\]
\( \rho_{\kappa}(x, y) \leq \rho_{j^n(\kappa)}(x, y) \) for every \( \kappa \in \Psi \) and \( x, y \in X \). Consequently condition 4 of Theorem 5 is also fulfilled.

We need the following

**Lemma 6.** Let \( y(t), x(t) \in C(\mathbb{R}; \mathbb{R}^n), g(t) : [0, \infty) \to \mathbb{R} \) is a measurable function, \(-\infty < g(t) \leq t\). Then \( \|x(t)\|_g - \|y(t)\|_g \leq \|x(t) - y(t)\|_g \).
The proof of Lemma 6 is obtained as a consequence of Minkowski’s inequality.

Let \( p > 1 \). Define \( \Phi_K : [0, \infty) \to [0, \infty) \) by the formula:

\[
\Phi_K(y) = \begin{cases} 
(\lambda q)^{-1/p} y \| \omega \|_{L^p([0,\sup K])}, & \text{if } \sup K > 0 \\
0, & \text{if } \sup K \leq 0,
\end{cases}
\]

where \( 1/p + 1/q = 1 \), or \( q = 1 \), if \( p = \infty \), and \( \lambda \) is fixed such that \((\lambda q)^{-1/q} \| \omega \|_{L^p([0,\infty))} < 1 \). Then \( \Phi_K \) is a continuous, non-decreasing function, \( \Phi_K(y) < y \) for every \( y > 0 \) and \( \Phi_K(y)/y \) does not depend on \( y \), in particular it is non-decreasing. We have \( \Phi_K(y) = \Phi_j^m(K)(y) = \overline{\Phi}_K(y) \) for all \( m = 1, 2, \ldots \), consequently \( \overline{\Phi}_K(y)/y \) is non-decreasing (i.e. condition 2 of Theorem 4).

We are able to prove that the operator \( T : X \to X \) is a \( \Phi \)-contraction on \( X \), i.e. \( \rho_K(T(x), T(y)) \leq \Phi_K(\rho_j(K)(x, y)) \) for every \( x, y \in X \), and \( K \in \Psi \).

If \( \sup K \leq 0 \), then \( T(x)(t) - T(y)(t) = \varphi(t) - \varphi(t) = 0 \) for every \( t \in K \).

For \( t \in K \cap (0, \infty) \neq \emptyset \), we have

\[
\| T(x)(t) - T(y)(t) \| \leq \int_0^t \| f(\tau, x(\tau), \| x(\tau) \|_g) - f(\tau, y(\tau), \| y(\tau) \|_g) \| d\tau \\
\leq \int_0^t \Omega(\tau, \| x(\tau) - y(\tau) \|, \| x(\tau) \|_g - \| y(\tau) \|_g) d\tau \\
\leq \int_0^t \Omega(\tau, \sup_{0 \leq s \leq \tau} \| x(s) - y(s) \|, \sup_{0 \leq s \leq \tau} \| x(s) - y(s) \|) d\tau \\
\leq \int_0^t \Omega(\tau, e^{\lambda \tau} \sup_{0 \leq s \leq \tau} (e^{-\lambda s} \| x(s) - y(s) \|), e^{\lambda \tau} \sup_{0 \leq s \leq \tau} (e^{-\lambda s} \| x(s) - y(s) \|)) d\tau \\
\leq \rho_j(K)(x, y) \int_0^t e^{\lambda \tau} \omega(\tau) d\tau \leq \rho_j(K)(x, y) \| \omega \|_{L^p[0, \infty]} \left( \int_0^t e^{\lambda \tau} d\tau \right)^{\frac{1}{p}} \\
\leq \rho_j(K)(x, y) \| \omega \|_{L^p[0, \sup K]} e^{\lambda \tau} \| \omega \|_{L^p[0, \infty]} \left( \int_0^t e^{\lambda \tau} d\tau \right)^{\frac{1}{p}} = e^{\lambda \tau} \Phi_K(\rho_j(K)(x, y)).
\]

Consequently

\[
\rho_K(T(x), T(y)) = \sup \{ e^{-\lambda \tau} \| T(x)(t) - T(y)(t) \| : t \in K \} \\
= \sup \{ e^{-\lambda \tau} \| T(x)(t) - T(y)(t) \| : t \in K \cap (0, \infty) \} \\
\leq \Phi_K(\rho_j(K)(x, y))
\]

for every \( x, y \in X \). Hence condition 1 of Theorem 4 is fulfilled. Therefore \( T \) has a unique fixed point in \( X \), which is a solution of the IVP(2).

Let \( p = 1 \). Extending \( \omega \) as 0 on \( (-\infty, 0] \) and denote again by \( \omega \) the resulting extension, we obtain a function \( \omega \in L^1(\mathbb{R}) \). Then \( \forall \varepsilon > 0 \exists h = h_\varepsilon \in C_0^\infty(\mathbb{R}) \) such that (\cite{3}, p.71)

\[
\int_{-\infty}^{+\infty} |\omega(\tau) - h(\tau)| d\tau < \varepsilon.
\]
Fixing $\varepsilon \in (0, \frac{1}{2})$ and $\lambda \geq 2 \int_{-\infty}^{+\infty} h^2(\tau) \, d\tau$, we define $\Phi_K : [0, \infty) \to [0, \infty)$ as follows:

$$
\Phi_K(y) = \begin{cases} 
  y \left( \varepsilon + \left( \frac{1}{2\lambda} \int_0^{\sup K} h^2(\tau) \, d\tau \right)^\frac{1}{2} \right), & \sup K > 0 \\
  0, & \sup K \leq 0.
\end{cases}
$$

$\Phi_K$ is a continuous, non-decreasing function, $\Phi_K(y) \leq y(\varepsilon + \frac{1}{2}) < y$ for every $y > 0$; $\Phi_K(y)/y$ does not depend on $y$, in particular it is non-decreasing. $\Phi_K(y) = \Phi_{j_{m}^{+}(K)}(y) = \Phi_{j}(y)$ for every $m \in \mathbb{N}$, consequently $\Phi_K(y)/y$ is non-decreasing (i.e. condition 2 of Theorem 4).

For $t \in K \cap (0, \infty) \neq \emptyset$, we have

$$
\|T(x)(t) - T(y)(t)\| \leq \int_0^t \Omega(\tau, e^{\lambda\tau} \rho_j(K)(x,y), e^{\lambda\tau} \rho_j(K)(x,y)) \, d\tau \\
\leq \rho_j(K)(x,y) \int_0^t e^{\lambda\tau} \omega(\tau) \, d\tau \\
\leq \rho_j(K)(x,y) \left( \int_0^t e^{\lambda\tau} [\omega(\tau) - \delta(\tau)] \, d\tau + \left( \int_0^t e^{2\lambda\tau} \, d\tau \right) \frac{1}{2} \left( \int_0^t h^2(\tau) \, d\tau \right)^\frac{1}{2} \right) \\
\leq e^{\lambda\tau} \rho_j(K)(x,y) \left( \varepsilon + \left( \frac{1}{2\lambda} \int_0^{\sup K} h^2(\tau) \, d\tau \right)^\frac{1}{2} \right) = e^{\lambda\tau} \Phi_K(\rho_j(K)(x,y)).
$$

Thus $T$ is a $\Phi$-contraction on $X$, which is a condition 1 of Theorem 4.

Therefore $T$ has a unique fixed point in $X$. The proof of the Theorem 2 is complete.

In what follows we consider a maximum equation

$$
L I(t) + M \| I(t) \|_h = k \frac{I^3(t)}{1 + I^2(t)},
$$

where the unknown function $I(t)$ is electric current, $L \neq 0, M, k$ are constants and $\|I(t)\|_h = \max \{ |I(s)| : t - h \leq s \leq t \}$, with some $h > 0$. It is derived treating the original automatic regulation phenomenon ([2]) without linearization. Then we can formulate an initial-value problem for the above equation as follows:

$$
\begin{cases}
  \dot{I}(t) = f(I(t), \| I(t) \|_h), & t > 0 \\
  I(t) = \varphi(t), & t \leq 0,
\end{cases}
$$

where $\varphi$ is a prescribed initial continuous function, and

$$
f(u, v) = L^{-1}(k \frac{u^3}{1 + u^2} - Mu).
$$
We check conditions of the Theorem 2: $\varphi : (-\infty, 0] \to \mathbb{R}$ is a continuous function – that is the condition (ii) of the Theorem 2.

$$|f(u_1, v_1) - f(u_2, v_2)| \leq |L|^{-1}\left(\frac{9}{8}k|u_1 - u_2| + |M||v_1 - v_2|\right)$$

$$= \Omega(|u_1 - u_2|, |v_1 - v_2|).$$

Here $\Omega(u, v) = |L|^{-1}(C_k u + |M| v)$ is a homogeneous polynomial of the non-negative variables $u, v$. $\Omega(v, v) = |L|^{-1}(C_k + |M|)v = \omega v$, where $\omega$ does not depend on $t$ and in particular $\omega \in L^\infty([0, \infty); [0, \infty))$. Thus condition (i) of the Theorem 2 is also fulfilled, which implies an existence of solution of (3).

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References


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