ON A-STATISTICAL CLUSTER POINTS

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Abstract. In this paper we study the concepts of statistical cluster points and statistical core of a sequence for $C_A$ methods defined by deleting a set of rows from the Cesaro matrix $C_1$. Also we get necessary conditions on the matrices $A$ and $B$ so that $A$ and $B$ are equivalent in the statistical convergence sense and, study the equality $\Gamma_A(x) = \Gamma_B(x)$, where $\Gamma_A(x)$ is the set of A-statistical cluster points of the real number sequence $x$.

1. INTRODUCTION AND NOTATIONS

In [5] Fridy introduced the concepts of statistical limit points and statistical cluster points of a number sequence. These concepts are compared to the usual concept of limit point of a sequence. In [6] Fridy and Orhan introduced the concepts of statistical limit superior and inferior. They have also given the definition of the statistical core of a real number sequence which is based on the idea of the statistical cluster points of the sequence, and proved the statistical core theorem. Those results have also been extended [7] to the complex case by them, too. In [2] Connor and Kline extended the concept of a statistical limit (cluster) point of a number sequence to a A-statistical limit (cluster) point where $A$ is a nonnegative regular summability matrix. In [3] the present author extended the concepts of statistical limit superior and inferior (as introduced by Fridy and Orhan) to A-statistical limit superior and inferior and given some A-statistical analogue of properties of statistical limit superior and inferior for a sequence of real numbers. Also in [3] the concept of statistical core is extended to A-statistical core.

In this paper we study the concepts of statistical cluster points and statistical core of a sequence for $C_A$ methods, defined by deleting a set of rows from

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the Cesàro matrix $C_1$. Also we get necessary conditions on the matrices $A$ and $B$ so that $A$ and $B$ are equivalent in the statistical convergence sense and, study the equality $\Gamma_A(x) = \Gamma_B(x)$, where $\Gamma_A(x)$ is the set of $A$– statistical cluster points of the real number sequence $x$.

First we introduce some notation. Let $A = (a_{nk})$ denote a summability matrix which transforms a number sequence $x = (x_k)$ into the sequence $Ax$ whose n-th term is given by $(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k$. As usual, $\mathbb{N}$ and $\mathbb{C}$ denote the sets of positive integers and complex numbers, respectively.

If $K$ is a set of positive integers, $|K|$ will denote the cardinality of $K$. The natural density of $K$ [11] is given by
\[
\delta(K) := \lim_{n \to \infty} \frac{1}{n} \left| \{ k \leq n : k \in K \} \right|,
\]
if it exists, where $C_1$ is the Cesàro mean of order one and $\chi_K$ is the characteristic function of the set $K$.

We recall the following elementary result concerning natural density (See [11, page 222]):

Let $E$ be an infinite subset of $\mathbb{N}$ and consider $E$ as strictly increasing sequence of positive integers, say $E = \{ \lambda(n) \}_{n=1}^{\infty}$. Then
\[
\delta(E) = \lim_{n \to \infty} \frac{n}{\lambda(n)}
\]
provided this limit exists. Because $\delta(E)$ does not exists for all subsets of $\mathbb{N}$, it is convenient to use the upper asymptotic density $\delta^*(E)$, which is defined by
\[
\delta^*(E) = \limsup_{n \to \infty} \frac{1}{n} \left| \{ k \leq n : k \in E \} \right|
\]
(See [9, p.xvii]). For convenience we state here some properties of $\delta^*$. For arbitrary subsets $E$ and $F$ of $\mathbb{N}$ we have

(i) if $\delta(E)$ exists then $\delta(E) = \delta^*(E)$;
(ii) $\delta(E) \neq 0$ if and only if $\delta^*(E) > 0$;
(iii) if $E \subseteq F$, then $\delta^*(E) \leq \delta^*(F)$.

Natural density can be generalized by using a nonnegative regular summability matrix $A$ in place of $C_1$.

Following Freedman and Sember [4] we say that a set $K \subseteq \mathbb{N}$ has $A$–density if
\[
\delta_A(K) = \lim_{n \to \infty} \frac{1}{n} \sum_{k \in K} a_{nk} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\infty} a_{nk} \chi_K(k) = \lim_{n \to \infty} (A\chi_K)_n
\]
exists where $A$ is a nonnegative regular summability matrix.

The number sequence $x = (x_k)$ is $A$–statistically convergent to $L$ provided that for every $\epsilon > 0$ the set $K_\epsilon := \{ k \in \mathbb{N} : |x_k - L| \geq \epsilon \}$ has $A$–density zero [2, 10]. In this case we write $st_A - \lim x = L$. 

By \( st_A \) we denote the set of all \( A \)-statistically convergent sequences.

The number \( \gamma \) is a \( A \)-statistical cluster point of the number sequence \( x = (x_k) \) provided that for every \( \epsilon > 0, \delta_A(K_\epsilon) \neq 0 \) where \( K_\epsilon := \{ k \in \mathbb{N} : |x_k - \gamma| < \epsilon \} \) [2]. Note that the statement \( \delta_A(K) \neq 0 \) means that either \( \delta_A(K) > 0 \) or \( K \) fails to have \( A \)-density.

By \( \Gamma_A(x) \) we denote the set of all \( A \)-statistical cluster points of \( x \). When \( A = C_1 \) we shall simply write \( \delta \) instead of \( \delta_{C_1} \) and \( \Gamma \) instead of \( \Gamma_{C_1} \).

The sequence \( x = (x_k) \) is the \( A \)-statistically bounded if it has a bounded subsequence \( \{x_k\}_{k \in E} \) such that \( \delta_A(E) = 1; st_A - \limsup x \) and \( st_A - \liminf x \) are the greatest and least \( A \)-statistical cluster point of such an \( x \) [3]. Also \( A \)-statistically bounded sequence \( x \) is \( A \)-statistically convergent if and only if \( st_A - \liminf x = st_A - \limsup x \) [3].

Note that \( A \)-statistically boundedness implies that \( st_A - \limsup \) and \( st_A - \liminf \) are finite [3]. Some results on statistical limit points may be found in [2, 5, 6, 13].

For any complex number sequence \( x = (x_k) \) the \( A \)-statistical core of \( x \) is given by

\[
st_A - \text{core} \{ x \} = \bigcap_{H \in \mathbf{H}(x)} H,
\]

where \( \mathbf{H}(x) \) is the collection of all closed half-planes \( H \) that satisfy \( \delta_A \{ k \in \mathbb{N} : x_k \in H \} = 1 \) (see [3]).

In [3, Theorem 6] it is shown that for every \( A \)-statistically bounded complex number sequence \( x = (x_k) \)

\[
st_A - \text{core} \{ x \} = \bigcap_{z \in \mathbb{C}} B_x(z),
\]

where

\[
B_x(z) := \left\{ w \in \mathbb{C} : |w - z| \leq st_A - \limsup_k |x_k - z| \right\}.
\]

When \( A = C_1 \) we shall simply write \( st \)-core instead of \( st_{C_1} \)-core (see [6, 7]).

2. \( C_\lambda \)-statistical cluster points

In [1] Armitage and Maddox introduced the summability method \( C_\lambda \) defined by deleting a set of rows from the Cesàro matrix. They gave some inclusion theorems for \( C_\lambda \) methods. This method has also been studied in [12].

Let \( E \) be an infinite subset of \( \mathbb{N} \) and consider \( E \) as strictly increasing sequence of positive integers, say \( E = \{\lambda(n)\}_{n=1}^{\infty} \). The summability method \( C_\lambda \), as introduced in [1], is defined as

\[
(C_\lambda x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k,
\]
where $x = (x_k)$ is a sequence of real or complex numbers and $n = 1, 2, \ldots$. It is clear that $C_\lambda$ is regular for any $\lambda$.

Note that if $A = C_\lambda$, then $\gamma \in \Gamma_{C_\lambda}(x)$ if, for every $\varepsilon > 0$,

$$\delta_{C_\lambda}(K) = \lim_{n} (C_\lambda x)_n = \lim_{n} \frac{1}{\lambda(n)} \{ k \leq k(n) : |x_k - \gamma| < \varepsilon \} \neq 0.$$

In the particular case when $\lambda(n) = n$ we see that $(C_\lambda x)_n$ is the $C_1$ mean of $x$.

In this section we establish inclusion relations between $\Gamma_{C_\lambda}(x)$ and $\Gamma_{C_\mu}(x)$ and between $\Gamma(C_\lambda x)$ and $\Gamma(C_\mu x)$ for $C_\lambda$ methods. Also we study $C_\lambda$-statistical core for a bounded complex sequence.

**Theorem 2.1.** Let $F = \{ \lambda(n) \}$ and $E = \{ \mu(n) \}$ be infinite subsets of $\mathbb{N}$. If $E \setminus F$ is finite and $\lim_{n} \frac{\lambda(n)}{\mu(n)} = d \neq 0$, then

$$\delta_{C_\lambda}(K) \neq 0 \text{ implies } \delta_{C_\mu}(K) \neq 0 \text{ for every } K \subseteq \mathbb{N}.$$

**Proof.** If $E \setminus F$ is finite, then there exists $N$ such that $\{ \mu(n) : n \geq N \} \subseteq F$. For $n \geq N$ let $j(n)$ be such that $\mu(n) = \lambda_{j(n)}$. Then $(j(n))$ increases and $j(n) \to \infty$, (as $n \to \infty$). If $\delta_{C_\lambda}(K) \neq 0$, then

$$\delta_{C_\lambda}(K) = \lim_{n} \sup \frac{|\{ i \leq \lambda(n) : i \in K \}|}{\lambda(n)} > 0.$$

Since $\lim_{n} \sup \lambda_n(x_n y_n) \leq (\lim_{n} x_n)(\lim_{n} y_n)$ provided that the right hand side exists, and

$$\frac{\lambda(n)}{\lambda_{j(n)}} \frac{|\{ i \leq \lambda(n) : i \in K \}|}{\lambda(n)} \leq \frac{|\{ i \leq \lambda_{j(n)} : i \in K \}|}{\lambda_{j(n)}},$$

we get

$$\delta_{C_\mu}(K) = \lim_{n} \sup \frac{|\{ i \leq \mu(n) : i \in K \}|}{\mu(n)} > 0.$$

Hence $\delta_{C_\mu}(K) \neq 0$.

Since $E \Delta F = (E \setminus F) \cup (F \setminus E), (C_\mu x)_n = (C_1 x)_{\mu(n)}$ and $(C_\lambda x)_n = (C_1 x)_{\lambda(n)}$, we immediately get the following from Theorem 2.1.

**Theorem 2.2.** Let $F = \{ \lambda(n) \}$ and $E = \{ \mu(n) \}$ be infinite subsets of $\mathbb{N}$.

(i) If $E \setminus F$ is finite and $\lim_{n} \frac{\lambda(n)}{\mu(n)} = d \neq 0$, then $\Gamma_{C_\lambda}(x) \subseteq \Gamma_{C_\mu}(x)$.

(ii) If $E \Delta F$ is finite and $\lim_{n} \frac{\lambda(n)}{\mu(n)} = d \neq 0$, then $\Gamma_{C_\lambda}(x) = \Gamma_{C_\mu}(x)$.

(iii) If $E \setminus F$ is finite and $\lim_{n} \frac{\lambda(n)}{\mu(n)} = d \neq 0$, then $\Gamma(C_\mu x) \subseteq \Gamma(C_\lambda x)$.

(iv) If $E \Delta F$ is finite and $\lim_{n} \frac{\lambda(n)}{\mu(n)} = d \neq 0$, then $\Gamma(C_\mu x) = \Gamma(C_\lambda x)$.

When $\lambda(n) = n$ the following may be deduced from (i) and (iii) of Theorem 2.2.
**Theorem 2.3.** Let $E = \{\mu(n)\}$ be infinite subset of $\mathbb{N}$.

(i) If $\lim_{n} \frac{\lambda(n)}{\mu(n)} = d \neq 0$, then $\Gamma(x) \subseteq \Gamma_{\mu}(x)$.

(ii) If $\lim_{n} \frac{\lambda(n)}{\mu(n)} = d \neq 0$, then $\Gamma(C_{\mu}x) \subseteq \Gamma(C_{1}x)$.

It is clear from (i) of Theorem 2.2 that for every bounded complex sequence $x = (x_k)$

$$st_{C_{\lambda}} - \lim sup |x| \leq st_{C_{\mu}} - \lim sup |x|.$$ 

So it follows that, for any $z \in \mathbb{C}$,

$$\left\{ w \in \mathbb{C} : |w - z| \leq st_{C_{\lambda}} - \lim sup |x_k - z| \right\} \subseteq$$

$$\subseteq \left\{ w \in \mathbb{C} : |w - z| \leq st_{C_{\mu}} - \lim sup |x_k - z| \right\}.$$ 

Now Theorem 6 of [3] implies that

$$\bigcap_{z \in \mathbb{C}} \left\{ w \in \mathbb{C} : |w - z| \leq st_{C_{\lambda}} - \lim sup |x_k - z| \right\} \subseteq$$

$$\subseteq \bigcap_{z \in \mathbb{C}} \left\{ w \in \mathbb{C} : |w - z| \leq st_{C_{\mu}} - \lim sup |x_k - z| \right\},$$

i.e.,

$$st_{C_{\lambda}} - \text{core } \{x\} \subseteq st_{C_{\mu}} - \text{core } \{x\}.$$ 

Thus we have

**Corollary 2.4.** Let $F = \{\lambda(n)\}$ and $E = \{\mu(n)\}$ be infinite subsets of $\mathbb{N}$. If $E \setminus F$ is finite and $\lim_{n} \frac{\lambda(n)}{\mu(n)} = d \neq 0$, then $st_{C_{\lambda}} - \text{core } \{x\} \subseteq st_{C_{\mu}} - \text{core } \{x\}$ for every bounded complex sequence $x$.

We immediately get the next corollary from (ii),(iii) and (iv) of Theorem 2.2 while the latter from Theorem 2.3 for every bounded complex sequence $x$.

**Corollary 2.5.** Let $F = \{\lambda(n)\}$ and $E = \{\mu(n)\}$ be infinite subsets of $\mathbb{N}$. Then, for every bounded complex sequence $x$,

(i) if $E \Delta F$ is finite and $\lim_{n} \frac{\lambda(n)}{\mu(n)} = d \neq 0$, then $st_{C_{\lambda}} - \text{core } \{x\} = st_{C_{\mu}} - \text{core } \{x\}$;

(ii) if $E \setminus F$ is finite and $\lim_{n} \frac{\lambda(n)}{\mu(n)} = d \neq 0$, then $st_{C_{\mu}} \setminus \text{core } \{C_{1}x\} \subseteq st - \text{core } \{C_{\lambda}x\};$

(iii) if $E \Delta F$ is finite and $\lim_{n} \frac{\lambda(n)}{\mu(n)} = d \neq 0$, then $st - \text{core } \{C_{\mu}x\} = st - \text{core } \{C_{\lambda}x\}$. 
Corollary 2.6. Let \( E = \{\mu(n)\} \) be infinite subset of \( \mathbb{N} \). Then, for every bounded complex sequence \( x \),

(i) if \( \lim_{n} \frac{n}{\mu(n)} = d \neq 0 \), then \( \text{st} \) \( \text{core} \{x\} \subseteq \text{st} \text{core} \{C_{\mu}x\} \);

(ii) if \( \lim_{n} \frac{n}{\mu(n)} = d \neq 0 \), then \( \text{st} \) \( \text{core} \{C_{\mu}x\} \subseteq \text{st} \text{core} \{C_{1}x\} \).

3. Consistency of \( A \)-statistical convergence

In this section we consider the concept of \( A \)-statistical convergence and recall definitions of inclusion and consistency in the statistical convergence sense as introduced by Fridy and Khan [8]. Also we get necessary conditions on the matrices \( A \) and \( B \) so that \( A \) and \( B \) are equivalent in the statistical convergence sense and \( \Gamma_{A}(x) = \Gamma_{B}(x) \) for a real number sequence \( x \) where \( A \) and \( B \) are nonnegative regular summability matrices.

We begin by giving two definitions.

**Definition 3.1.** If \( \text{st} A \supset \text{st} B \), \( A \) is said to be stronger than \( B \) in the statistical convergence sense.

**Definition 3.2.** Matrices \( A \) and \( B \) are called consistent in the statistical convergence sense if \( \text{st} A = \text{st} B \) whenever \( x \in \text{st} A \cap \text{st} B \). If \( A \) is stronger than \( B \) in the statistical convergence sense and consistent with \( B \) in the statistical convergence sense we then write \( A \supset B \) [8]. If \( A \supset B \) and \( B \supset A \), \( A \) and \( B \) are called equivalent in the statistical convergence sense (denoted by \( A \cong B \)).

Throughout this section \( A = (a_{nk}) \) and \( B = (b_{nk}) \) will denote nonnegative regular summability matrices.

**Theorem 3.3.** If the condition

\[
\limsup_{n} \sum_{k=1}^{\infty} |a_{nk} - b_{nk}| = 0 \tag{*}
\]

holds, then \( \delta_{A}(K) = 0 \) if and only if \( \delta_{B}(K) = 0 \) for every \( K \subseteq \mathbb{N} \).

**Proof.** (Necessity). If \( \delta_{A}(K) = 0 \), then \( \lim_{n} \sum_{k \in K} a_{nk} = 0 \). Since

\[
|A_{K}x - B_{K}x| = \sum_{k \in K} |a_{nk} - b_{nk}| \leq \sum_{k=1}^{\infty} |a_{nk} - b_{nk}|,
\]

we have \( \limsup_{n} |A_{K}x - B_{K}x| = 0 \) by \( (*) \), which implies \( \delta_{B}(K) = \lim_{n} \sum_{k \in K} b_{nk} = 0 \).

Sufficiency follows from the symmetry. \( \square \)

Hence we can get the following results from Theorem 3.3.
Theorem 3.4. If $A$ and $B$ satisfy the condition $(*)$, then

(i) $st_A = st_B$
(ii) $\Gamma_A(x) = \Gamma_B(x)$

for a real number sequence $x$.

The statistical limits in (i) of Theorem 3.4 agree (i.e., $st_B - \lim x = L$ implies $st_A - \lim x = L$). Therefore, if $A$ and $B$ satisfy condition $(*)$ of Theorem 3.3, then $A$ and $B$ are consistent in the statistical convergence sense.

Note that the support sets generated by nonnegative summability methods $A$ and $B$ can be used to determine when, if a sequence $x$ is both $A$- and $B$-statistically convergent, the $A$-statistical and $B$-statistical limits of $x$ agree. In [2] Connor and Kline, using the “$\beta N$ program” have shown that $A$ and $B$ assign the same statistical limit to $x$ if $K_A \cap K_B \neq \phi$ where the sets $K_A$ and $K_B$ are the support sets of the nonnegative regular summability matrices $A$ and $B$.

The next corollary shows that we have the same result under different conditions.

Corollary 3.5. If $A$ and $B$ satisfy the conditions $(*)$ of Theorem 3.3, then $A \preceq B$.

Recall that $A$-statistical boundedness implies that $st_A - \lim sup$ and $st_A - \lim inf$ are finite and $st_A - \lim sup x$ and $st_A - \lim inf x$ are the greatest and least $A$-statistical cluster points of such an $x$ [3]. Also

$$st_A - \text{core}\{x\} = [st_A - \lim inf x, st_A - \lim sup x]$$

for any $A$-statistically bounded real number sequence $x$ [3]. Hence we can get the following from (ii) of Theorem 3.4.

Corollary 3.6. If $A$ and $B$ satisfy the condition $(*)$, then $st_A - \text{core}\{x\} = st_B - \text{core}\{x\}$ for every bounded real sequence $x$.

Note that the converse of Corollary 3.6 does not hold. This is seen by the following example.

Example 3.7. Consider the matrices $A = (a_{nk})$ and $B = (b_{nk})$ defined by

$$a_{nk} = \begin{cases} \frac{n}{3(n+1)}, & k = n^2 \\ 1 - \frac{n}{3(n+1)}, & k = n^2 + 1 \\ 0, & \text{otherwise;} \end{cases}$$
and
\[
b_{nk} = \begin{cases} 
  \frac{n}{5(n+1)}, & k = n^2 \\
  1 - \frac{n}{5(n+1)}, & k = n^2 + 1 \\
  0, & \text{otherwise.}
\end{cases}
\]

It is clear that $A$ and $B$ are nonnegative regular matrix summability methods.

Let us define the sequence $x = (x_k)$ by
\[
x_k = \begin{cases} 
  1, & k = n^2 \\
  0, & \text{otherwise.}
\end{cases}
\]

If we write $E_1 := \{k = n^2 : n = 1, 2, \ldots\}$ and $E_2 := \{k \neq n^2 : n = 1, 2, \ldots\}$, then we have $\delta_A(E_1) = \frac{1}{5}$, $\delta_A(E_2) = \frac{2}{3}$, $\delta_B(E_1) = \frac{1}{5}$, $\delta_B(E_2) = \frac{4}{5}$. Thus $\Gamma_A(x) = \Gamma_B(x) = \{0, 1\}$. Also, $st_A - \text{core}\{x\} = st_B - \text{core}\{x\} = [0, 1]$.

Observe that
\[
\limsup_n \sum_{k=1}^{\infty} |a_{nk} - b_{nk}| = \frac{4}{15}.
\]

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