GLASNIK MATEMATIČKI
Vol. 37(57)(2002), 303 – 319

ON APPROXIMATION BY MODIFIED SZASZ-MIRAKYAN OPERATORS

Zbigniew Walczak
Poznań University of Technology, Poland

Abstract. We consider certain modified Szasz-Mirakyan operators $S_n(f;a_n,b_n)$ and $T_n(f;a_n,b_n)$ in spaces C_p and L of continuous and Lebesgue integrable functions, respectively. We study approximation properties of these operators.

1. Introduction

1.1. Let S_n be the Szasz-Mirakyan operator and let T_n be the Szasz-Mirakyan-Kantorovitch operator, i. e.

$$S_n(f;x) := \sum_{k=0}^{\infty} \varphi_k(nx)\, f\left(\frac{k}{n}\right), \tag{1.1}$$

$$T_n(f;x) := n\sum_{k=0}^{\infty} \varphi_k(nx) \int_{k/n}^{(k+1)/n} f(t)dt, \tag{1.2}$$

$x \in R_0 := [0,+\infty)$, $n \in N$, where

$$\varphi_k(t) := e^{-t}\frac{t^k}{k!} \qquad \text{for} \qquad t \in R_0, \quad k \in N_0 = N \cup \{0\}. \tag{1.3}$$

Approximation properties of S_n were examined in [1] for functions $f \in C_p$, $p \in N_0$, where C_p is a polynomial weighted space with the weight function w_p,

$$w_0(x) := 1, \quad w_p(x) := (1+x^p)^{-1} \qquad \text{if} \qquad p \geq 1, \tag{1.4}$$

2000 *Mathematics Subject Classification.* 41A36.
Key words and phrases. Szasz-Mirakyan operator, Szasz-Mirakyan-Kantorovitch operator, degree of approximation.

and C_p is the set of all real-valued functions f for which fw_p is uniformly continuous and bounded on R_0 and the norm is defined by

$$\|f\|_p \equiv \|f(\cdot)\|_p := \sup_{x\in R_0} w_p(x)\,|f(x)|. \tag{1.5}$$

In papers [2] and [3] the approximation properties of operators T_n for functions $f \in L$ were examined, where $L \equiv L\,(R_0)$ is the space of all real-valued functions f Lebesgue integrable on R_0 and the norm

$$\|f\|_L := \int_0^{+\infty} |f(t)|dt. \tag{1.6}$$

1.2. In this paper we modify operators S_n and T_n given by (1.1) and (1.2), i. e. we consider operators

$$S_n(f; a_n, b_n; x) := \sum_{k=0}^{\infty} \varphi_k(a_n x)\, f\left(\frac{k}{b_n}\right), \qquad x \in R_0, \quad n \in N, \tag{1.7}$$

for $f \in C_p$, $p \in N_0$, and

$$T_n(f; a_n, b_n; x) := b_n \sum_{k=0}^{\infty} \varphi_k(a_n x) \int_{k/b_n}^{(k+1)/b_n} f(t)dt, \quad x \in R_0,\ n \in N, \tag{1.8}$$

for $f \in L$, where $(a_n)_1^\infty$ and $(b_n)_1^\infty$ are given increasing and unbounded numerical sequences such that $a_n \geq 1$, $b_n \geq 1$ and $(a_n/b_n)_1^\infty$ is non-decreasing and

$$\frac{a_n}{b_n} = 1 + o\left(\frac{1}{b_n}\right). \tag{1.9}$$

If $a_n = b_n = n$ for all $n \in N$, then we have operators (1.1) and (1.2).

Operator s S_n defined by (1.7) have some application to differential equations, similarly as operators Szasz-Mirakyan (1.1).

In our paper we shall study approximation properties of operators (1.7) and (1.8). In Section 2 we shall examine operators $S_n(f; a_n, b_n)$ for $f \in C_p$ and in Section 3 operators $T_n(f; a_n, b_n)$ for $f \in L$. We shall prove approximation theorems which are similar to some results given in [1, 2, 3] for operators (1.1) and (1.2).

2. Operators $S_n(f; a_n, b_n)$

2.1. First we shall give some auxiliary results. From (1.7) and (1.3) we derive the following formulas

$$S_n(1; a_n, b_n; x) = 1, \tag{2.1}$$

$$S_n(t - x; a_n, b_n; x) = \left(\frac{a_n}{b_n} - 1\right) x, \tag{2.2}$$

$$S_n((t - x)^2; a_n, b_n; x) = \left(\frac{a_n}{b_n} - 1\right)^2 x^2 + \frac{a_n x}{b_n^2}. \tag{2.3}$$

for $x \in R_0$ and $n \in N$, which by (1.9) imply

$$\lim_{n\to\infty} b_n S_n(t-x; a_n, b_n; x) = 0,$$
$$\lim_{n\to\infty} b_n S_n((t-x)^2; a_n, b_n; x) = x, \tag{2.4}$$

for every $x \in R_0$.

By elementary calculations we can prove also the following

Lemma 2.1. *For every $x \in R_0$*

$$\lim_{n\to\infty} b_n^2 S_n((t-x)^4; a_n, b_n; x) = 3x^2.$$

Using the mathematical induction we can prove

Lemma 2.2. *Let $r \in N$ be fixed number. Then there exist positive numerical coefficients $\lambda_{r,j}$, $1 \le j \le r$, depending only on r and j such that*

$$S_n(t^r; a_n, b_n; x) = \frac{1}{b_n^r}\sum_{j=1}^{r} \lambda_{r,j}(a_n x)^j,$$

for all $x \in R_0$ and $n \in N$. Moreover we have $\lambda_{r,1} = 1 = \lambda_{r,r}$.

Applying Lemma 2.2, we shall prove two lemmas.

Lemma 2.3. *For given $p \in N_0$ and $(a_n)_1^\infty$ and $(b_n)_1^\infty$ there exists a positive constant $M_1(b_1, p)$ such that*

$$\left\| S_n\left(1/w_p(t); a_n, b_n; \cdot\right)\right\|_p \le M_1(b_1, p), \qquad n \in N. \tag{2.5}$$

Moreover for every $f \in C_p$

$$\left\| S_n\left(f; a_n, b_n; \cdot\right)\right\|_p \le M_1(b_1, p)\left\|f\right\|_p, \qquad n \in N. \tag{2.6}$$

The formulas (1.7) and (1.3) and the inequality (2.6) show that S_n, $n \in N$, is a positive linear operator from the space C_p into C_p, $p \in N_0$.

Proof. First we shall prove (2.5).

If $p = 0$, then (2.5) follows by (1.4), (1.5) and (2.1). If $p \ge 1$, then by (1.4), (1.7), (1.9) and Lemma 2.2 we get

$$\begin{aligned} w_p(x) S_n\left(1/w_p(t); a_n, b_n; x\right) &= w_p(x)\left\{1 + S_n\left(t^p; a_n, b_n; x\right)\right\} \\ &= \frac{1}{1+x^p} + \sum_{j=1}^{p} \lambda_{p,j}\frac{1}{b_n^{p-j}}\left(\frac{a_n}{b_n}\right)^j \frac{x^j}{1+x^p} \\ &\le 1 + \sum_{j=1}^{p}\lambda_{p,j}\frac{1}{b_1^{p-j}} = M_1(b_1, p), \end{aligned}$$

for all $x \in R_0$ and $n \in N$. From this follows (2.5).

By (1.7) and (1.5) we have

$$\left\| S_n\left(f; a_n, b_n; \cdot\right)\right\|_p \le \left\|f\right\|_p \left\| S_n\left(1/w_p(t); a_n, b_n; \cdot\right)\right\|_p,$$

for every $f \in C_p$, $p \in N_0$ and $n \in N$, which by (2.5) yields (2.6). $\square$

LEMMA 2.4. *For every $p \in N_0$ there exists a positive constant $M_2(b_1, p)$ such that*

$$w_p(x)S_n\left(\frac{(t-x)^2}{w_p(t)}; a_n, b_n; x\right) \le M_2(b_1,p)\left[\left(\frac{a_n}{b_n}-1\right)^2 x^2 + \frac{x}{b_n}\right] \tag{2.7}$$

for all $x \in R_0$ and $n \in N$.

PROOF. If $p = 0$, then (2.7) follows by (2.3).

Let $S_n(f;x) \equiv S_n(f; a_n, b_n; x)$. By (1.4) and (1.7) we have

$$S_n\left((t-x)^2/w_p(t); x\right) = S_n\left((t-x)^2; x\right) + S_n\left(t^p(t-x)^2; x\right), \tag{2.8}$$

for $p \ge 1$ and $n \in N$. Since

$$\begin{aligned} S_n\left((t-x)^3; x\right) &= \left(\frac{a_n}{b_n}-1\right)^3 x^3 + \left(\frac{a_n}{b_n}-1\right)\frac{3a_n x^2}{b_n^2} + \frac{a_n x}{b_n^3}, \\ S_n\left(t(t-x)^2; x\right) &= S_n\left((t-x)^3; x\right) + xS_n\left((t-x)^2; x\right), \end{aligned}$$

we immediately obtain (2.7) for $p = 1$ by (2.3) and (2.8).

If $p \ge 2$, then by Lemma 2.2 we get

$$\begin{aligned} &w_p(x)S_n\left(t^p(t-x)^2; x\right) = \\ &\quad = w_p(x)\left\{S_n\left(t^{p+2}; x\right) - 2xS_n\left(t^{p+1}; x\right) + x^2 S_n\left(t^p; x\right)\right\} \\ &\quad = w_p(x)\left\{\frac{1}{b_n^{p+2}}\sum_{j=1}^{p+2}\lambda_{p+2,j}(a_n x)^j - \frac{2x}{b_n^{p+1}}\sum_{j=1}^{p+1}\lambda_{p+1,j}(a_n x)^j + \right. \\ &\qquad \left. + \frac{x^2}{b_n^p}\sum_{j=1}^{p}\lambda_{p,j}(a_n x)^j\right\} = \frac{x^{p+2}}{1+x^p}\left(\frac{a_n}{b_n}\right)^p\left(\frac{a_n}{b_n}-1\right)^2 + \\ &\qquad + \frac{x}{b_n}\left\{\frac{1}{b_n^{p+1}}\sum_{j=1}^{p+1}\lambda_{p+2,j}a_n^j\frac{x^{j-1}}{1+x^p} - \frac{2}{b_n^p}\sum_{j=1}^{p}\lambda_{p+1,j}a_n^j\frac{x^j}{1+x^p} + \right. \\ &\qquad \left. + \frac{1}{b_n^{p-1}}\sum_{j=1}^{p-1}\lambda_{p,j}a_n^j\frac{x^{j+1}}{1+x^p}\right\}, \end{aligned}$$

and by $0 < \frac{a_n}{b_n} \leq 1$ for $n \in N$ it follows that

$$w_p(x) S_n\left(t^p(t-x)^2; x\right) \leq \left(\frac{a_n}{b_n} - 1\right)^2 x^2 + \frac{x}{b_n}\left\{\sum_{j=1}^{p+1} \lambda_{p+2,j} \frac{1}{b_1^{p+1-j}} + \right.$$

$$\left. + 2\sum_{j=1}^{p} \lambda_{p+1,j} \frac{1}{b_1^{p-j}} + \sum_{j=1}^{p-1} \lambda_{p,j} \frac{1}{b_1^{p-1-j}}\right\}$$

$$\leq M_2(b_1, p)\left\{\left(\frac{a_n}{b_n} - 1\right)^2 x^2 + \frac{x}{b_n}\right\}$$

for $x \in R_0$, $n \in N$. From this and by (2.8) and (2.3) and (1.4) we obtain (2.7) for $p \geq 2$. Thus the proof is completed. $\square$

2.2. Now we shall prove three approximation theorems for $S_n(f; a_n, b_n)$, using the modulus of continuity $\omega_1(f; C_p)$ and the modulus of smoothness $\omega_2(f; C_p)$ of function $f \in C_p$, $p \in N_0$, i. e.

$$\omega_1(f; C_p; t) := \sup_{0 \leq h \leq t} \|\Delta_h f(\cdot)\|_p, \qquad \omega_2(f; C_p; t) := \sup_{0 \leq h \leq t} \|\Delta_h^2 f(\cdot)\|_p,$$

for $t \geq 0$, where

$$\Delta_h f(x) := f(x+h) - f(x), \qquad \Delta_h^2 f(x) := f(x) - 2f(x+h) + f(x+2h).$$

Let

$$\Psi_n(x) := \left(\frac{a_n}{b_n} - 1\right)^2 x^2 + \frac{x}{b_n}, \qquad x \in R_0, \quad n \in N. \tag{2.9}$$

THEOREM 2.5. *Suppose that $f \in C_p^2$ with a fixed $p \in N_0$. Then there exists a positive constant $M_3(b_1, p)$ such that*

(2.10)

$$w_p(x) \left|S_n(f; a_n, b_n; x) - f(x)\right| \leq \|f'\|_p \left|\frac{a_n}{b_n} - 1\right| x + M_3(b_1, p)\|f''\|_p \Psi_n(x)$$

for all $x \in R_0$ and $n \in N$.

PROOF. From (1.7) we get

$$S_n(f; a_n, b_n; 0) = f(0), \qquad n \in N, \tag{2.11}$$

which implies (2.10) for $x = 0$. Let $x > 0$ and let $S_n(f; x) \equiv S_n(f; a_n, b_n; x)$. For $f \in C_p^2$ and $t \in R_0$ we have

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t \int_x^s f''(u) du ds$$

and consequently

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t \int_u^t f''(u) ds du$$
$$= f(x) + f'(x)(t-x) + \int_x^t (t-u) f''(u) du.$$

From this and by (2.1) we get

$$S_n(f(t);x) = f(x) + f'(x) S_n(t-x;x) + S_n\left(\int_x^t (t-u) f''(u) du; x\right),$$

for $n \in N$. But by (1.4) and (1.5) we have

$$\left|\int_x^t (t-u) f''(u) du\right| \le \|f''\|_p \left(\frac{1}{w_p(t)} + \frac{1}{w_p(x)}\right)(t-x)^2.$$

From the above and by (2.2), (2.3), (2.7) and (2.9) we obtain

$$\begin{aligned} w_p(x)|S_n(f(t);x) - f(x)| \le \\ &\le \|f'\|_p \, |S_n(t-x;x)| + \\ &\quad + \|f''\|_p \left\{ w_p(x) S_n\left(\frac{(t-x)^2}{w_p(t)}; x\right) + S_n\left((t-x)^2; x\right)\right\} \\ &\le \|f'\|_p \left|\frac{a_n}{b_n} - 1\right| x + M_3(b_1,p) \|f''\|_p \Psi_n(x), \qquad n \in N. \end{aligned}$$

Thus the proof is completed. ∎

From Theorem 2.5 we derive the following

COROLLARY 2.6. *Let $\rho(x) = \left(1+x^2\right)^{-1}$, $x \in R_0$. If assumptions of Theorem 2.5 are satisfied then there exists a positive constant $M_4(b_1,p)$ such that*

$$\|[S_n(f;a_n,b_n) - f]\,\rho\|_p \le \left(1 - \frac{a_n}{b_n}\right) \|f'\|_p + M_4(b_1,p) \|f''\|_p b_n^{-1}, \quad n \in N.$$

THEOREM 2.7. *Suppose that $f \in C_p$ with a fixed $p \in N_0$. Then there exists a positive constant $M_5(b_1,p)$ such that*

$$\begin{aligned} w_p(x)|S_n(f;a_n,b_n;x) - f(x)| \le \\ &\le \left|\frac{a_n}{b_n} - 1\right| x \left(\Psi_n(x)\right)^{-1/2} \omega_1\left(f; C_p; \sqrt{\Psi_n(x)}\right) + \\ &\quad + M_5(b_1,p) \omega_2\left(f; C_p; \sqrt{\Psi_n(x)}\right), \end{aligned} \tag{2.12}$$

for all $x > 0$ and $n \in N$, where $\Psi_n(\cdot)$ is defined by (2.9). For $x = 0$ follows (2.11).

PROOF. Similarly as in [1] we shall apply the Stieklov function f_h for $f \in C_p$:

$$f_h(x) := \frac{4}{h^2} \int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} [f(x+s+t) - f(x+2(s+t))] ds dt,$$

$x \in R_0$, $h > 0$, for which we have

$$f_h'(x) = \frac{1}{h^2} \int_0^{\frac{h}{2}} \left[8\Delta_{h/2} f(x+s) - 2\Delta_h f(x+2s) \right] ds,$$

$$f_h''(x) = \frac{1}{h^2} \left[8\Delta_{h/2}^2 f(x) - \Delta_h^2 f(x) \right].$$

Hence, for $h > 0$, we have

$$\|f_h - f\|_p \leq \omega_2 \left(f, C_p; h, \right), \tag{2.13}$$

$$\|f_h'\|_p \leq 5h^{-1} \omega_1 \left(f, C_p; h \right) \frac{w_p(x)}{w_p(x+h)}, \tag{2.14}$$

$$\|f_h''\|_p \leq 9h^{-2} \omega_2 \left(f, C_p; h \right), \tag{2.15}$$

which show that $f_h \in C_p^2$ if $f \in C_p$. By denoting $S_n(f; a_n, b_n; x)$ as $S_n(f; x)$, we can write

$$\begin{aligned} w_p(x) \left| S_n(f;x) - f(x) \right| \leq w_p(x) \{ &\left| S_n \left(f - f_h; x \right) \right| + \left| S_n \left(f_h; x \right) - f_h(x) \right| \\ &+ \left| f_h(x) - f(x) \right| \} := A_1 + A_2 + A_3, \end{aligned}$$

for $x > 0$, $h > 0$ and $n \in N$. By (2.6) and (2.13) we have

$$A_1 \leq M_1(b_1, p) \left\| f - f_h \right\|_p \leq M_1(b_1, p) \omega_2 \left(f, C_p; h \right),$$

$$A_3 \leq \omega_2 \left(f, C_p; h \right).$$

Applying Theorem 2.5 and (2.14) and (2.15), we get

$$\begin{aligned} A_2 &\leq \|f_h'\|_p \left| \frac{a_n}{b_n} - 1 \right| x + M_3(b_1, p) \|f_h''\|_p \Psi_n(x) \\ &\leq \frac{w_p(x)}{w_p(x+h)} \left| \frac{a_n}{b_n} - 1 \right| \frac{5x}{h} \omega_1(f; C_p; h) + 9M_3(b_1, p) h^{-2} \Psi_n(x) \omega_2(f; C_p; h). \end{aligned}$$

Combining these and setting $h = \sqrt{\Psi_n(x)}$, for fixed $x > 0$ and $n \in N$, we obtain the desired estimate (2.12). $\square$

Analogously we obtain the following

THEOREM 2.8. *Let $f \in C_p$, $p \in N_0$, and let $\rho(x) = \left(1 + x^2\right)^{-1}$ for $x \in R_0$. Then there exists a positive constant $M_6(b_1, p)$ such that*

$$\begin{aligned} \left\| \left[S_n(f; a_n, b_n) - f \right] \rho \right\|_p \leq &\left(1 - \frac{a_n}{b_n} \right) \sqrt{b_n} \omega_1 \left(f; C_p; 1/\sqrt{b_n} \right) + \\ &+ M_6(b_1, p) \omega_2 \left(f; C_p; 1/\sqrt{b_n} \right), \quad n \in N. \end{aligned}$$

From Theorems 2.7 and 2.8 we derive

Corollary 2.9. *Let $f \in C_p$, $p \in N_0$. Then for S_n defined by (1.7) we have*

$$\lim_{n\to\infty} S_n(f;a_n,b_n;x) = f(x), \quad x \in R_0. \tag{2.16}$$

The convergence (2.16) is uniform on every interval $[x_1,x_2]$, $x_2 > x_1 \geq 0$.

2.3. Now we shall prove the analogy of (2.16) for the first order derivative. Moreover we shall give other properties of derivatives of $S_n(f;a_n,b_n)$.

Theorem 2.10. *Let $f \in C_p$, $p \in N_0$, and let $x_0 > 0$ be a point for which there exists finite derivative $f'(x_0)$. Then*

$$\lim_{n\to\infty} \left(S_n(f;a_n,b_n)\right)'(x_0) = f'(x_0). \tag{2.17}$$

Proof. By assumptions we can write

$$f(t) = f(x_0) + f'(x_0)(t-x_0) + \varepsilon_1(t,x_0)(t-x_0), \tag{2.18}$$

for $t \in R_0$, where

$$\varepsilon_1(t) \equiv \varepsilon_1(t;x_0) = \begin{cases} \frac{f(t)-f(x_0)}{t-x_0} - f'(x_0) & \text{if } t \neq x_0, \\ 0 & \text{if } t = x_0, \end{cases}$$

is continuous function at x_0 and $\varepsilon_1 \in C_p$. From (1.7) we get for $S_n(f;x) \equiv S_n(f;a_n,b_n;x)$

$$\left(S_n(f(t))\right)'(x) = (b_n-a_n)S_n(f(t);x) + \frac{b_n}{x}S_n((t-x)f(t);x), \tag{2.19}$$

for $x > 0$ and $n \in N$. By (2.18), (2.19) and (2.1) and by elementary calculations we obtain

$$\begin{aligned} &(S_n(f(t)))'(x_0) = \\ &= f(x_0)\left\{b_n - a_n + \frac{b_n}{x_0}S_n(t-x_0;x_0)\right\} + \\ &\quad + f'(x_0)\left\{(b_n-a_n)S_n(t-x_0;x_0) + \frac{b_n}{x_0}S_n\left((t-x_0)^2;x_0\right)\right\} + \\ &\quad + (b_n-a_n)S_n\left(\varepsilon_1(t)(t-x_0;x_0\right) + \frac{b_n}{x_0}S_n\left(\varepsilon_1(t)(t-x_0)^2;x_0\right). \end{aligned} \tag{2.20}$$

Applying Corollary 2.9 and properties of ε_1, we get

$$\lim_{n\to\infty} S_n\left(\varepsilon_1(t)(t-x_0);x_0\right) = 0, \tag{2.21}$$

$$\lim_{n\to\infty} S_n\left(\varepsilon_1^2(t);x_0\right) = \varepsilon_1^2(x_0) = 0. \tag{2.22}$$

By the Hölder inequality we have

$$\left|S_n\left(\varepsilon_1(t)(t-x_0)^2;x_0\right)\right| \leq \left\{S_n\left(\varepsilon_1^2(t);x_0\right)\right\}^{\frac{1}{2}}\left\{S_n\left((t-x_0)^4;x_0\right)\right\}^{\frac{1}{2}},$$

for $n \in N$ which by (2.22) and Lemma 2.1 implies

$$\lim_{n\to\infty} b_n S_n\left(\varepsilon_1(t)(t-x_0)^2; x_0\right) = 0. \tag{2.23}$$

Using (2.2) - (2.4), (1.9), (2.21) and (2.22) to (2.20), we immediately obtain (2.17). ∎

THEOREM 2.11. *Suppose that $f \in C_p$, $p \in N_0$. Then for every $r \in N$ there exists the r-th derivative of $S_n(f; a_n, b_n)$ on R_0. Moreover $(S_n(f; a_n, b_n))^{(r)} \in C_p$ and*

$$\left\|(S_n(f; a_n, b_n))^{(r)}\right\|_p \le M_1(b_1, p) a_n^r \left\|\Delta^r_{1/b_n} f(\cdot)\right\|_p \qquad \textit{for} \quad r, n \in N, \tag{2.24}$$

where $M_1(b_1, p)$ is a positive constant given in Lemma 2.3 and

$$\Delta_h^r f(x) := \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} f(x+kh). \tag{2.25}$$

PROOF. Let $S_n(f; x) \equiv S_n(f; a_n, b_n; x)$. From (1.7) we get

$$\begin{aligned}\left(S_n(f(t))\right)'(x) &= -a_n S_n(f(t); x) + a_n S_n(f(t+1/b_n); x)\\ &= a_n S_n(\Delta_{1/b_n} f(t); x), \qquad x \in R_0, \quad n \in N.\end{aligned}$$

Hence, for every $r \in N$, we obtain the formula

$$\left(S_n(f(t))\right)^{(r)}(x) = a_n^r S_n(\Delta^r_{1/b_n} f(t); x), \qquad \text{for} \quad x \in R_0, \quad n \in N, \tag{2.26}$$

where $\Delta_h^r f(x) := \Delta_h\left(\Delta_h^{r-1} f(x)\right)$ for $r \ge 2$ and from this follows (2.25). Applying Lemma 2.3, we derive the inequality (2.24) from (2.26). ∎

Theorem 2.11 and Lemma 2.3 imply

COROLLARY 2.12. *S_n, $n \in N$, defined by (1.7) is positive linear operator from the space C_p, $p \in N_0$, into C_p^∞.*

THEOREM 2.13. *Suppose that $f \in C_p$, $p \in N_0$. Then:*

(i) *if f is increasing (decreasing) on R_0, then the function $S_n(f; a_n, b_n; \cdot)$, $n \in N$, is also increasing (decreasing) on R_0;*
(ii) *if f is convex (concave) on R_0, then $S_n(f; a_n, b_n; \cdot)$, $n \in N$, is also convex (concave) on R_0.*

PROOF. Properties (i) and (ii) we derive from (2.26) with $r = 1, 2$ and by Corollary 2.12 and classical theorems of mathematical analysis. ∎

3. Operators $T_n(f;a_n,b_n)$

It is obvious that operators $T_n(f;a_n,b_n)$, $n\in N$, defined by (1.8) we can consider for functions $f\in C_p$, $p\in N_0$. For these operators and $f\in C_p$ we can prove lemmas and theorems similar to Theorems 2.5-2.13.

In this section we shall study properties of $T_n(f;a_n,b_n)$ for functions $f\in L(R_0)$. We shall give theorem on point-convergence of the sequence $(T_n(f;a_n,b_n))_1^\infty$ and theorem on the degree of approximation of $f\in L(R_0)$ by these operators.

Below we shall denote by $\sum_{k=A}^{B} x_k$ the sum of all x_k with $k\in N_0$ and $A\le k\le B$.

3.1. First we shall give some auxiliary results. From (1.8) we get

$$T_n(1;a_n,b_n;x)=1, \qquad x\in R_0, \quad n\in N. \tag{3.1}$$

Lemma 3.1. *$T_n(f;a_n,b_n)$, $n\in N$, defined by (1.8) is positive linear operator from the space $L(R_0)$ into $L(R_0)$ and*

$$\|T_n(f;a_n,b_n)\|_L\le \frac{b_1}{a_1}\|f\|_L \qquad for \quad n\in N. \tag{3.2}$$

Proof. By (1.8) it is obvious that T_n, $n\in N$, is positive linear operator well-defined for $f\in L(R_0)$ and

$$|T_n(f;a_n,b_n;x)|\le \|f\|_L\, b_n\sum_{k=0}^{\infty}\varphi_k(a_nx)=b_n\|f\|_L \qquad \text{for} \quad x\in R_0, n\in N.$$

Moreover by (1.8) and (1.6) we get

$$\begin{aligned}\|T_n(f;a_n,b_n)\|_L&=\int_0^{+\infty}\left|b_n\sum_{k=0}^{\infty}\varphi_k(a_nx)\int_{k/b_n}^{(k+1)/b_n}f(t)dt\right|dx\\&\le b_n\sum_{k=0}^{\infty}\int_{k/b_n}^{(k+1)/b_n}|f(t)|\,dt\int_0^{+\infty}\varphi_k(a_nx)dx.\end{aligned}$$

But

$$\int_0^{+\infty}\varphi_k(a_nx)dx=\frac{1}{a_n} \qquad \text{for} \quad k\in N_0, \quad n\in N. \tag{3.3}$$

Hence for $n\in N$ we have

$$\|T_n(f;a_n,b_n)\|_L\le\frac{b_n}{a_n}\sum_{k=0}^{\infty}\int_{k/b_n}^{(k+1)/b_n}|f(t)|\,dt=\frac{b_n}{a_n}\|f\|_L\,.$$

Since the sequence $(b_n/a_n)_1^\infty$ is non-increasing and convergent to 1, we obtain the inequality (3.2). $\square$

LEMMA 3.2. *Let $f \in L(R_0)$ and let*

$$F(x) := \int_0^x f(t)dt, \qquad x \in R_0. \tag{3.4}$$

Then F belongs to the space C_p with $p = 0$ and for every $n \in N$ there exists the operator $S_n(F; a_n, b_n; \cdot)$ defined by (1.7). Moreover,

$$\left(S_n(F; a_n, b_n)\right)'(x) = \frac{a_n}{b_n} T_n(f; a_n, b_n; x) \tag{3.5}$$

for all $x \in R_0$ and $n \in N$.

PROOF. It is know that if $f \in L(R_0)$, then F is a function continuous on R_0 and

$$|F(x)| \le \|f\|_L \qquad \text{for} \quad x \in R_0,$$

which shows that $F \in C_0$. Hence by Lemma 2.3 there exists $S_n(F; a_n, b_n; \cdot)$, $n \in N$, and by (2.26) and (1.8) we get

$$\frac{d}{dx} S_n(F; a_n, b_n; x) = a_n S_n\left(\Delta_{1/b_n} F; a_n, b_n; x\right) = \frac{a_n}{b_n} T_n(f; a_n, b_n; x),$$

for $x \in R_0$, $n \in N$. ◻

LEMMA 3.3. *Let $s \ge 0$ be a fixed number and let*

$$y_s(t) := \begin{cases} 1 & \text{if } t \le s, \\ 0 & \text{if } t > s. \end{cases} \tag{3.6}$$

Then

$$\int_0^{+\infty} \left|T_n\left(y_s(t); a_n, b_n; x\right) - y_s(x)\right| dx \le \left(\frac{b_n}{a_n} - 1\right) s + \sqrt{\frac{2b_1}{\pi a_1}} \sqrt{\frac{s}{b_n}} \tag{3.7}$$

for $n \in N$.

PROOF. Let $T_n(f; x) \equiv T_n(f; a_n, b_n; x)$. From (1.8) and (3.6) we get

$$T_n\left(y_s(t); x\right) = \sum_{k=0}^{sb_n - 1} \varphi_k(a_n x) \qquad \text{for} \quad x \in R_0, \quad n \in N.$$

Hence

$$\int_0^{+\infty} \left|T_n\left(y_s(t); x\right) - y_s(x)\right| dx =$$
$$= \left(\int_0^s + \int_s^{+\infty}\right) \left|\sum_{k=0}^{sb_n - 1} \varphi_k(a_n x) - y_s(x)\right| dx := I_1 + I_2.$$

Similarly as in [3] and by (3.3) we get for I_1 and I_2:

$$I_1 = \int_0^s \left| \sum_{k=0}^{sb_n-1} \varphi_k(a_n x) - 1 \right| dx = \int_0^s \left(\sum_{k=sb_n}^{\infty} \varphi_k(a_n x) \right) dx,$$

$$\begin{aligned} I_2 &= \int_s^{+\infty} \left(\sum_{k=0}^{sb_n-1} \varphi_k(a_n x) \right) dx \\ &= \sum_{k=0}^{sb_n-1} \left\{ \int_0^{+\infty} \varphi_k(a_n x) dx - \int_0^s \varphi_k(a_n x) dx \right\} \\ &= \sum_{k=0}^{sb_n-1} \frac{1}{a_n} - \int_0^s \left(\sum_{k=0}^{sb_n-1} \varphi_k(a_n x) \right) dx \\ &= \left(\frac{b_n}{a_n} - 1 \right) s + \int_0^s \left(\sum_{k=sb_n}^{\infty} \varphi_k(a_n x) \right) dx. \end{aligned}$$

Consequently

$$\begin{aligned} \int_0^{+\infty} \left| T_n\left(y_s(t); x\right) - y_s(x) \right| dx &\le \left(\frac{b_n}{a_n} - 1 \right) s + 2 \int_0^s \left(\sum_{k=sb_n}^{\infty} \varphi_k(a_n x) \right) dx \\ &\le \left(\frac{b_n}{a_n} - 1 \right) s + 2 \int_0^s \left(\sum_{k=sa_n}^{\infty} \varphi_k(a_n x) \right) dx. \end{aligned}$$

Similarly as in [3] we can assume without loss of generality that sa_n is an integer. Denoting by I_3 the last integral, we get

$$\begin{aligned} I_3 &:= \sum_{k=sa_n}^{\infty} \frac{a_n^k}{k!} \int_0^s e^{-a_n x} x^k dx = \sum_{k=sa_n}^{\infty} \left\{ \frac{e^{-sa_n}}{-a_n} \sum_{j=0}^{k} \frac{(sa_n)^j}{j!} + \frac{1}{a_n} \right\} \\ &= \frac{e^{-sa_n}}{a_n} \sum_{k=sa_n}^{\infty} \sum_{j=k+1}^{\infty} \frac{(sa_n)^j}{j!} = \frac{e^{-sa_n}}{a_n} \sum_{k=sa_n+1}^{\infty} (k - sa_n) \frac{(sa_n)^k}{k!} \\ &= s e^{-sa_n} \frac{(sa_n)^{sa_n}}{(sa_n)!} \end{aligned}$$

and by the Stirling formula we get

$$I_3 \le \frac{s}{\sqrt{2\pi s a_n}} \le \sqrt{\frac{b_1}{2\pi a_1}} \sqrt{\frac{s}{b_n}}, \qquad n \in N.$$

Combining these, we obtain (3.7). ∎

Arguing similarly as in the proof of Lemma 3.3, we shall prove the main lemma.

Lemma 3.4. *Let $y_x(t)$, $x \in R_0$, be the function defined by (3.6) and let*

$$\rho_1 := \frac{1}{1+x}, \qquad x \in R_0. \tag{3.8}$$

Then there exists a positive constant $M_7(a_1, b_1)$ such that for all $n \in N$

$$\begin{aligned} R_n(t) &:= \int_0^{+\infty} \left| T_n\left(y_x(t); a_n, b_n; x\right) - y_x(t) \right| \rho_1(x) dx \\ &\le M_7(a_1, b_1) \frac{1}{\sqrt{b_n}}. \end{aligned} \tag{3.9}$$

Proof. From (3.6) it follows that

$$y_x(t) + y_t(x) = 1 \qquad \text{for} \quad t, x \in R_0.$$

Hence, for $T_n(f; x) \equiv T_n(f; a_n, b_n; x)$, we have as in [3]

$$T_n(y_x(t); x) - y_x(t) = T_n(1 - y_t(x); x) - 1 = y_t(x) - T_n(y_t(u); x).$$

By Lemma 3.3 and by $\frac{b_n}{a_n} - 1 = o\left(\frac{1}{b_n}\right)$ we get for $t \le 2$

$$R_n(t) := \int_0^{+\infty} \left| T_n\left(y_t(u); x\right) - y_t(x) \right| \rho_1(x) dx \le M_8(a_1, b_1) \frac{1}{\sqrt{b_n}}.$$

If $t > 2$, then similarly as in [3] we can write

$$\begin{aligned} R_n(t) &:= \int_0^t \rho_1(x) \left(\sum_{k=tb_n}^{\infty} \varphi_k(a_n x) \right) dx + \int_t^{+\infty} \rho_1(x) \left(\sum_{k=0}^{tb_n} \varphi_k(a_n x) \right) dx \\ &:= I_1 + I_2. \end{aligned}$$

Analogously as in the proof of Lemma 3.3 we get

$$\begin{aligned} I_2 &\le \frac{1}{1+t} \sum_{k=0}^{tb_n} \frac{a_n^k}{k!} \int_t^{+\infty} e^{-a_n x} x^k dx = \frac{1}{1+t} \sum_{k=0}^{tb_n} \sum_{j=0}^{k} \frac{e^{-ta_n}}{a_n} \frac{(ta_n)^j}{j!} \\ &= \frac{1}{1+t} \frac{e^{-ta_n}}{a_n} \left\{ \sum_{k=0}^{ta_n} \sum_{j=0}^{k} \frac{(ta_n)^j}{j!} + \sum_{k=ta_n+1}^{tb_n} \sum_{j=0}^{k} \frac{(ta_n)^j}{j!} \right\} \\ &\le \frac{1}{1+t} \frac{e^{-ta_n}}{a_n} \left\{ \sum_{j=0}^{ta_n} (ta_n - j + 1) \frac{(ta_n)^j}{j!} + te^{ta_n}(b_n - a_n) \right\} \\ &\le \frac{1}{1+t} \left\{ te^{-ta_n} \frac{(ta_n)^{ta_n}}{(ta_n)!} + \frac{1}{a_n} + \left(\frac{b_n}{a_n} - 1 \right) t \right\} \end{aligned}$$

and by the Stirling formula and properties of a_n and b_n it follows that

$$I_2 \le \frac{1}{\sqrt{2\pi t a_n}} + \frac{b_1}{a_1} \frac{1}{b_n} + \frac{b_1}{a_1} \left(1 - \frac{a_n}{b_n} \right) \le M_9(a_1, b_1) \frac{1}{\sqrt{b_n}}, \qquad n \in N.$$

Arguing as in [3], p. 169, we have for $t > 2$

$$I_1 \leq \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^{t} \right) \rho_1(x) \left(\sum_{k=ta_n}^{\infty} \varphi_k(a_n x) \right) dx := I_{11} + I_{12}.$$

Since the function $\varphi_k(a_n x)$ is increasing for $x \in [0, k/a_n]$ and decreasing for $x \in [k/a_n, \infty)$, we have

$$\begin{aligned} I_{11} &\leq \frac{t}{2} e^{-ta_n/2} \sum_{k=ta_n}^{\infty} \frac{(ta_n/2)^k}{k!} \leq \frac{t}{2} e^{-ta_n/2} \frac{(ta_n/2)^{ta_n}}{(ta_n)!} \sum_{k=0}^{\infty} \frac{1}{2^k} \\ &= t \left(\frac{e}{4} \right)^{ta_n/2} e^{-ta_n} \frac{(ta_n)^{ta_n}}{(ta_n)!} \end{aligned}$$

and similarly as for I_2 we get

$$I_{11} \leq M_{10}(a_1, b_1) \frac{1}{\sqrt{b_n}}, \qquad n \in N.$$

For I_{12} we get

$$I_{12} \leq \frac{1}{1 + t/2} \int_0^t e^{-a_n x} \left(\sum_{k=ta_n}^{\infty} \frac{(a_n x)^k}{k!} \right) dx$$

and applying the estimation obtained for I_3 in the proof of Lemma 3.3, we can write

$$I_{12} \leq \frac{2}{2+t} \sqrt{\frac{b_1}{2\pi a_1}} \sqrt{\frac{t}{b_n}} \leq \sqrt{\frac{2b_1}{\pi a_1}} \frac{1}{\sqrt{b_n}}, \qquad n \in N.$$

Hence, for $t > 2$ and $n \in N$, we obtain also

$$R_n(t) \leq M_{11}(a_1, b_1) \frac{1}{\sqrt{b_n}}.$$

This completes the proof of (3.9). $\square$

3.2. Now we shall prove main theorems on operators T_n defined by (1.8).

THEOREM 3.5. *Suppose that $f \in L(R_0)$ and F is defined by (3.4). Then*

$$\lim_{n \to \infty} T_n(f; a_n, b_n; x) = f(x) \tag{3.10}$$

at every point $x \in R_0$ where

$$F'(x) = f(x). \tag{3.11}$$

Hence (3.10) follows almost everywhere on R_0.

PROOF. By properties of F given in Lemma 3.2 and by Theorem 2.10 we deduce that

$$\lim_{n \to \infty} \left(S_n(F; a_n, b_n) \right)'(x) = F'(x) \tag{3.12}$$

at every $x \in R_0$ where $F'(x)$ there exists. Next, by (3.5) and (1.9) and (3.12), we obtain

$$\lim_{n\to\infty} T_n(f; a_n, b_n; x) = F'(x) = f(x)$$

at every $x \in R_0$ where the condition (3.11) is satisfied.

Since (3.11) follows almost everywhere on R_0, we obtain the desired assertion. □

Now we shall prove approximation theorem for T_n and $f \in L(R_0)$. We shall apply the integral modulus of continuity of $f \in L(R_0)$, i. e.

$$\omega_1(f; L; t) := \sup_{0\le h\le t} \|\Delta_h f(\cdot)\|_L, \qquad t \ge 0. \tag{3.13}$$

Theorem 3.6. *Suppose that $f \in L(R_0)$ and $f' \in L(R_0)$ and ρ_1 is the function defined by (3.8). Then there exists a positive constant $M_{12}(a_1, b_1)$ such that*

$$\|[T_n(f; a_n, b_n) - f]\,\rho_1\|_L \le M_{12}(a_1, b_1)\|f'\|_L \frac{1}{\sqrt{b_n}} \tag{3.14}$$

for all $n \in N$.

Proof. Let $T_n(f; x) \equiv T_n(f; a_n, b_n; x)$. Analogously as in [3] we can write

$$f(x) - f(0) = \int_0^x f'(t)dt = \int_0^{+\infty} f'(t)y_x(t)dt$$

and

$$T_n(f; x) - f(x) = \int_0^{+\infty} f'(t)\{T_n(y_x(t); x) - y_x(t)\}\,dt,$$

for $x \in R_0$ and $n \in N$. Hence

$$\begin{aligned}\|[T_n(f) - f]\,\rho_1\|_L &\le \int_0^{+\infty}\left|\int_0^{+\infty} f'(t)\{T_n(y_x(t); x) - y_x(t)\}\,dt\right|\rho_1(x)dx\\ &\le \int_0^{+\infty} |f'(t)|\,R_n(t)dt,\end{aligned}$$

where $R_n(t)$ is defined in (3.9). Applying Lemma 3.4, we get

$$\int_0^{+\infty} |f'(t)|\,R_n(t)dt \le M_7(a_1, b_1)\|f'\|_L \frac{1}{\sqrt{b_n}}, \qquad n \in N,$$

and we complete the proof of (3.14). □

Theorem 3.7. *Suppose that $f \in L(R_0)$ and ρ_1 is defined by (3.8). Then there exists a positive constant $M_{13}(a_1, b_1)$ such that*

$$\|[T_n(f; a_n, b_n) - f]\,\rho_1\|_L \le M_{13}(a_1, b_1)\omega_1\left(f; L; \frac{1}{\sqrt{b_n}}\right) \tag{3.15}$$

for all $n \in N$.

Proof. We shall apply the Stieklov function

$$f_h(x)=\frac{1}{h}\int_0^h f(x+u)du, \qquad x\in R_0, \quad h>0,$$

for $f\in L(R_0)$. For f_h we have

$$\begin{aligned}\|f-f_h\|_L &= \frac{1}{h}\int_0^{+\infty}\left|\int_0^h (f(x+u)-f(x))du\right|dx \\ &\le \frac{1}{h}\int_0^h\left(\int_0^{+\infty}|\Delta_u f(x)|\,dx\right)du \le \omega_1(f;L;h)\,,\end{aligned} \tag{3.16}$$

where ω_1 is defined by (3.13). Moreover we have

$$f_h'(x)=h^{-1}\Delta_h f(x), \qquad x\in R_0, \quad h>0,$$

which implies

$$\|f_h'(\cdot)\|_L \le h^{-1}\omega_1(f;L;h)\,. \tag{3.17}$$

Hence $f_h\in L(R_0)$ and $f_h'\in L(R_0)$ if $f\in L(R_0)$ and $h>0$. From this and by Lemma 3.1 we get for $T_n(f;x)\equiv T_n(f;a_n,b_n;x)$

$$|T_n(f;x)-f(x)|\le|T_n(f-f_h;x)|+|T_n(f_h;x)-f_h(x)|+|f_h(x)-f(x)|\,,$$

for $x\in R_0$, $n\in N$ and $h>0$. Consequently

$$\|[T_n(f)-f]\rho_1\|_L\le\|T_n(f-f_h)\rho_1\|_L+\|[T_n(f_h)-f_h]\rho_1\|_L+\|(f_h-f)\rho_1\|_L\,,$$

for $n\in N$. By (3.2) and (3.8) and (3.16) we get

$$\begin{aligned}\|T_n(f-f_h)\rho_1\|_L &\le \|T_n(f-f_h)\|_L \le \frac{b_1}{a_1}\|f-f_h\|_L \\ &\le \frac{b_1}{a_1}\omega_1(f;L;h)\,, \quad n\in N,\\ \|(f_h-f)\rho_1\|_L &\le \|f-f_h\|_L\le\omega_1(f;L;h)\,.\end{aligned}$$

Applying Theorem 3.6 and (3.17), we obtain

$$\begin{aligned}\|[T_n(f_h)-f_h]\rho_1\|_L &\le M_{12}(a_1,b_1)b_n^{-1/2}\|f_h'\|_L \\ &\le M_{12}(a_1,b_1)h^{-1}b_n^{-1/2}\omega_1(f;L;h)\,, \qquad n\in N.\end{aligned}$$

Combining these and setting $h=1/\sqrt{b_n}$ for every fixed n, we immediately obtain (3.15). ◻

Finally we remark that Theorems 2.5-3.7 for operators defined by (1.7) and (1.8) are similar to certain results obtained for operators (1.1) and (1.2) in papers [1, 2, 3].

References

[1] M. Becker, *Global approximation theorems for Szasz - Mirakjan and Baskakov operators in polynomial weight spaces*, Indiana Univ. Math. J. **27**(1) (1978), 127-142.

[2] P. L. Butzer, *On the extensions of Bernstein polynomials to the infinite interval*, Proc. Amer. Math. Soc. **5** (1954), 547-553.

[3] T. Herman, *On the Szasz-Mirakian operator*, Acta. Math. Acad. Sci. Hung. **32**(1-2) (1978), 163-173.

Institute of Mathematics
Poznań University of Technology
Piotrowo 3A
60-965 Poznań, Poland
E-mail address: zwalczak@math.put.poznan.pl

Received: 05.12.2000.