GROWTH OF MAXIMUM MODULUS OF POLYNOMIALS WITH PRESCRIBED ZEROS

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Abstract. Let \( P(z) \) be a polynomial of degree \( n \) not vanishing in \(|z| < k \) where \( k \geq 1 \). It is shown that

\[
\max_{|z|=R>1} |P(z)| < \frac{(R + k)^n}{(R + k)^n + (1 + Rk)^n} \times \left\{ (R^n + 1) \max_{|z|=1} |P(z)| - \left( R^n - \left( \frac{1 + Rk}{R + k} \right)^n \right) \min_{|z|=k} |P(z)| \right\}.
\]

Among other things our result includes a refinement of a theorem due to Ankeny and Rivlin as a special case. We shall also prove an another result of similar nature.

Let \( P(z) \) be a polynomial of degree \( n \), then

\[
(1) \quad \max_{|z|=R>1} |P(z)| \leq R^n \max_{|z|=1} |P(z)|.
\]

Inequality (1) is a simple deduction from Maximum Modulus Principle (see [6, vol. 1, p. 137, problem III 269] or [7, p. 346]). It was shown by Ankeny and Rivlin [1] (see also [5, p. 442]), that if \( P(z) \neq 0 \) in \(|z| < 1\), then (1) can be replaced by

\[
(2) \quad \max_{|z|=R>1} |P(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)|.
\]

Inequality (2) is sharp, with equality for \( P(z) = az^n + \beta \), \(|\alpha| = |\beta| = 1\). For the class of polynomials not vanishing in the disk \(|z| < k\), \( k \geq 1 \), Aziz and Mohammad [4] proved the following generalization of inequality (2).

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Theorem 1. Let $P(z)$ be a polynomial of degree $n$ having no zeros in the disk $|z| < k$, where $k > 1$, then
\[
\max_{|z|=R>1} |P(z)| \leq \frac{(R^n+1)(R+k)^n}{(R+k)^n+(1+Rk)^n} \max_{|z|=1} |P(z)|.
\]

Theorem 1 does not appear to be sharp for $k > 1$ with the exception $n = 1$. However Aziz [2] (see also [3]) have proved the following sharp result which is an interesting generalization of inequality (2).

Theorem 2. Let $P(z)$ be a polynomial of degree $n$ which does not vanish in the disk $|z| < 1$, then
\[
\max_{|z|=R>1} |P(z)| \leq \left(\frac{R^n+1}{2}\right) \max_{|z|=1} |P(z)| - \left(\frac{R^n-1}{2}\right) \min_{|z|=1} |P(z)|.
\]

Here the result is best possible and equality holds for $P(z) = az^n + \beta$ where $|\beta| \geq |a|$

In this paper we first prove the following more general result which provides a refinement of Theorem 1 and includes Theorem 2 as a special case.

Theorem 3. If $P(z)$ is a polynomial of degree $n$ which does not vanish in $|z| < k$ where $k \geq 1$, then
\[
\max_{|z|=R>1} |P(z)| < \frac{(R+k)^n}{(R+k)^n+(1+Rk)^n} \times \left\{ (R^n+1) \max_{|z|=1} |P(z)| - \left(\frac{R^n-1}{R+k}\right) \min_{|z|=1} |P(z)| \right\}.
\]

For $k = 1$, this reduces to Theorem 2.

If $P(z)$ does not vanish in $|z| < k$, where $k \geq 1$ then it is known (see [4, inequality (6)]) that
\[
\max_{|z|=R} |P(z)| \leq \left(\frac{R+k}{1+k}\right)^n \max_{|z|=1} |P(z)| \quad \text{for } 1 \leq R^2 \leq k.
\]

The result is best possible and equality in (4) holds for $P(z) = ((z+k)/(1+k))^n$. Here we present the following refinement of (4).

Theorem 4. If $P(z)$ is a polynomial of degree $n$ having no zeros in the disk $|z| < k$ where $k \geq 1$ then for $1 \leq R \leq k^2$ we have
\[
\max_{|z|=R} |P(z)| \leq \left(\frac{R+k}{1+k}\right)^n \max_{|z|=1} |P(z)| - \left\{ \left(\frac{R+k}{1+k}\right)^n - 1 \right\} \min_{|z|=k} |P(z)|.
\]

Remark 5. Theorem 3 in general provides much better information than Theorem 1 regarding $\max_{|z|=R>1} |P(z)|$. We illustrate this with the help of following examples.
Example 6. Let

\[ P(z) = (z^2 + 9)(z - 19). \]

Then \( P(z) \) is a polynomial of degree 3 which does not vanish in \( |z| < t \), where \( 0 < t \leq 3 \). Clearly

\[ |P(z)| \geq \left\{ 9 - |z|^2 \right\} \left\{ 19 - |z| \right\} \]

which in particular gives

\[ \min_{|z|=2} |P(z)| \geq 85 \quad \text{and} \quad \max_{|z|=1} |P(z)| = 200 \]

Using Theorem 1 with \( k = t = 3, R = 2 \), it follows that

(5) \[ \max_{|z|=2} |P(z)| \leq 480.8 \]

where as using Theorem 3 with \( k = 2, R = 2 \), we get

\[ \max_{|z|=2} |P(z)| \leq 435.5 \]

which is much better than (5).

Example 7. Let

\[ P(z) = z^3 + 3z^3, \]

then \( P(z) \) does not vanish in \( |z| < t \), where \( 0 < t \leq 3 \). Clearly

\[ \min_{|z|=2} |P(z)| \geq 19 \quad \text{and} \quad \max_{|z|=1} |P(z)| = 28. \]

Using Theorem 1 with \( k = t = 3, R = 2 \), it follows that

(6) \[ \max_{|z|=2} |P(z)| \leq 67.4. \]

We use Theorem 3 with \( k = t = 2, R = 2 \), we get

\[ \max_{|z|=2} |P(z)| \leq 46.5 \]

which is much better than (6).

Similar remarks apply to Theorem 4 also. For the proof of Theorem 3 we need the following lemma.

Lemma 8. If \( P(z) \) is a polynomial of degree \( n \) which does not vanish for \( |z| < k, k > 0 \) then for all \( R \geq 1, r \leq k \) and for every \( \theta, 0 \leq \theta < 2\pi \)

(7) \[ |P(R r e^{i \theta})| < \left( \frac{R r + k}{r + R k} \right)^n R^n P \left( \frac{r e^{i \theta}}{R} \right) - \left\{ \left( \frac{R r + k}{r + R k} \right)^n \right\} \min_{|z|=k} |P(z)|. \]

Proof. The result is obvious for \( R = 1 \). So we assume \( R > 1 \). By hypothesis, the polynomial \( P(z) \) has all its zeros in \( |z| \geq k \) and \( m = \min_{|z|=k} |P(z)| \), therefore, \( m \leq |P(z)| \) for \( |z| \leq k \). We show for any given complex number \( \alpha \) with \( |\alpha| \leq 1 \), the polynomial \( F(z) = P(z) + a m \) has all its zeros in \( |z| \geq k \). This is obvious if \( m = 0 \) that is if \( P(z) \) has a zero on \( |z| = k \). We now suppose
that all the zeros of $P(z)$ lie in $|z| > k$ so that $m = \min_{|z|=k} |P(z)| > 0$. Hence $\frac{m}{P(z)}$ is analytic for $|z| \leq k$ and $\left| \frac{m}{P(z)} \right| \leq 1$ for $|z| = k$. Since $\frac{m}{P(z)}$ is not a constant, it follows by Maximum Modulus Principle that

\begin{equation}
\tag{8}
m < |P(z)| \quad \text{for } |z| < k.
\end{equation}

Now assume that $F(z) = P(z) + \alpha m$ has a zero in $|z| < k$, say at $z = z_0$ with $|z_0| < k$, then $P(z_0) + \alpha m = F(z_0) = 0$.

This implies

$$|P(z_0)| = |\alpha m| \leq m,$$

which is a contradiction to (8). Hence we conclude that in any case $F(z) = P(z) + \alpha m$ has all its zeros in $|z| \geq k$. Let

$$R_1 e^{i\theta_1}, R_2 e^{i\theta_2}, \ldots, R_n e^{i\theta_n}$$

be the zeros of $F(z)$. Then $R_j \geq k$, $j = 1, 2, \ldots, n$ and we have

$$F(z) = \prod_{j=1}^{n} (z - R_j e^{i\theta_j}),$$

therefore, for all $R \geq 1$, $r \leq k$ and for every $\theta$, $0 \leq \theta < 2\pi$, we have

\begin{equation}
\tag{9}
\left| \frac{F(Rre^{i\theta})}{R^n F \left( \frac{re^{i\theta}}{R} \right)} \right| = \prod_{j=1}^{n} \left| \frac{Rre^{i\theta} - R_j e^{i\theta_j}}{re^{i\theta} - RR_j e^{i\theta_j}} \right| = \prod_{j=1}^{n} \left| \frac{Rre^{i(\theta - \theta_j)} - R_j}{re^{i(\theta - \theta_j)} - RR_j} \right|.
\end{equation}

Since $R_j \geq k \geq r$ and $R \geq 1$, therefore, it can be easily verified after a short calculation that

$$\left| \frac{Rre^{i(\theta - \theta_j)} - R_j}{re^{i(\theta - \theta_j)} - RR_j} \right| = \left( \frac{R^2r^2 + R_j^2 - 2RrR_j \cos(\theta - \theta_j)}{r^2 + R^2R_j^2 - 2RrR_j \cos(\theta - \theta_j)} \right)^{1/2} \leq \left( \frac{Rr + R}{r + RR_j} \right) \leq \left( \frac{Rr + k}{r + RK} \right).$$

The first estimate is obtained by observing that the function

$$f(t) = \frac{Rr^2 + R_j^2 - 2RrR_j t}{r^2 + R^2R_j^2 - 2RrR_j t}$$

is a decreasing function of $t$ on $[-1, 1]$, which follows from taking a derivative and using the hypothesis $R_j \geq r$. The function $f$, therefore, has a maximum
at $t = -1$ and the first estimate follows. The estimate (10) also follows by noting that the function

$$g(R_j) = \frac{R_r + R_j}{R + RR_j}$$

is a decreasing function of $R_j$ which can be verified by using derivative again and the fact that $R_j \geq k$. Thus $g(k)$ is maximum. Using (10) in (9), it follows that

$$|F(Rre^{i\theta})| \leq \left( \frac{Rr + k}{r + Rk} \right)^n R^n F \left( \frac{r e^{i\theta}}{R} \right)$$

for every $\theta$, $0 \leq \theta < 2\pi$, $R > 1$, $k \geq r$. Replacing $F(z)$ by $P(z) + \alpha m$, we get

$$|P(Rre^{i\theta}) + \alpha m| \leq \left( \frac{Rr + k}{r + Rk} \right)^n |R^n P \left( \frac{r e^{i\theta}}{R} \right) + R^n \alpha m|$$

for every $\alpha$ with $|\alpha| \leq 1$, $0 \leq \theta < 2\pi$, $R > 1$ and $k \geq r$. Since $r/R \leq k$, we choose argument of $\alpha$ with $|\alpha| = 1$ on the R. H. S of (11) such that for $|z| = 1$,

$$|P(Rrz)| + \alpha m = |P \left( \frac{rz}{R} \right) - m$$

which is possible by (8). Using (12) in (11), we attain for $|z| = 1$, $R > 1$ and $k > r$,

$$|P(Rrz)| - m \leq \left( \frac{Rr + k}{r + Rk} \right)^n \left| R^n P \left( \frac{r}{R} \right) - \left( \frac{Rr + k}{r + Rk} \right)^n R^n m. \right.$$

This implies

$$|P(Rrz)| \leq \left( \frac{Rr + k}{r + Rk} \right)^n \left| R^n P \left( \frac{r}{R} \right) \right| - \left\{ \left( \frac{Rr + k}{r + Rk} \right)^n R^n - 1 \right\} \min_{|z|=k} |P(z)|$$

for $|z| = 1$, $R \geq 1$ and $r \leq k$, which is the desired result. This completes the proof of Lemma 8.

We also need the following lemma:

**Lemma 9.** If $P(z)$ is a polynomial of degree $n$, then

$$|P(Rre^{i\theta})| + |Q(Re^{i\theta})| \leq (R^n + 1) \max_{|z|=1} |P(z)|, \quad 0 \leq \theta \leq 2\pi,$$

where

$$Q(z) = z^n \overline{P(1/z)} \quad \text{and} \quad R \geq 1.$$

Lemma 9 is due to Aziz and Mohammad [4]. However, for the sake of completeness, we give here a brief outline of the proof. In fact, we deduce it from Lemma 8, and thereby present an independent proof of Lemma 9. Let $M = \max_{|z|=1} |P(z)|$, then

$$|P(z)| \leq M \quad |z| = 1.$$
By Rouche's theorem, it follows that for every real or complex number \( \lambda \), with \(|\lambda| > 1\), the polynomial

\[
F(z) = P(z) - \lambda M
\]

does not vanish in \(|z| < 1\). Applying Lemma 8, to the polynomial \( F(z) \) with \( k = 1 = r \), it follows that for every \( \theta \), \( 0 \leq \theta < 2\pi \), \( R > 1 \),

\[
|F(Re^{i\theta})| \leq R^n \left| F\left(\frac{e^{i\theta}}{R}\right)\right| - (R^n - 1) \min_{|z|=1} |F(z)|
\]

(14)

\[
\leq \left| R^n F\left(\frac{e^{i\theta}}{R}\right)\right|.
\]

If \( G(z) = z^n F(1/z) \), then we have \( G(z) = Q(z) - \bar{\lambda} z^n M \) and

\[
|G(Re^{i\theta})| = \left| R^n e^{i\theta} F\left(\frac{e^{i\theta}}{R}\right)\right| = \left| R^n F\left(\frac{e^{i\theta}}{R}\right)\right|.
\]

Using this in (14), it follows that for every \( R \geq 1 \), and \( 0 \leq \theta < 2\pi \),

\[
|P(Re^{i\theta}) - \lambda M| = |F(Re^{i\theta})| \leq |G(Re^{i\theta})| = |Q(Re^{i\theta}) - \bar{\lambda} R^n e^{i\theta} M|\]

choosing the argument of \( \lambda \) in R. H. S of this inequality suitably, we get

\[
|P(Re^{i\theta})| - |\lambda| M \leq |\lambda| R^n - |Q(Re^{i\theta})|.
\]

Or

\[
|P(Re^{i\theta})| + |Q(Re^{i\theta})| \leq (R^n + 1)|\lambda| M
\]

for every \( \theta \), \( 0 \leq \theta < 2\pi \), and \( k \geq 1 \), letting \(|\lambda| \to 1\), we get the assertion of Lemma 9.

**Proof of Theorem 3.** Since all the zeros of \( P(z) \) lie in \(|z| \geq k \geq 1\), using Lemma 8, it follows from (7) with \( r = 1 \), that

\[
|P(Re^{i\theta})| \leq \left(\frac{R + k}{1 + Rk}\right)^n R^n P\left(\frac{e^{i\theta}}{R}\right) - \left\{ \left(\frac{R + k}{1 + Rk}\right)^n R^n - 1 \right\} m
\]

(15)

for every \( \theta \), \( 0 \leq \theta < 2\pi \) and \( R \geq 1 \). Since

\[
Q(z) = z^n P(1/z)
\]

therefore,

\[
|Q(Re^{i\theta})| = |R^n P\left(\frac{e^{i\theta}}{R}\right)|.
\]

Using (16) in (15), we get

\[
|P(Re^{i\theta})| \leq \left(\frac{R + k}{1 + Rk}\right)^n |Q(Re^{i\theta})| - \left\{ \left(\frac{R + k}{1 + Rk}\right)^n R^n - 1 \right\} m.
\]
This implies
\[
\frac{(1 + Rk)^n + (R + k)^n}{(1 + Rk)^n} |P(Re^{i\theta})| \leq \left( \frac{R + k}{1 + Rk} \right)^n \left\{ \left| P(Re^{i\theta}) \right| + \left| Q(Re^{i\theta}) \right| \right\} - \left\{ \left( \frac{R + k}{1 + Rk} \right)^n R^n - 1 \right\} m.
\]

Inequality (17) yields with the help of Lemma 9 that
\[
\frac{(1 + Rk)^n + (R + k)^n}{(1 + Rk)^n} |P(Re^{i\theta})| \leq \left( \frac{R + k}{1 + Rk} \right)^n \frac{(R + k)^n(R^n + 1)}{(1 + Rk)^n} \max_{|z|=1} |P(z)| - \left\{ \left( \frac{R + k}{1 + Rk} \right)^n R^n - 1 \right\} \min_{|z|=1} |P(z)| = \left( \frac{R + k}{1 + Rk} \right)^n \left[ (R^n + 1) \max_{|z|=1} |P(z)| - \left\{ R^n - \left( \frac{1 + Rk}{R + k} \right)^n \right\} \min_{|z|=k} |P(z)| \right].
\]

From (18) it follows that
\[
|P(Re^{i\theta})| \leq \frac{(R + k)^n}{(1 + Rk)^n + (R + k)^n} \times \left[ (R^n + 1) \max_{|z|=1} |P(z)| - \left\{ R^n - \left( \frac{1 + Rk}{R + k} \right)^n \right\} \min_{|z|=1} |P(z)| \right]
\]
for every $\theta$, $0 \leq \theta < 2\pi$ and $R \geq 1$. Which is equivalent to the desired result. This completes the proof of Theorem 3.

**Proof of Theorem 4.** Let $m = \min_{|z|=k} |P(z)|$, then we have
\[
m \leq |P(z)| \quad \text{for} \quad |z| = k.
\]
Since $P(z)$ does not vanish in $|z| < k$, and it follows as in the proof of Lemma 8 that for every real or complex number $\alpha$ with $|\alpha| \leq 1$, the polynomial $F(z) = P(z) + am$ has all its zeros in $|z| \geq k$. If
\[
R_1e^{i\theta_1}, R_2e^{i\theta_2}, \ldots, R_ne^{i\theta_n}
\]
be the zeros of $F(z)$, then $R_j \geq k$, $j = 1, 2, \ldots, n$ and we have
\[
F(z) = \prod_{j=1}^n (z - R_je^{i\theta_j}).
\]
It can be easily seen for $1 \leq R \leq k^2$ and $0 \leq \theta < 2\pi$
\[
\frac{|P(Re^{i\theta})|}{P(e^{i\theta})} = \prod_{j=1}^n \left| \frac{Re^{i\theta} - R_je^{i\theta_j}}{e^{i\theta} - R_je^{i\theta_j}} \right| \leq \prod_{j=1}^n \left( \frac{R + R_j}{1 + R_j} \right) \leq \prod_{j=1}^n \left( \frac{R + k}{1 + k} \right)^n.
\]
This implies

\[ |F(Re^{i\theta})| \leq \left(\frac{R + k}{1 + k}\right)^n |F(e^{i\theta})| \]

for every \( \theta, 0 \leq \theta < 2\pi \) and \( 1 \leq R \leq k^2 \). Replacing \( F(z) \) by \( P(z) + am \) in (20), we get

\[ |P(Re^{i\theta}) + am| \leq \left(\frac{R + k}{1 + k}\right)^n |P(e^{i\theta}) + am| \]

for every \( \alpha \) with \( |\alpha| \leq 1, 0 \leq \theta < 2\pi \) and \( 1 \leq R \leq k^2 \). Since \( P(z) \) does not vanish for \( |z| < k \), by Maximum Modulus Principle it follows from (19) that

\[ m \leq |P(z)| \text{ for } |z| \leq k \text{ where } k \geq 1. \]

Taking in particular \( z = e^{i\theta}, 0 \leq \theta < 2\pi \) in (22), then

\[ |z| = |e^{i\theta}| = 1 \leq k \]

and we get

\[ m \leq |P(e^{i\theta})| \text{ for } 0 \leq \theta < 2\pi. \]

Choosing the argument \( \alpha \) with \( |\alpha| = 1 \) on the R. H. S of (21) such that for \( |z| = 1 \),

\[ |P(z) + am| = |P(z)| - m \]

which is possible by (23), we obtain from (21) that

\[ |P(Re^{i\theta})| - m \leq \left(\frac{R + k}{1 + k}\right)^n \{P(e^{i\theta}) - m\} \]

for every \( \theta, 0 \leq \theta < 2\pi, 1 \leq R \leq k^2 \). This gives

\[ |P(Rz)| \leq \left(\frac{R + k}{1 + k}\right)^n |P(z)| - \left\{ \left(\frac{R + k}{1 + k}\right)^n - 1 \right\} m \]

for \( |z| = 1 \) and \( 1 \leq R \leq k^2 \), from which it immediately follows that

\[ \max_{|z|=1} |P(z)| \leq \left(\frac{R + k}{1 + k}\right)^n \max_{|z|=1} |P(z)| - \left\{ \left(\frac{R + k}{1 + k}\right)^n - 1 \right\} \min_{|z|=k} |P(z)| \]

for \( |z| = 1 \) and \( 1 \leq R \leq k^2 \). This completes the proof of Theorem 4.

**References**


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