INTEGRAL INEQUALITIES FOR POLYNOMIALS HAVING A ZERO OF ORDER $m$ AT THE ORIGIN

V. K. JAIN
Kharagpur, India

ABSTRACT. For a polynomial $p(z)$ of degree $n$, it is known that

$$\left( \int_0^{2\pi} |p'(e^{i\theta})|^s d\theta \right)^{1/s} \leq n \left( \int_0^{2\pi} |p(e^{i\theta})|^s d\theta \right)^{1/s}, \quad s \geq 1.$$  

We have obtained inequalities in the reverse direction for the polynomials having a zero of order $m$ at the origin.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $p(z)$ be a polynomial of degree $n$. Zygmund [3] has shown that for $s \geq 1$

$$\left( \int_0^{2\pi} |p'(e^{i\theta})|^s d\theta \right)^{1/s} \leq n \left( \int_0^{2\pi} |p(e^{i\theta})|^s d\theta \right)^{1/s}. \quad (1.1)$$

In this paper, we have obtained similar type of integral inequalities, but in the reverse direction, for polynomials having a zero of order $m$ at the origin. More precisely, we prove

**Theorem 1.1.** Let $p(z)$ be a polynomial of degree $n$, having a zero of order $m$ at $z = 0$. Then for $s \geq 1$

$$\left( \int_0^{2\pi} |p'(e^{i\theta})|^s d\theta \right)^{1/s} \geq m \left( \int_0^{2\pi} |p(e^{i\theta})|^s d\theta \right)^{1/s}. \quad (1.2)$$

2000 Mathematics Subject Classification. 30C10, 30A10.

Key words and phrases. Inequalities, zero of order $m$, polynomials.
By letting $s \to \infty$ in (1.2), we obtain

**Corollary 1.2.** Let $p(z)$ be a polynomial of degree $n$, having a zero of order $m$ at $z = 0$. Then

$$\max_{|z|=1} |p'(z)| \geq m \max_{|z|=1} |p(z)|.$$

**Theorem 1.3.** Let $p(z)$ be a polynomial of degree $n$, having all its zeros in $|z| \leq k$, $k \leq 1$, with a zero of order $m$ at $z = 0$. Then for $s \geq 1$

$$\left( \int_0^{2\pi} \left| p'(e^{i\theta}) + \frac{mm'\beta e^{i(m-1)\theta}}{k^n} \right|^s d\theta \right)^{1/s} \geq \{n - (n - m)C_s^{(k)}\} \left( \int_0^{2\pi} |p(e^{i\theta})|^{s} d\theta \right)^{1/s},$$

where

$$m' = \min_{|z|=k} |p(z)|,$$  

$$C_s^{(k)} = k \left/ \left( \frac{1}{2\pi} \int_0^{2\pi} |1 + ke^{i\alpha}|^s d\alpha \right) \right.^{1/s}.$$

By taking $k = 1$ and $\beta = 0$ in Theorem 1.3, we obtain

**Corollary 1.4.** If $p(z)$ is a polynomial of degree $n$, having all its zeros in $|z| \leq 1$, with a zero of order $m$ at $z = 0$, then for $s \geq 1$

$$\left( \int_0^{2\pi} \left| p'(e^{i\theta}) \right|^s d\theta \right)^{1/s} \geq \{n - (n - m)D_s\} \left( \int_0^{2\pi} |p(e^{i\theta})|^s d\theta \right)^{1/s},$$

where

$$D_s = 1 \left/ \left( \frac{1}{2\pi} \int_0^{2\pi} |1 + e^{i\alpha}|^s d\alpha \right) \right.^{1/s}.$$

Inequality (1.6), with $m = 0$, is also true for self-inversive polynomials. In other words we have

**Theorem 1.5.** If $p(z)$ is a polynomial of degree $n$ such that

$$p(z) = z^n \overline{p(1/z)},$$
then for \( s \geq 1 \),
\[
\left( \int_0^{2\pi} |p'(e^{i\theta})|^s \, d\theta \right)^{1/s} \geq n(1 - D_s) \left( \int_0^{2\pi} |p(e^{i\theta})|^s \, d\theta \right)^{1/s},
\]
where \( D_s \) is as in Corollary 1.4.

By letting \( s \to \infty \) in Theorem 1.3, we obtain

**Corollary 1.6.** Let \( p(z) \) be a polynomial of degree \( n \), having all its zeros in \(|z| \leq k, k \leq 1\), with a zero of order \( m \) at \( z = 0 \). Then for \( \beta \) with \( |\beta| < k^{n-m} \)
\[
\max_{|z|=1} |p'(z)| + \frac{mn'}{kn} \beta z^{m-1} \geq \left( \frac{n + mk}{1 + k} \right) \max_{|z|=1} |p(z)| + \frac{m'}{kn} \beta z^m,
\]
where \( m' \) is as in Theorem 1.3.

By choosing argument of \( \beta \) suitably and letting \( |\beta| \to k^{n-m} \) in Corollary 1.6, we obtain

**Corollary 1.7.** If \( p(z) \) is a polynomial of degree \( n \), having all its zeros in \(|z| \leq k, k \leq 1\), with a zero of order \( m \) at \( z = 0 \), then
\[
\max_{|z|=1} |p'(z)| \geq \left( \frac{n + mk}{1 + k} \right) \max_{|z|=1} |p(z)| + \left( \frac{n - m}{1 + k} \right) \frac{m'}{kn},
\]
where \( m' \) is as in Theorem 1.3.

## 2. Lemmas

For the proofs of the theorems, we require the following lemmas.

**Lemma 2.1.** If \( p(z) \) is a polynomial of degree \( n \), having no zeros in \(|z| < k\), \( k \geq 1 \), then for \( s \geq 1 \)
\[
\left( \int_0^{2\pi} |p'(e^{i\theta})|^s \, d\theta \right)^{1/s} \leq nE_s^{(k)} \left( \int_0^{2\pi} |p(e^{i\theta})|^s \, d\theta \right)^{1/s},
\]
where
\[
E_s^{(k)} = 1 \left( \frac{1}{2\pi} \int_0^{2\pi} |k + e^{i\alpha}|^s \, d\alpha \right)^{1/s}.
\]
This lemma is due to Govil and Rahman [2].

**Lemma 2.2.** If \( p(z) \) is a polynomial of degree \( n \) such that
\[
p(z) = z^m p(1/z),
\]
then for $s \geq 1$
\[
\left( \int_0^{2\pi} |p'(e^{i\theta})|^s d\theta \right)^{1/s} \leq nD_s \left( \int_0^{2\pi} |p(e^{i\theta})|^s d\theta \right)^{1/s},
\]

where $D_s$ is as in Corollary 1.4.

This lemma is due to Dewan and Govil [1].

3. PROOFS OF THE THEOREMS

PROOF OF THEOREM 1.1. We obviously have
\[
(3.1) \quad p(z) = z^m \phi(z),
\]
where $\phi(z)$ is a polynomial of degree $n - m$, with the property that
\[
\phi(0) \neq 0.
\]

Then
\[
(3.2) \quad q(z) = z^n \overline{p(1/z)} = z^{n-m} \overline{\phi(1/z)},
\]
is also a polynomial of degree $n - m$. Hence we have for $s \geq 1$,
\[
(3.3) \quad \left( \int_0^{2\pi} |q'(e^{i\theta})|^s d\theta \right)^{1/s} \leq (n - m) \left( \int_0^{2\pi} |q(e^{i\theta})|^s d\theta \right)^{1/s}.
\]

But by (3.2), we have for $0 \leq \theta \leq 2\pi$
\[
|q'(e^{i\theta})| = |np(e^{i\theta}) - e^{i\theta} p'(e^{i\theta})|,
\]
\[
|q(e^{i\theta})| = |p(e^{i\theta})|,
\]
which, by (3.3), imply that for $s \geq 1$,
\[
(3.4) \quad \left( \int_0^{2\pi} |np(e^{i\theta}) - e^{i\theta} p'(e^{i\theta})|^s d\theta \right)^{1/s} \leq (n - m) \left( \int_0^{2\pi} |p(e^{i\theta})|^s d\theta \right)^{1/s}.
\]
Now, by Minkowski inequality, we have for $s \geq 1$

\[
\left( \int_0^{2\pi} |p(e^{i\theta})|^{s} d\theta \right)^{1/s} \leq \leq \left( \int_0^{2\pi} |np(e^{i\theta}) - e^{i\theta} p'(e^{i\theta})|^{s} d\theta \right)^{1/s} + \left( \int_0^{2\pi} |e^{i\theta} p'(e^{i\theta})|^{s} d\theta \right)^{1/s}
\]

\[
\leq (n - m) \left( \int_0^{2\pi} |p(e^{i\theta})|^{s} d\theta \right)^{1/s} + \left( \int_0^{2\pi} |p'(e^{i\theta})|^{s} d\theta \right)^{1/s}, \quad \text{(by (3.4))}
\]

and Theorem 1.1 follows.

**Proof of Theorem 1.3.** The polynomial $q(z)$, given by (3.2) will have no zeros in $|z| < \frac{1}{k}$. Now if

\[
(3.5) \quad m_0 = \min_{|z| = \frac{1}{k}} |q(z)| = \min_{|z| = \frac{1}{k}} \left| z^n p(1/z) \right| = \frac{m'}{kn}, \quad \text{by (1.4)).}
\]

then, by Rouché’s theorem, the polynomial

\[
q(z) + m_0 z^{n-m}, \quad |\beta| < k^{n-m},
\]

of degree $n - m$, will also have no zeros in $|z| < \frac{1}{k}, \frac{1}{k} \geq 1$. Hence, by Lemma 2.1, we have for $s \geq 1$ and $|\beta| < k^{n-m}$

\[
\left( \int_0^{2\pi} \left| q'(e^{i\theta}) + \frac{m'}{kn} \beta e^{i(n-m-1)\theta} (n - m) \right|^s d\theta \right)^{1/s} \leq (n - m) C_s^{(k)} \left( \int_0^{2\pi} \left| q(e^{i\theta}) + \frac{m'}{kn} \beta e^{i(n-m)\theta} \right|^s d\theta \right)^{1/s},
\]

i.e.,

\[
\left( \int_0^{2\pi} \left| np(e^{i\theta}) - e^{i\theta} p'(e^{i\theta}) + \frac{m'}{kn} (n - m) e^{im\theta} \right|^s d\theta \right)^{1/s} \leq (n - m) C_s^{(k)} \left( \int_0^{2\pi} \left| p(e^{i\theta}) + \frac{m'}{kn} e^{im\theta} \right|^s d\theta \right)^{1/s}, \quad \text{(by (3.2)).}
\]
Now by Minkowski inequality, we have for $s \geq 1$ and $|\beta| < k^{n-m}$

\[
n \left( \int_0^{2\pi} \left| p \left( e^{i\theta} \right) + \frac{m'}{k^n} \beta e^{im\theta} \right|^s \, d\theta \right)^{1/s} \leq 
\]

\[
\left( \int_0^{2\pi} \left| n p \left( e^{i\theta} \right) + \frac{m'}{k^n} \beta (n-m) e^{im\theta} - e^{i\theta} p' \left( e^{i\theta} \right) \right|^s \, d\theta \right)^{1/s} + 
\]

\[
\left( \int_0^{2\pi} \left| e^{i\theta} p' \left( e^{i\theta} \right) + m \frac{m'}{k^n} \beta e^{im\theta} \right|^s \, d\theta \right)^{1/s},
\]

and Theorem 1.3 follows, by (3.6).

Proof of Theorem 1.5. The polynomial

\[ q(z) = z^n p(1/z) \]

is a polynomial of degree $n$, with the property

\[ q(z) = z^n q(1/z), \quad (by \ (1.8)). \]

Hence, by Lemma 2.2, we have for $s \geq 1$

\[
\left( \int_0^{2\pi} \left| q' \left( e^{i\theta} \right) \right|^s \, d\theta \right)^{1/s} < n D_s \left( \int_0^{2\pi} \left| q \left( e^{i\theta} \right) \right|^s \, d\theta \right)^{1/s}.
\]

Now Theorem 1.5 follows on lines, similar to those of Theorem 1.1.

References


Mathematics Department,
I.I.T., Kharagpur-721302,
India

Received: 09.02.2000