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LOCAL CONNECTEDNESS AND UNICOHERENCE AT SUBCONTINUA

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Abstract. Let X be a continuum and Y a subcontinuum of X. The purpose of this paper is to investigate the relation between the conditions "X is unicoherent at Y" and "Y is unicoherent". We say that X *is strangled by* Y if the closure of each component of $X \setminus Y$ intersects Y in one single point. We prove: If X is strangled by Y and Y is unicoherent then X is unicoherent at Y. We also prove the converse for a locally connected (not necessarily metric) continuum X.

1. Introduction

In this paper continuum means a compact, connected and metric space. A subcontinuum of a space X is a subspace of X which is a continuum. In section 3 we also consider compact, connected and Hausdorff spaces (not necessarily metric). These spaces will be called Hausdorff continua.

The continuum X is said to be *unicoherent* if every pair of subcontinua of X whose union is X has connected intersection. The concept of unicoherence at a subcontinuum of a metric continuum is due to M. A. Owens [8]. The same definition may include the nonmetric case. It is said that X is *unicoherent at a subcontinuum* Y of X if for every pair H and K of subcontinua of X whose union is X, the intersection $H \cap K \cap Y$ is a subcontinuum of X.

First of all, observe that neither one of the following implications is true:
1) X is unicoherent at a subcontinuum $Y \Rightarrow Y$ is unicoherent.
2) Y is unicoherent $\Rightarrow X$ is unicoherent at Y .

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Indeed, as a counterexample to the first one, let X consists of a circumference Y and a spiral (homeomorphic copy of $[0,\infty)$) converging to Y. For the second, take X as a two dimensional cell and Y any subarc of X. (Notice that, in this example, X is unicoherent and locally connected).

The purpose of this paper is to investigate under which additional properties, the concepts "X is unicoherent at a subcontinuum Y" and "Y is unicoherent" are equivalent.

We define: A continuum X is *strangled* by a subcontinuum Y if the intersection of Y with the closure of each component of $X \setminus Y$ consists of a single point. When X is a semi-locally connected continuum, X is strangled by Y if and only if Y is the union of cyclic elements (see [9,IV, Theorem 3.3, p.67]). We also observe that every locally connected (metric) continuum is semi-locally connected [9,I,Corollary 13.21, p.20]

We prove the following result. Assume that X is strangled by Y and Y is unicoherent. Then X is unicoherent at Y. Since the converse is not true (Example 1) we discuss the problem under properties concerning local connectedness.

In section 3 we prove Theorem 11 which characterizes those subcontinua Y of a locally connected (not necessarily metric) continuum X such that X is unicoherent at Y.

A *dendrite* is a locally connected and hereditarily unicoherent metric continuum. Characterizations of dendrites in terms of unicoherence at subcontinua are given in [1, 3, 7, 8]. As a corollary of our results, we prove the following generalization of Theorem 1 in [3]: If a locally connected metric continuum X is unicoherent at a one-dimensional subcontinuum Y then Y is a dendrite. (Theorem 13)

Recently, some papers have been written about unicoherence at subcontinua ([3, 4, 10]). In particular, these papers deal with some questions posed in [2] about mappings preserving unicoherence at subcontinua.

We will use the following notation in this paper:

$\mathcal{P}(X)$ denotes the family of subsets of X, $C(X)$ is the set of all subcontinua of X and $\Gamma(X) = C(X) \setminus (\{X\} \cup \{\{x\} : x \in X\})$. If Z is a subset of X, the set of components of Z will be denoted by $\mathcal{K}(Z)$.

2. Strangled.

We prove here that under the condition "X is strangled by Y", Y is unicoherent implies X is unicoherent at Y, the converse is discussed in this section.

The following results will be used below:

Theorem 2.1. $[6, ChV., 48, VIII, Theorem\ 5, p.220]$ *If a space K is irreducibly connected between the closed sets M and N then $K \setminus (M \cup N)$ is connected and dense in K.*

Theorem 2.2. $[6, ChV., 48, IX, Theorem\ 3, p.223]$ *If an indecomposable continuum X is irreducibly connected between two closed sets M and N, then there exists a composant L such that $L \cap (M \cup N) = \emptyset$.*

If X is strangled by Y and $C \in \mathcal{K}(X \setminus Y)$ we call the unique point in $Cl(C) \cap Y$ the *attaching point* of C, and we denote it by $att(C)$.

Lemma 2.3. *Let X be a metric continuum which is strangled by $Y \in C(X)$. Then*
i) For every $H \in C(X)$, $H \cap Y$ is connected.
ii) $att(C) \in H$ whenever $C \in \mathcal{K}(X \setminus Y)$, and $H \cap C \neq \emptyset \neq H \cap (X \setminus C)$.
iii) $H \cap Cl(C)$ is connected for every $C \in \mathcal{K}(X \setminus Y)$

Proof. *i)* Suppose $H \cap Y$ is not connected, so that $H \cap Y = M \mid N$. Let $K \in C(H)$ be irreducible between M and N. We consider the following two cases:
Case 1) K is decomposable. Since K is irreducibly connected between the closed sets $M \cap K$ and $N \cap K$, it follows from Theorem 1 that $K \setminus (M \cup N)$ $= K \setminus Y$ is a connected subset which is dense in K. Therefore $K \setminus (M \cup N)$ is contained in $C \in \mathcal{K}(X \setminus Y)$ and $K = Cl(K \setminus (M \cup N)) \subset Cl(C)$. But this is a contradiction since $Cl(C) \cap Y$ contains a single point, while $K \cap Y$ contains at least two points (one in M and one in N).
Case 2) K is indecomposable. Then by Theorem 2 there exists a composant L of K contained in $K \setminus (M \cup N)$. Being a composant, L is a connected subset which is dense in K . Since L is contained in $K \setminus Y$, L is contained in some $C \in \mathcal{K}(X \setminus Y)$ so that $K \cap Y = Cl(L) \cap Y$ is contained in $Cl(C) \cap Y$ and this again is a contradiction.
ii) The hypothesis imply that $H \cap Y \neq \emptyset$. Let $K \in C(H)$ be irreducible between $H \cap Y$ and a point $p \in H \cap C$. We consider the same two cases than in the proof of i) and proceed in the same way with $\{p\} = M$ and $H \cap Y = N$.
Case1) K is decomposable. Then the set $L = K \setminus (M \cup N)$ is a connected subset of K which is dense in K. Since L is contained in $X \setminus Y$ then it is contained in $D \in \mathcal{K}(X \setminus Y)$. Therefore $K = Cl(L) \subset Cl(D) = D \cup att(D)$. Since $p \in K$, $D = C$ so that $att(C) \in K \subset H$.
Case 2) K is indecomposable. The composant L is contained in $D \in \mathcal{K}(X \setminus Y)$. On the other hand since $K = Cl(L) \subset Cl(D) = D \cup att(D)$, then $D = C$ and $att(C) \in K \subset H$.
iii) Suppose that $H \cap Cl(C)$ is not connected, so it is clear that $H \cap C \neq \emptyset$. Therefore, by *ii)* $att(C) \in H$. Write $H \cap Cl(C) = M|N$ and suppose that $att(C) \in N$. Let $K \in C(H)$ be irreducible between M and Y. As above, we consider two cases and again we get a connected and dense subset L of $K \setminus Y$ which intersects C. Therefore $L \subseteq C$ which implies $K \subseteq Cl(C)$. This shows that $H \cap Cl(C)$ contains a connected set intersecting M and N contrary to the assumption. $\square$

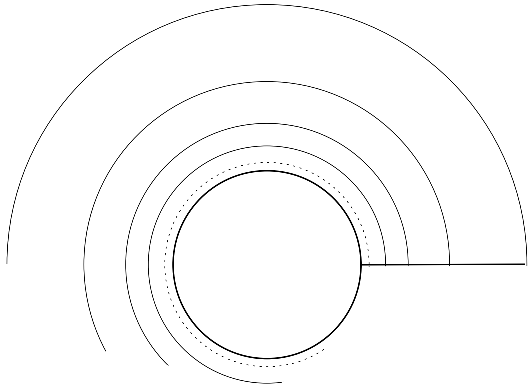

Figure 1

THEOREM 2.4. *Let X be a continuum and $Y \in C(X)$. Suppose that X is strangled by Y and Y is unicoherent. Then X is unicoherent at Y.*

PROOF. Follows from Lemma 2.3 *i)* $\square$

The following example shows that the converse of Theorem 3 is not true.

EXAMPLE 2.1 (See Figure 1). *Let $Y \subseteq \mathbb{R}^2$ be the union of $\mathbf{S}^1$ and the arc $[1,2] \times \{0\}$. Let $C_n = \left\{(1+\frac{1}{2^n})(\cos\theta, \sin\theta) : \theta \in \left[0, (2-\frac{1}{2^n})\pi\right]\right\}$ and $X = Y \cup \bigcup_{n \in \mathbb{N}} C_n$. It is easy to verify that X is strangled by Y. In order to prove that X is unicoherent at Y, suppose that $X = H \cup K$. Then, for infinitely many indices $n \in \mathbb{N}$, $C_n \subset H$. Since $\mathbf{S}^1 \subset Cl(\bigcup C_n)$ then we can assume that for some $a \leq 2$, $H \cap Y = \mathbf{S}^1 \cup ([1,a] \times \{0\})$, so that $H \cap K \cap Y = K \cap (\mathbf{S}^1 \cup [1,a] \times \{0\})$ which is a connected set by Lemma 2.3 i).*

Nevertheless we have the following Theorem

THEOREM 2.5. *Let X be a continuum and $Y \in C(X)$. Assume that X is locally connected at each point of $Bd(Y)$. If X is strangled by Y and X is unicoherent at Y then Y is unicoherent.*

PROOF. Suppose that Y is not unicoherent so that $Y = H \cup K$ where H, $K \in C(X)$ and $H \cap K$ is not connected. Let $\tilde{H} = H \cup \nabla_H$ where ∇_H is the closure of the union of all components of $X \setminus Y$ whose attaching point is in H. Similarly define $\tilde{K} = K \cup \nabla_K$

We want to prove that $\tilde{H} \cap \tilde{K} \cap Y$ is not connected, contrary to the hypothesis.

Let $x \in (H \cap \nabla_K) \setminus K$. Then $x = \lim x_n$ where $x_n \in C_n \in \mathcal{K}(X \setminus Y)$ and $att(C_n) \in K$. Since $x \notin K$ there is an open and connected subset U of X such that $x \in U \subseteq Cl(U) \subseteq H \setminus K$ so that $x_n \in Cl(U)$ for n large enough. This implies that for some fixed $n \in \mathbb{N}$, $Cl(U) \cap C_n \neq \emptyset$ and $Cl(U) \cap X \setminus C_n \neq \emptyset$. Therefore, by Lemma 2.3 *ii)* $att(C_n) \in Cl(U)$ and this is a contradiction.

This proves that $H \cap \nabla_K \subseteq H \cap K$. Similarly $K \cap \nabla_H \subseteq H \cap K$. On the other hand, since $Y = H \cup K$, then $(\nabla_H \cap \nabla_K) \cap Y \subseteq H \cap K$.
Therefore the equality
$\tilde{H} \cap \tilde{K} \cap Y = (H \cap K) \cup (H \cap \nabla_K) \cup (K \cap \nabla_H) \cup (\nabla_H \cap \nabla_K) \cap Y$
becomes
$\tilde{H} \cap \tilde{K} \cap Y = H \cap K$ and this proves that $\tilde{H} \cap \tilde{K} \cap Y$ is not connected, as desired. ☐

Question: Let X be a continuum and $Y \in C(X)$. Assume that X is locally connected at each point of $Bd(Y)$ and X is unicoherent at Y. Is it true that Y intersects the closure of each component of $X \setminus Y$ in a connected set? Is X strangled by Y?

Example 2.1 shows that, in Theorem 2.5, the hypothesis X is locally connected at every point of $Bd(Y)$ cannot be changed by X is locally connected at some points of $Bd(Y)$. Indeed It is easy to verify that X is locally connected at every point in $(1, 2] \times \{0\}$.

Theorem 2.6. *Let X be strangled by a subcontinuum Y. Assume that X contains two open and connected disjoint subsets U_1, U_2 such that $Y \setminus U_i$ is connected, $i = 1, 2$ but $Y \setminus (U_1 \cup U_2)$ is not connected. Then X is not unicoherent at Y.*

Proof. Let $H_i = (Y \setminus U_i) \cup Cl(\{Cl(C) : C \in \mathcal{K}(X \setminus Y), att(C) \in Y \setminus U_i\})$. Clearly, $H_i \in C(X)$, $i = 1, 2$ and $X = H_1 \cup H_2$. It follows from Lemma 2.3 *ii)*, that $H_i \cap U_i = \emptyset$ whenever $i, j \in \{1, 2\}$, $i \neq j$. This implies that $H_1 \cap H_2 \cap Y = Y \setminus (U_1 \cup U_2)$ which is not connected. ☐

In particular, suppose that X is strangled by $\mathbf{S}^1$ and that there exist $y_1, y_2 \in \mathbf{S}^1$ and connected disjoint neighborhoods of y_1 and y_2. Then it follows from Lemma 2.3 *i)*, that the hypothesis of the last theorem are satisfied.

3. Locally Connected Hausdorff Continua

In this section we consider Hausdorff continua, which are locally connected. For such spaces X we prove Theorem 11 which characterizes those $Y \in C(X)$ such that X is unicoherent at Y.

Recall that Hausdorff continuum means compact, connected and Hausdorff space (not necessarily metric). We will use the following definitions and results:

A *chain* in a space X is a finite family $\{U_1, ..., U_m\}$ of open subsets of X (called *links* of the chain) such that $U_i \cap U_j \neq \emptyset$ iff $| i - j | \leq 1$

Theorem 3.1. $[5, Theorem\ 3.4, p.108]$ *Let $\mathcal{W} \subseteq \mathcal{P}(X)$ be an open cover of a connected space X. Then for every $u, v \in X$ there is a chain from u to v whose links are elements of $\mathcal{W}$.*

THEOREM 3.2. $[6, ChV., 47, I, Theorem\ 3, p.168]$ *Let X be a Hausdorff continuum and $C \in C(X)$. Suppose that $X \setminus C = A \cup B$ is a separation of $X \setminus C$ (A and B are open and nonempty subsets of $X \setminus C$ and they are disjoint). Then $C \cup A$ and $C \cup B$ are Hausdorff continua*

THEOREM 3.3. $[6, ChV., 47, III, Theorem\ 2, p.172]$ *Let E be a proper and non-empty subset of a Hausdorff continuum X. If $U \in \mathcal{K}(E)$ then $Cl(U) \cap Bd(E) \neq \emptyset$.*

In what follows X stands for a locally connected, Hausdorff continuum.

LEMMA 3.4. *Let $Y \in C(X)$ and $\mathcal{U} \subset \mathcal{K}(X \setminus Y)$. Then*

$$Cl(\bigcup\{U : U \in \mathcal{U}\}) \setminus \bigcup\{U : U \in \mathcal{U}\} \subset Bd(Y)$$

PROOF. Let $x \in Cl(\bigcup\{U : U \in \mathcal{U}\}) \setminus \bigcup\{U : U \in \mathcal{U}\}$ and suppose $x \notin Bd(Y)$. Then $x \in X \setminus Y$, so that $x \in U_0$ for some $U_0 \in \mathcal{K}(X \setminus Y)$. Therefore $U_0 \notin \mathcal{U}$ and since $x \in Cl(\bigcup\{U : U \in \mathcal{U}\})$ and U_0 is open, $U_0 \cap (\bigcup\{U : U \in \mathcal{U}\}) \neq \emptyset$. But this is is impossible since the components are disjoint. $\square$

A similar version of Lemma 3.5, below, was proved in section 2 (Lemma 2.3 *i*). In the present case we do not require that X be metric and we only require connectedness for the subset V of X. Instead, X is assumed to be locally connected.

LEMMA 3.5. *Let $Y \in C(X)$. Suppose that X is strangled by Y. Then for each connected subset V of X, $V \cap Y$ is also a connected subset of X.*

PROOF. We may assume that $V \cap Y \neq \emptyset$. Let $V \cap Y = A \cup B$ where $Cl(A) \cap B = \emptyset = Cl(B) \cap A$. It follows immediately that $Cl(A) \cap Cl(B) \subset Y \setminus V$.

Define $M^* = Cl(\bigcup\{U \in \mathcal{K}(X \setminus Y) : att(U) \in A\})$ and $M = M^* \cap V$. Analogously, let $N^* = Cl(\bigcup\{U \in \mathcal{K}(X \setminus Y) : att(U) \in B\}$ and $N = N^* \cap V$. In what follows it will be proved that V is the union of the sets $M \cup A$ and $N \cup B$ and that these two sets are separated. Therefore one of them shall be empty, say $N \cup B = \emptyset$. This implies $B = \emptyset$ and proves that $V \cap Y$ is connected.

We assert that $V \setminus Y \subseteq M \cup N$. Indeed, if $x \in V \setminus Y$ then $x \in U \in \mathcal{K}(X \setminus Y)$ so that $x \in V \cap U$. On the other hand $V \cap (X \setminus U) \neq \emptyset$ because $V \cap Y \neq \emptyset$. Then, since V is connected, $V \cap\, Bd(U) \neq \emptyset$ and therefore $Bd(U) = att(U) = V \cap Bd(U) \subset V \cap Y = A \cup B$ and it follows that $Cl(U) \subseteq M^* \cup N^*$. Hence $Cl(U) \cap V \subset M \cup N$ and therefore $x \in (M \cup N)$.

It follows now that $V = (V \cap Y) \cup (V \setminus Y) = (A \cup B) \cup (M \cup N) = (M \cup A) \cup (N \cup B)$. In order to verify that these two sets are separated it will be enough to prove that $Cl(M \cup A) \cap (N \cup B) = \emptyset$. (Similarly $(M \cup A) \cap Cl(N \cup B) = \emptyset$).

We assert that $M^* \cap Bd(Y) \subset Cl(A)$ (1)

Let $x \in M^* \cap Bd(Y)$. Any open set containing x, contains an open and connected set W containing x. Since $x \in M^*$, then $W \cap U \neq \emptyset$ for some U

in the set defining M^*. Hence $W \cap Bd(U) \neq \emptyset$ so that $W \cap A \neq \emptyset$ and this proves that $x \in Cl(A)$.

Similarly $N^* \cap Bd(Y) \subset Cl(B)$ (2)

Now we consider the equality:

$$Cl(M \cup A) \cap (N \cup B) = (Cl(M) \cap N) \cup (Cl(M) \cap B) \\ \cup (Cl(A) \cap N) \cup (Cl(A) \cap B)$$

and prove that each one of the parenthesis on its right side is an empty set.

Let $x \in Cl(M) \cap N$. Then $x \in M^* \setminus \bigcup \{U \in \mathcal{K}(X \setminus Y) : att(U) \in A\}$. By Lemma 3.4, $x \in Bd(Y)$ and by (1), $x \in Cl(A)$. Similarly, $x \in Cl(B)$, so that $x \in Cl(A) \cap Cl(B) \subseteq Y \setminus V$. This contradicts that $x \in V$ and proves that $Cl(M) \cap N = \emptyset$.

Now, let $x \in Cl(M) \cap B$. Again, by Lemma 3.4, $x \in Bd(Y)$ and by (1), $x \in Cl(A)$. But this is a contradiction since $Cl(A) \cap B = \emptyset$.

Since $Cl(A) \cap B = \emptyset$, it only remains to prove that $Cl(A) \cap N = \emptyset$. Let $x \in Cl(A) \cap N$. Then $x \in Bd(Y)$. By (2) $x \in Cl(B)$. Therefore $x \in Cl(A) \cap Cl(B)$ so that $x \notin V$. $\square$

Theorem 3.6. *Assume that X is unicoherent at $Y \in C(X)$. Then, X is strangled by Y.*

Proof. Let U be any component of $X \setminus Y$, then we have to prove that $Cl(U) \cap Y$ is a single point. Since the boundary of every nonempty, proper subset of a connected space X is nonempty, we only need to prove that $Bd(U)$ contains no more than one point. Since X is a regular and locally connected space, then U is open [5,Theorem 3.2,p.106] and for each $u \in U$ there is an open and connected subset W_u of U such that $u \in W_u \subset Cl(W_u) \subset U$. Let us suppose that there are two different points p and q in $Bd(U)$ and let P and Q be open and connected neighborhoods of p and q respectively such that $Cl(P) \cap Cl(Q) = \emptyset$. Let $u \in P \cap U$ and $v \in Q \cap U$. Since U is connected, there exists, by Theorem 3.1, a finite set $F \subset U$ such that the set $\{W_x : x \in F\}$ is a chain from u to v. Therefore $H = (\bigcup_{x \in F} Cl(W_x)) \cup Cl(P) \cup Cl(Q)$ is a subcontinuum of X (in the metric case, by [7,Theorem 8.26 p.132] an arc from u to v can be taken instead of $\bigcup_{x \in F} Cl(W_x)$). Now we consider two cases:

i) $X \setminus H$ is connected. Then $Cl(X \setminus H)$ is a subcontinuum of X and $X = H \cup Cl(X \setminus H)$. It follows from the definition of H, that $H \cap Cl(X \setminus H) \cap Y = (Bd(P) \cup Bd(Q)) \cap Y$, so that $H \cap Cl(X \setminus H) \cap Y = Bd_Y(P) \cup Bd_Y(Q)$. Since Y is connected then $Bd_Y(P)$ and $Bd_Y(Q)$ are nonempty subsets of Y. Moreover each one of them is closed and they are disjoint. This proves that $H \cap Y \cap Cl(X \setminus H)$ is not connected.

ii) $X \setminus H$ is not connected. Let $X \setminus H = A \cup B$ be a separation of $X \setminus H$. Then, by Theorem 3.1, $X = (A \cup H) \cup (B \cup H)$ is a decomposition of X into two of its subcontinua. On the other hand $(A \cup H) \cap (B \cup H) \cap Y = H \cap Y = (Cl(P) \cap Y) \cup (Cl(Q) \cap Y)$ gives a separation of the set $(A \cup H) \cap (B \cup H) \cap Y$, so that it is not connected. $\square$

The following example shows that the converse of Theorem 3.6 fails to be true.

EXAMPLE 3.1. *Let X be a figure eight. In other words, X is the union of two circumferences intersecting in exactly one point p. Let Y be one of the two circumferences. Then X is a locally connected continuum which is not unicoherent at $Y \in \Gamma(X)$. Nevertheless, the boundary of the connected set $X \setminus Y$ is the singleton $\{p\}$.*

We recall that a *cut point* of a connected space X is a point $p \in X$ such that $X \setminus \{p\}$ is not connected.

COROLLARY 3.7. *Assume that X has no cut points. Then X is not unicoherent at any $Y \in \Gamma(X)$.*

PROOF. We notice that the boundary of a nonempty and open subset U of X whose complement contains more than one point, contains at least two points. Indeed, $Bd(U)$ is nonempty since X is connected. On the other hand if $Bd(U) = \{p\}$ then $Bd(X \setminus U) = \{p\}$ and $X \setminus \{p\} = U \cup (X \setminus (U \cup \{p\}))$ is a separation of $X \setminus \{p\}$, so that p is a cut point of X. Now, since $X \setminus Y$ is open and X is locally connected, then each $U \in \mathcal{K}(X \setminus Y)$ is an open set. Since $Y \in \Gamma(X)$ then U is a proper subset of X whose complement is not a single point. Therefore, $Bd(U)$ has more than one point and hence, by Theorem 3.6, X is not unicoherent at Y. □

THEOREM 3.8. *Suppose that X is unicoherent at $Y \in C(X)$. Then Y is unicoherent.*

PROOF. Let H and K be subcontinua of Y such that $Y = H \cup K$. We need to prove that $H \cap K$ is connected.

Let $\tilde{H}$ (resp. $\tilde{K}$) be the family of $U \in \mathcal{K}(X \setminus Y)$ such that $U \cap H \neq \emptyset$ (resp. $U \cap K \neq \emptyset$).

Let $M = H \cup \bigcup \left\{ U : U \in \tilde{H} \right\}$ and $N = K \cup \bigcup \left\{ U : U \in \tilde{K} \right\}$. It is clear that M and N are connected subsets of X. It follows from Theorem 3.3 that $X = M \cup N$. To prove that M is a closed set, take $x \in Cl(M) \setminus M$, hence $x \in Cl(\bigcup \left\{ U : U \in \tilde{H} \right\}) \setminus \bigcup \left\{ U : U \in \tilde{H} \right\}$. Hence, by Lemma 3.4, $x \in Bd(Y)$. Since $x \notin M$ then $x \notin H$. Let W be an open set such that $x \in W \subseteq X \setminus H$. There exists $U_0 \in \tilde{H}$ such that $W \cap U_0 \neq \emptyset$. Since U_0 is an open set of X then $W \cap U_0$ is an open and nonempty subset of W and it is also a proper subset of W since $x \in W \setminus U_0$. On the other hand, by Theorem 3.6, $Bd(U_0) = att(U_0) \in H$, so that $W \cap U_0$ is a closed subset of W. Hence W is not a connected set. This contradicts that X is a locally connected space and proves that $X = M \cup N$ is a decomposition of X into two of its subcontinua. Therefore, by hypothesis, $M \cap N \cap Y$ is a connected set and since $H \cap K = M \cap N \cap Y$ then $H \cap K$ is connected. □

THEOREM 3.9. *Let X be a locally connected and Hausdorff continuum. Then X is unicoherent at Y if and only if the following two conditions are satisfied:*

i) X is strangled by Y and

ii) Y is unicoherent.

PROOF. The necessity follows from Theorems 3.6 and 3.8 For the sufficiency, let H and K be subcontinua of X such that $X = H \cup K$. By Lemma 3.5, $H \cap Y$ and $K \cap Y$ are subcontinua of Y and since $Y = (H \cap Y) \cup (K \cap Y)$ and Y is unicoherent then $H \cap K \cap Y$ is connected, so that X is unicoherent at Y. $\square$

The following example shows that, if local connectedness is dropped in the last theorem then conditions *i)* and *ii)* are not necessarily satisfied.

EXAMPLE 3.2. *let X consists of a circumference Y contained in the Euclidean plane and a spiral (homeomorphic copy of a ray) converging to Y. Then X is unicoherent at Y but neither i) nor ii) are satisfied.*

THEOREM 3.10. *Let X be a locally connected continuum. X strangled by Y and X is unicoherent, then X is unicoherent at Y.*

PROOF. Let H and K be subcontinua of X such that $X = H \cup K$. Then $H \cap K$ is connected and, by Lemma 3.4, $H \cap K \cap Y$ is connected, so that X is unicoherent at Y.

Nevertheless, the converse is not true. Indeed, let X be the union of a circumference C and an arc Y such that $C \cap Y$ is one of the end points of Y. Then X is unicoherent at Y but X is not unicoherent.

As a consequence of Theorem 3.1 and Lemma 3.5 we have the following Theorem.

THEOREM 3.11. *Let X be a locally connected continuum which is unicoherent at $Y \in C(X)$. Then Y is locally connected.*

The following Theorem generalizes Theorem 1 in [3].

THEOREM 3.12. *Let X be a locally connected metric continuum. Suppose that X is unicoherent at $Y \in C(X)$ and Y is one dimensional. Then Y is a dendrite.*

PROOF. By Theorems 3.8 and 3.11, Y is locally connected and unicoherent. Every one dimensional, locally connected and unicoherent metric continuum is a dendrite, [6,VIII,57,III,Corollary8,p.442] , so Y is a dendrite. $\square$

The following characterization of dendrites follows immediately from Theorem 3.4:

A locally connected metric continuum X is a dendrite iff X is unicoherent at Y for every $Y \in C(X)$. A stronger version of this characterization is proved in [8,Theorem 3.7 p.155]

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