LOCAL CONNECTEDNESS AND UNICOHERENCE AT SUBCONTINUA

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Abstract. Let $X$ be a continuum and $Y$ a subcontinuum of $X$. The purpose of this paper is to investigate the relation between the conditions “$X$ is unicoherent at $Y$” and “$Y$ is unicoherent”. We say that $X$ is strangled by $Y$ if the closure of each component of $X \setminus Y$ intersects $Y$ in one single point. We prove: If $X$ is strangled by $Y$ and $Y$ is unicoherent then $X$ is unicoherent at $Y$. We also prove the converse for a locally connected (not necessarily metric) continuum $X$.

1. Introduction

In this paper continuum means a compact, connected and metric space. A subcontinuum of a space $X$ is a subspace of $X$ which is a continuum. In section 3 we also consider compact, connected and Hausdorff spaces (not necessarily metric). These spaces will be called Hausdorff continua.

The continuum $X$ is said to be unicoherent if every pair of subcontinua of $X$ whose union is $X$ has connected intersection. The concept of unicoherence at a subcontinuum of a metric continuum is due to M. A. Owens [8]. The same definition may include the nonmetric case. It is said that $X$ is unicoherent at a subcontinuum $Y$ of $X$ if for every pair $H$ and $K$ of subcontinua of $X$ whose union is $X$, the intersection $H \cap K \cap Y$ is a subcontinuum of $X$.

First of all, observe that neither one of the following implications is true:

1) $X$ is unicoherent at a subcontinuum $Y$ $\Rightarrow$ $Y$ is unicoherent.
2) $Y$ is unicoherent $\Rightarrow$ $X$ is unicoherent at $Y$.

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Indeed, as a counterexample to the first one, let $X$ consists of a circumference $Y$ and a spiral (homeomorphic copy of $[0, \infty)$) converging to $Y$. For the second, take $X$ as a two dimensional cell and $Y$ any subarc of $X$. (Notice that, in this example, $X$ is unicoherent and locally connected).

The purpose of this paper is to investigate under which additional properties, the concepts “$X$ is unicoherent at a subcontinuum $Y$” and “$Y$ is unicoherent” are equivalent.

We define: A continuum $X$ is strangled by a subcontinuum $Y$ if the intersection of $Y$ with the closure of each component of $X \setminus Y$ consists of a single point. When $X$ is a semi-locally connected continuum, $X$ is strangled by $Y$ if and only if $Y$ is the union of cyclic elements (see [9,IV, Theorem 3.3, p.67]). We also observe that every locally connected (metric) continuum is semi-locally connected [9,I,Corollary 13.21, p.20]

We prove the following result. Assume that $X$ is strangled by $Y$ and $Y$ is unicoherent. Then $X$ is unicoherent at $Y$. Since the converse is not true (Example 1) we discuss the problem under properties concerning local connectedness.

In section 3 we prove Theorem 11 which characterizes those subcontinua $Y$ of a locally connected (not necessarily metric) continuum $X$ such that $X$ is unicoherent at $Y$.

A dendrite is a locally connected and hereditarily unicoherent metric continuum. Characterizations of dendrites in terms of unicoherence at subcontinua are given in [1,3,7,8]. As a corollary of our results, we prove the following generalization of Theorem 1 in [3]: If a locally connected metric continuum $X$ is unicoherent at a one-dimensional subcontinuum $Y$ then $Y$ is a dendrite. (Theorem 13)

Recently, some papers have been written about unicoherence at subcontinua ([3,4,10]). In particular, these papers deal with some questions posed in [2] about mappings preserving unicoherence at subcontinua.

We will use the following notation in this paper:

$P(X)$ denotes the family of subsets of $X$, $C(X)$ is the set of all subcontinua of $X$ and $\Gamma(X) = C(X) \setminus \{\{x\} : x \in X\}$. If $Z$ is a subset of $X$, the set of components of $Z$ will be denoted by $K(Z)$.

2. Strangled.

We prove here that under the condition “$X$ is strangled by $Y$”, $Y$ is unicoherent implies $X$ is unicoherent at $Y$, the converse is discussed in this section.

The following results will be used below:

Theorem 2.1. [6, ChV, 48, VIII, Theorem 5, p.220] If a space $K$ is irreducibly connected between the closed sets $M$ and $N$ then $K \setminus (M \cup N)$ is connected and dense in $K$. 
Theorem 2.2. [6, ChV., 48, IX, Theorem 3, p.223] If an indecomposable continuum \( X \) is irreducibly connected between two closed sets \( M \) and \( N \), then there exists a composant \( L \) such that \( L \cap (M \cup N) = \emptyset \).

If \( X \) is strangled by \( Y \) and \( C \in K(X \setminus Y) \) we call the unique point in \( Cl(C) \cap Y \) the attaching point of \( C \), and we denote it by \( att(C) \).

Lemma 2.3. Let \( X \) be a metric continuum which is strangled by \( Y \in C(X) \). Then

i) For every \( H \in C(X) \), \( H \cap Y \) is connected.

ii) \( att(C) \in H \) whenever \( C \in K(X \setminus Y) \), and \( H \cap C \neq \emptyset \neq H \cap (X \setminus C) \).

iii) \( H \cap Cl(C) \) is connected for every \( C \in K(X \setminus Y) \)

Proof. i) Suppose \( H \cap Y \) is not connected, so that \( H \cap Y = M \mid N \). Let \( K \in C(H) \) be irreducible between \( M \) and \( N \). We consider the following two cases:

Case 1) \( K \) is decomposable. Since \( K \) is irreducibly connected between the closed sets \( M \cap K \) and \( N \cap K \), it follows from Theorem 1 that \( K \setminus (M \cup N) = K \setminus Y \) is a connected subset which is dense in \( K \). Therefore \( K \setminus (M \cup N) \) is contained in \( C \in K(X \setminus Y) \) and \( K = Cl(K \setminus (M \cup N)) \subset Cl(C) \). But this is a contradiction since \( Cl(C) \cap Y \) contains a single point, while \( K \cap Y \) contains at least two points (one in \( M \) and one in \( N \)).

Case 2) \( K \) is indecomposable. Then by Theorem 2 there exists a composant \( L \) of \( K \) contained in \( K \setminus (M \cup N) \). Being a composant, \( L \) is a connected subset which is dense in \( K \). Since \( L \) is contained in \( K \setminus Y \), \( L \) is contained in some \( C \in K(X \setminus Y) \) so that \( K \cap Y = Cl(L) \cap Y \) is contained in \( Cl(C) \cap Y \) and this again is a contradiction.

ii) The hypothesis imply that \( H \cap Y \neq \emptyset \). Let \( K \in C(H) \) be irreducible between \( H \cap Y \) and a point \( p \in H \cap C \). We consider the same two cases than in the proof of i) and proceed in the same way with \( \{p\} = M \) and \( H \cap Y = N \). Case 1) \( K \) is decomposable. Then the set \( L = K \setminus (M \cup N) \) is a connected subset of \( K \) which is dense in \( K \). Since \( L \) is contained in \( X \setminus Y \) then it is contained in \( D \in K(X \setminus Y) \). Therefore \( K = Cl(L) \subset Cl(D) = D \cup att(D) \).

Since \( p \in K \), \( D = C \) so that \( att(C) \in K \subset H \).

Case 2) \( K \) is indecomposable. The composant \( L \) is contained in \( D \in K(X \setminus Y) \). On the other hand since \( K = Cl(L) \subset Cl(D) = D \cup att(D) \), then \( D = C \) and \( att(C) \in K \subset H \).

iii) Suppose that \( H \cap Cl(C) \) is not connected, so it is clear that \( H \cap C \neq \emptyset \).

Therefore, by ii) \( att(C) \in H \). Write \( H \cap Cl(C) = M \mid N \) and suppose that \( att(C) \in N \). Let \( K \in C(H) \) be irreducible between \( M \) and \( Y \). As above, we consider two cases and again we get a connected and dense subset \( L \) of \( K \setminus Y \) which intersects \( C \). Therefore \( L \subset C \) which implies \( K \subset Cl(C) \). This shows that \( H \cap Cl(C) \) contains a connected set intersecting \( M \) and \( N \) contrary to the assumption. \( \blacksquare \)
Figure 1

**Theorem 2.4.** Let $X$ be a continuum and $Y \in C(X)$. Suppose that $X$ is strangled by $Y$ and $Y$ is unicoherent. Then $X$ is unicoherent at $Y$.

**Proof.** Follows from Lemma 2.3 i)

The following example shows that the converse of Theorem 3 is not true.

**Example 2.1 (See Figure 1).** Let $Y \subseteq \mathbb{R}^2$ be the union of $S^1$ and the arc $[1, 2] \times \{0\}$. Let $C_n = \{(1 + \frac{1}{n})(\cos \theta, \sin \theta) : \theta \in [0, (2 - \frac{1}{n})\pi]\}$ and $X = Y \cup \bigcup_{n \in \mathbb{N}} C_n$. It is easy to verify that $X$ is strangled by $Y$. In order to prove that $X$ is unicoherent at $Y$, suppose that $X = H \cup K$. Then, for infinitely many indices $n \in \mathbb{N}$, $C_n \subseteq H$. Since $S^1 \subseteq Cl(\bigcup C_n)$ then we can assume that for some $a \leq 2$, $H \cap Y = S^1 \cup ([1, a] \times \{0\})$, so that $H \cap K \cap Y = K \cap (S^1 \cup [1, a] \times \{0\})$ which is a connected set by Lemma 2.3 i).

Nevertheless we have the following Theorem

**Theorem 2.5.** Let $X$ be a continuum and $Y \in C(X)$. Assume that $X$ is locally connected at each point of $Bd(Y)$. If $X$ is strangled by $Y$ and $X$ is unicoherent at $Y$ then $Y$ is unicoherent.

**Proof.** Suppose that $Y$ is not unicoherent so that $Y = H \cup K$ where $H$, $K \in C(X)$ and $H \cap K$ is not connected. Let $ar{H} = H \cup \nabla_H$ where $\nabla_H$ is the closure of the union of all components of $X \setminus Y$ whose attaching point is in $H$. Similarly define $\bar{K} = K \cup \nabla_K$.

We want to prove that $\bar{H} \cap \bar{K} \cap Y$ is not connected, contrary to the hypothesis.

Let $x \in (H \cap \nabla_K) \setminus K$. Then $x = \lim x_n$ where $x_n \in C_n \in K(X \setminus Y)$ and $att(C_n) \in K$. Since $x \notin K$ there is an open and connected subset $U$ of $X$ such that $x \in U \subseteq Cl(U) \subseteq H \setminus K$ so that $x_n \in Cl(U)$ for $n$ large enough. This implies that for some fixed $n \in \mathbb{N}$, $Cl(U) \cap C_n \neq \emptyset$ and $Cl(U) \cap X \setminus C_n \neq \emptyset$. Therefore, by Lemma 2.3 ii) $att(C_n) \in Cl(U)$ and this is a contradiction.
This proves that $H \cap \nabla_K \subseteq H \cap K$. Similarly $K \cap \nabla_H \subseteq H \cap K$. On the other hand, since $Y = H \cup K$, then $(\nabla_H \cap \nabla_K) \cap Y \subseteq H \cap K$.

Therefore the equality

$\tilde{H} \cap \tilde{K} \cap Y = (H \cap K) \cup (H \cap \nabla_K) \cup (K \cap \nabla_H) \cup (\nabla_H \cap \nabla_K) \cap Y$

becomes

$\tilde{H} \cap \tilde{K} \cap Y = H \cap K$ and this proves that $\tilde{H} \cap \tilde{K} \cap Y$ is not connected, as desired.

uestion: Let $X$ be a continuum and $Y \in C(X)$. Assume that $X$ is locally connected at each point of $\text{Bd}(Y)$ and $X$ is unicoherent at $Y$. Is it true that $Y$ intersects the closure of each component of $X \setminus Y$ in a connected set? Is $X$ strangled by $Y$?

Example 2.1 shows that, in Theorem 2.5, the hypothesis $X$ is locally connected at every point of $\text{Bd}(Y)$ cannot be changed by $X$ is locally connected at some points of $\text{Bd}(Y)$. Indeed It is easy to verify that $X$ is locally connected at every point in $(1, 2) \times \{0\}$.

Theorem 2.6. Let $X$ be strangled by a subcontinuum $Y$. Assume that $X$ contains two open and connected disjoint subsets $U_1, U_2$ such that $Y \setminus U_i$ is connected, $i = 1, 2$ but $Y \setminus (U_1 \cup U_2)$ is not connected. Then $X$ is not unicoherent at $Y$.

Proof. Let $H_i = (\tilde{Y} \setminus U_i) \cup \text{Cl}\{\text{Cl}(C) : C \in K(X \setminus Y), \text{att}(C) \in Y \setminus U_i\}$. Clearly, $H_i \in C(X)$, $i = 1, 2$ and $X = H_1 \cup H_2$. It follows from Lemma 2.3 ii), that $H_i \cap U_i = \emptyset$ whenever $i, j \in \{1, 2\}$, $i \neq j$. This implies that $H_1 \cap H_2 \cap Y = Y \setminus (U_1 \cup U_2)$ which is not connected.

In particular, suppose that $X$ is strangled by $S^1$ and that there exist $y_1, y_2 \in S^1$ and connected disjoint neighborhoods of $y_1$ and $y_2$. Then it follows from Lemma 2.3 i), that the hypothesis of the last theorem are satisfied.

3. Locally Connected Hausdorff Continua

In this section we consider Hausdorff continua, which are locally connected. For such spaces $X$ we prove Theorem 11 which characterizes those $Y \in C(X)$ such that $X$ is unicoherent at $Y$.

Recall that Hausdorff continuum means compact, connected and Hausdorff space (not necessarily metric). We will use the following definitions and results:

A chain in a space $X$ is a finite family $\{U_1, \ldots, U_m\}$ of open subsets of $X$ (called links of the chain) such that $U_i \cap U_j \neq \emptyset$ iff $|i - j| \leq 1$.

Theorem 3.1. [5, Theorem 3.4, p.108] Let $\mathcal{W} \subseteq \mathcal{P}(X)$ be an open cover of a connected space $X$. Then for every $u, v \in X$ there is a chain from $u$ to $v$ whose links are elements of $\mathcal{W}$. 
Theorem 3.2. [6, ChV, 47.1, Theorem 3, p.168] Let $X$ be a Hausdorff continuum and $C \subseteq C(X)$. Suppose that $X \setminus C = A \cup B$ is a separation of $X \setminus C$ ($A$ and $B$ are open and nonempty subsets of $X \setminus C$ and they are disjoint). Then $C \cup A$ and $C \cup B$ are Hausdorff continua.

Theorem 3.3. [6, ChV, 47, III, Theorem 2, p.172] Let $E$ be a proper and non-empty subset of a Hausdorff continuum $X$. If $U \in \mathcal{K}(E)$ then $Cl(U) \cap Bd(E) \neq \emptyset$.

In what follows $X$ stands for a locally connected, Hausdorff continuum.

Lemma 3.4. Let $Y \subseteq C(X)$ and $U \subset \mathcal{K}(X \setminus Y)$. Then

$$Cl\left(\bigcup \{U : U \in \mathcal{U}\}\right) \setminus \bigcup \{U : U \in \mathcal{U}\} \subseteq Bd(Y)$$

Proof. Let $x \in Cl\left(\bigcup \{U : U \in \mathcal{U}\}\right) \setminus \bigcup \{U : U \in \mathcal{U}\}$ and suppose $x \notin Bd(Y)$. Then $x \in X \setminus Y$, so that $x \in U_0$ for some $U_0 \in \mathcal{K}(X \setminus Y)$. Therefore $U_0 \notin \mathcal{U}$ and since $x \in Cl(\bigcup \{U : U \in \mathcal{U}\})$ and $U_0$ is open, $U_0 \cap (\bigcup \{U : U \in \mathcal{U}\}) \neq \emptyset$. But this is impossible since the components are disjoint.

A similar version of Lemma 3.5, below, was proved in section 2 (Lemma 2.3). In the present case we do not require that $X$ be metric and we only require connectedness for the subset $V$ of $X$. Instead, $X$ is assumed to be locally connected.

Lemma 3.5. Let $Y \subseteq C(X)$. Suppose that $X$ is strangled by $Y$. Then for each connected subset $V$ of $X$, $V \cap Y$ is also a connected subset of $X$.

Proof. We may assume that $V \cap Y \neq \emptyset$. Let $V \cap Y = A \cup B$ where $Cl(A) \cap B = \emptyset = Cl(B) \cap A$. It follows immediately that $Cl(A) \cap Cl(B) \subseteq Y \setminus V$.

Define $M^* = Cl(\bigcup \{U \in \mathcal{K}(X \setminus Y) : att(U) \in A\})$ and $M = M^* \cap V$. Analogously, let $N^* = Cl(\bigcup \{U \in \mathcal{K}(X \setminus Y) : att(U) \in B\})$ and $N = N^* \cap V$. In what follows it will be proved that $V$ is the union of the sets $M \cup A$ and $N \cup B$ and that these two sets are separated. Therefore one of them shall be empty, say $N \cup B = \emptyset$. This implies $B = \emptyset$ and proves that $V \cap Y$ is connected.

We assert that $V \cap Y \subseteq M \cup N$. Indeed, if $x \in V \cap Y$ then $x \in \mathcal{K}(X \setminus Y)$ so that $x \in V \cap U$. On the other hand $V \cap (X \setminus U) \neq \emptyset$ because $V \cap Y \neq \emptyset$. Then, since $V$ is connected, $V \cap Bd(U) \neq \emptyset$ and therefore $Bd(U) = att(U) = V \cap Bd(U) \subset V \cap Y = A \cup B$ and it follows that $Cl(U) \subseteq M^* \cup N^*$. Hence $Cl(U) \cap V \subseteq M \cup N$ and therefore $x \in (M \cup N)$.

It follows now that $V = (V \cap Y) \cup (V \setminus Y) = (A \cup B) \cup (M \cup N) = (M \cup A) \cup (N \cup B)$. In order to verify that these two sets are separated it will be enough to prove that $Cl(M \cup A) \cap (N \cup B) = \emptyset$. (Similarly $(M \cup A) \cap Cl(N \cup B) = \emptyset$).

We assert that $M^* \cap Bd(Y) \subseteq Cl(A)$ (1)

Let $x \in M^* \cap Bd(Y)$. Any open set containing $x$, contains an open and connected set $W$ containing $x$. Since $x \in M^*$, then $W \cap U \neq \emptyset$ for some $U$.
in the set defining $M^*$. Hence $W \cap \text{Bd}(U) \neq \emptyset$ so that $W \cap A \neq \emptyset$ and this proves that $x \in \text{Cl}(A)$.

Similarly $N^* \cap \text{Bd}(Y) \subset \text{Cl}(B)$ \hfill (2)

Now we consider the equality:

$$\text{Cl}(M \cup A) \cap (N \cup B) = (\text{Cl}(M) \cap N) \cup (\text{Cl}(M) \cap B)$$

and prove that each one of the parenthesis on its right side is an empty set.

Let $x \in \text{Cl}(M) \cap N$. Then $x \in M^* \setminus \bigcup \{U \in K(X \setminus Y) : \text{att}(U) \in A\}$. By Lemma 3.4, $x \in \text{Bd}(Y)$ and by (1), $x \in \text{Cl}(A)$. Similarly, $x \in \text{Cl}(B)$, so that $x \in \text{Cl}(A) \cap \text{Cl}(B) \subset Y \setminus V$. This contradicts that $x \in V$ and proves that $\text{Cl}(M) \cap N = \emptyset$.

Now, let $x \in \text{Cl}(M) \cap B$. Again, by Lemma 3.4, $x \in \text{Bd}(Y)$ and by (1), $x \in \text{Cl}(A)$. But this is a contradiction since $\text{Cl}(A) \cap B = \emptyset$.

Since $\text{Cl}(A) \cap B = \emptyset$, it only remains to prove that $\text{Cl}(A) \cap N = \emptyset$. Let $x \in \text{Cl}(A) \cap N$. Then $x \in \text{Bd}(Y)$.

By (2) $x \in \text{Cl}(B)$. Therefore $x \in \text{Cl}(A) \cap \text{Cl}(B)$ so that $x \notin V$.

**Theorem 3.6.** Assume that $X$ is unicoherent at $Y \in \text{Cl}(X)$. Then, $X$ is strangled by $Y$.

**Proof.** Let $U$ be any component of $X \setminus Y$, then we have to prove that $\text{Cl}(U) \cap Y$ is a single point. Since the boundary of every nonempty, proper subset of a connected space $X$ is nonempty, we only need to prove that $\text{Bd}(U)$ contains no more than one point. Since $X$ is a regular and locally connected space, then $U$ is open [5, Theorem 3.2, p.106] and for each $u \in U$ there is an open and connected subset $W_u$ of $U$ such that $u \in W_u \subset \text{Cl}(W_u) \subset U$. Let us suppose that there are two different points $p$ and $q$ in $\text{Bd}(U)$ and let $P$ and $Q$ be open and connected neighborhoods of $p$ and $q$ respectively such that $\text{Cl}(P) \cap \text{Cl}(Q) = \emptyset$. Let $u \in P \cap U$ and $v \in Q \cap U$. Since $U$ is connected, there exists, by Theorem 3.1, a finite set $F \subset U$ such that the set $\{W_x : x \in F\}$ is a chain from $u$ to $v$. Therefore $H = \bigcup_{x \in F} \text{Cl}(W_x) \cup \text{Cl}(P) \cup \text{Cl}(Q)$ is a subcontinuum of $X$ (in the metric case, by [7, Theorem 8.26 p.132] an arc from $u$ to $v$ can be taken instead of $\bigcup_{x \in F} \text{Cl}(W_x)$). Now we consider two cases:

i) $X \setminus H$ is connected. Then $\text{Cl}(X \setminus H)$ is a subcontinuum of $X$ and $X = H \cup \text{Cl}(X \setminus H)$. It follows from the definition of $H$, that $H \cap \text{Cl}(X \setminus H) \cap Y = (\text{Bd}(P) \cup \text{Bd}(Q)) \cap Y$, so that $H \cap \text{Cl}(X \setminus H) \cap Y = \text{Bd}(P) \cup \text{Bd}(Q)$. Since $Y$ is connected then $\text{Bd}(P)$ and $\text{Bd}(Q)$ are nonempty subsets of $Y$. Moreover each one of them is closed and they are disjoint. This proves that $H \cap Y \cup \text{Cl}(X \setminus H)$ is not connected.

ii) $X \setminus H$ is not connected. Let $X \setminus H = A \cup B$ be a separation of $X \setminus H$.

Then, by Theorem 3.1, $X = (A \cup H) \cup (B \cup H)$ is a decomposition of $X$ into two of its subcontinua. On the other hand $(A \cup H) \cap (B \cup H) \cap Y = H \cap Y = (\text{Cl}(P) \cap Y) \cup (\text{Cl}(Q) \cap Y)$ gives a separation of the set $(A \cup H) \cap (B \cup H) \cap Y$, so that it is not connected.
The following example shows that the converse of Theorem 3.6 fails to be true.

**Example 3.1.** Let $X$ be a figure eight. In other words, $X$ is the union of two circumferences intersecting in exactly one point $p$. Let $Y$ be one of the two circumferences. Then $X$ is a locally connected continuum which is not unicoherent at $Y \in \Gamma(X)$. Nevertheless, the boundary of the connected set $X \setminus Y$ is the singleton $\{p\}$.

We recall that a cut point of a connected space $X$ is a point $p \in X$ such that $X \setminus \{p\}$ is not connected.

**Corollary 3.7.** Assume that $X$ has no cut points. Then $X$ is not unicoherent at any $Y \in \Gamma(X)$.

**Proof.** We notice that the boundary of a nonempty and open subset $U$ of $X$ whose complement contains more than one point, contains at least two points. Indeed, $\text{Bd}(U)$ is nonempty since $X$ is connected. On the other hand if $\text{Bd}(U) = \{p\}$ then $\text{Bd}(X \setminus U) = \{p\}$ and $X \setminus \{p\}$ is a separation of $X \setminus \{p\}$, so that $p$ is a cut point of $X$. Now, since $X \setminus Y$ is open and $X$ is locally connected, then each $U \in \mathcal{K}(X \setminus Y)$ is an open set. Since $Y \in \Gamma(X)$ then $U$ is a proper subset of $X$ whose complement is not a single point. Therefore, $\text{Bd}(U)$ has more than one point and hence, by Theorem 3.6, $X$ is not unicoherent at $Y$. ☐

**Theorem 3.8.** Suppose that $X$ is unicoherent at $Y \in C(X)$. Then $Y$ is unicoherent.

**Proof.** Let $H$ and $K$ be subcontinua of $Y$ such that $Y = H \cup K$. We need to prove that $H \cap K$ is connected.

Let $\hat{H}$ (resp. $\hat{K}$) be the family of $U \in \mathcal{K}(X \setminus Y)$ such that $U \cap H \neq \emptyset$ (resp. $U \cap K \neq \emptyset$).

Let $M = H \cup \bigcup \left\{U : U \in \hat{H}\right\}$ and $N = K \cup \bigcup \left\{U : U \in \hat{K}\right\}$. It is clear that $M$ and $N$ are connected subsets of $X$. It follows from Theorem 3.3 that $X = M \cup N$. To prove that $M$ is a closed set, take $x \in \text{Cl}(M) \setminus M$, hence $x \in \text{Cl}(\bigcup \left\{U : U \in \hat{H}\right\}) \setminus \bigcup \left\{U : U \in \hat{H}\right\}$. Hence, by Lemma 3.4, $x \in \text{Bd}(Y)$. Since $x \notin M$ then $x \notin H$. Let $W$ be an open set such that $x \in W \subseteq X \setminus H$. There exists $U_0 \in \hat{H}$ such that $W \cap U_0 \neq \emptyset$. Since $U_0$ is an open set of $X$ then $W \cap U_0$ is an open and nonempty subset of $W$ and it is also a proper subset of $W$ since $x \in W \setminus U_0$. On the other hand, by Theorem 3.6, $\text{Bd}(U_0) = \text{att}(U_0) \in H$, so that $W \cap U_0$ is a closed subset of $W$. Hence $W$ is not a connected set. This contradicts that $X$ is a locally connected space and proves that $X = M \cup N$ is a decomposition of $X$ into two of its subcontinua. Therefore, by hypothesis, $M \cap N \cap Y$ is a connected set and since $H \cap K = M \cap N \cap Y$ then $H \cap K$ is connected. ☐
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Theorem 3.9. Let $X$ be a locally connected and Hausdorff continuum. Then $X$ is unicoherent at $Y$ if and only if the following two conditions are satisfied:

i) $X$ is strangled by $Y$ and

ii) $Y$ is unicoherent.

Proof. The necessity follows from Theorems 3.6 and 3.8. For the sufficiency, let $H$ and $K$ be subcontinua of $X$ such that $X = H \cup K$. By Lemma 3.5, $H \cap Y$ and $K \cap Y$ are subcontinua of $Y$ and since $Y = (H \cap Y) \cup (K \cap Y)$ and $Y$ is unicoherent then $H \cap K \cap Y$ is connected, so that $X$ is unicoherent at $Y$.

The following example shows that, if local connectedness is dropped in the last theorem then conditions i) and ii) are not necessarily satisfied.

Example 3.2. Let $X$ consists of a circumference $Y$ contained in the Euclidean plane and a spiral (homeomorphic copy of a ray) converging to $Y$. Then $X$ is unicoherent at $Y$ but neither i) nor ii) are satisfied.

Theorem 3.10. Let $X$ be a locally connected continuum. $X$ strangled by $Y$ and $X$ is unicoherent, then $X$ is unicoherent at $Y$.

Proof. Let $H$ and $K$ be subcontinua of $X$ such that $X = H \cup K$. Then $H \cap K$ is connected and, by Lemma 3.4, $H \cap K \cap Y$ is connected, so that $X$ is unicoherent at $Y$.

Nevertheless, the converse is not true. Indeed, let $X$ be the union of a circumference $C$ and an arc $Y$ such that $C \cap Y$ is one of the end points of $Y$. Then $X$ is unicoherent at $Y$ but $X$ is not unicoherent.

As a consequence of Theorem 3.1 and Lemma 3.5 we have the following Theorem.

Theorem 3.11. Let $X$ be a locally connected continuum which is unicoherent at $Y \in C(X)$. Then $Y$ is locally connected.

The following Theorem generalizes Theorem 1 in [3].

Theorem 3.12. Let $X$ be a locally connected metric continuum. Suppose that $X$ is unicoherent at $Y \in C(X)$ and $Y$ is one dimensional. Then $Y$ is a dendrite.

Proof. By Theorems 3.8 and 3.11, $Y$ is locally connected and unicoherent. Every one dimensional, locally connected and unicoherent metric continuum is a dendrite, [6, VIII, 57, III, Corollary 8, p. 442], so $Y$ is a dendrite.

The following characterization of dendrites follows immediately from Theorem 3.4:

A locally connected metric continuum $X$ is a dendrite iff $X$ is unicoherent at $Y$ for every $Y \in C(X)$. A stronger version of this characterization is proved in [8, Theorem 3.7 p. 155]
References


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